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Sums, or amalgamations, of two abelian ordered groups with a subgroup amalgamated are constructed in two ways. These constructions are used to investigate the structure of the class of all amalgamations with the given groups and subgroup fixed, where the class is partially ordered in a natural way. In particular, necessary and sufficient conditions are found for there to be (a) exactly one amalgamation, up to equivalence, and (b) exactly one minimal amalgamation, up to equivalence.

It is known [5] that the class of abelian (totally) ordered groups (o-groups) has the amalgamation property. Relying on the injective property of η_α-groups, the indicated proof is existential in nature, and yields no information about the amalgamations. In this paper we present two ways to construct all amalgamations of a subgroup of two abelian o-groups. For the first way we merely consider a certain class of homomorphic images of the abelian amalgamated free product (Theorem 1.2). The technique quickly yields some general information about the structure of the class of all amalgamations of a given subgroup (Corollaries 1.3 and 3.3). The second way is more specific, involving the existence of certain embeddings of the groups into a Hahn group (Theorem 2.8). An amalgamation is called *minimal* if it admits no o-homomorphisms which are $1-1$ on the component groups. These turn out to be important because every amalgamation is a lexicographic extension of a non-essential group by a minimal amalgamation (Lemma 3.1). In § 4 we use the second construction to determine when there is precisely one minimal amalgamation (Theorem 4.3), and if there is more than one, how many there are (Theorem 4.6). In addition, we determine under what conditions there is exactly one amalgamation (Theorem 4.4).

The possibility exists that these techiques can be adapted to the class of abelian lattice-ordered groups (l-groups) since the author in [6] has shown that the class of abelian l-groups has the amalgamation property. The class of l-groups, however, does not have this property (see [6, Theorem 3.1]). The author in [6] and N. R. Reilly in [7] have determined some sufficient conditions for amalgamation to occur in this class.

NOTATION. All groups are additive and abelian. Z, Q, and R denote respectively the o-groups of integers, rational numbers, and real numbers. $|x|$ stands for the cardinality of x, and we let $|R| = c$. We sometimes denote a partially ordered group by $[H, P]$ where $P = H^+$ is the positive cone (partial order) of H. The lexicographic direct sum of the o-groups G and H is written as $G \oplus^* H$ (i.e., $0 < g + h$ if $0 < h$, or $h = 0$ and $0 < g$). The classes of groups and o-groups are denoted respectively by $\mathscr{G}$ and $\mathscr{O}$.

The reader should consult [3] for basic facts about o-groups. [4] is also a good reference.

1. Preliminaries; the free product construction. A class $\mathscr{K}$ of similar algebraic structures is said to have the *amalgamation property* if whenever G, H, and K are in $\mathscr{K}$ and $\sigma_1\colon G \to H$ and $\sigma_2\colon G \to K$ are embeddings, there exist $L \in \mathscr{K}$ and embeddings $\eta_1\colon H \to L$ and $\eta_2\colon K \to L$ such that $\sigma_1\eta_1 = \sigma_2\eta_2$. If L is generated by $H\eta_1 \cup K\eta_2$ then the triple (L, η_1, η_2) is called an *amalgamation* of G in H and K. For simplification, σ_1 and σ_2 will always be incusion maps, and we may just use L in place of (L, η_1, η_2). For fixed G, H, and K, we say that (L, η_1, η_2) is *freer than* (M, μ_1, μ_2) (denoted $L \succ M$) if there is a homomorphism $\theta\colon L \to M$ such that $\eta_i\theta = \mu_i$ $(i = 1, 2)$. If θ is an isomorphism then L and M are *equivalent* (denoted $L \approx M$). One easily shows that $\approx$ is an equivalence relation, $\succ$ is reflexive and transitive, and $L \approx M$ if and only if $L \succ M$ and $M \succ L$. $\mathscr{L}_{\mathscr{K}}(G, H, K)$ is a *representing set* of amalgamations of G in H and K if it consists of exactly one amalgamation out of every equivalence class. $\mathscr{L}_{\mathscr{K}}(G, H, K)$ is partially ordered by $\succ$, and any two such representing sets are canonically isomorphic. Our object is to determine much of the structure of $\mathscr{L}_{\mathscr{O}}(G, H, K)$.

If $\mathscr{L}_{\mathscr{K}}(G, H, K)$ has a greatest element, it is the *free product in $\mathscr{K}$ of H and K with G amalgamated.* Of interest here is the free product (F, μ_1, μ_2) in $\mathscr{G}$. It can be represented as

$$F = (H \oplus K)/G^*$$

where μ_1 and μ_2 are the natural embeddings and G^* consists of all pairs $(g\mu_1, -g\mu_2)$ $(g \in G)$.

Let X' denote the divisible closure of the o-group X. The order on X can be extended to X' by letting x' be positive if $nx' \in X^+$ for some positive integer n. Any o-homomorphism $\eta\colon X \to Y$ has a unique extension to $\eta'\colon X' \to Y'$, and η' is $1-1$ if and only if η is $1-1$. The next lemma justifies restricting our attention to the class of divisible o-groups. The proof is straightforward.

LEMMA 1.1. *If (L, η_1, η_2) is an amalgamation of G in the o-groups H and K, then (L', η_1', η_2') is an amalgamation of G' in H'*

and K', *and the induced map* $L \to L'$ *is an order isomorphism from* $\mathscr{L}_{\mathscr{O}}(G, H, K)$ *onto* $\mathscr{L}_{\mathscr{O}}(G', H', K')$.

From now on we will consider only divisible groups and subgroups. We will make frequent use of the fact that divisible subgroups are direct summands. Since division in an *o*-group is unique, we can alternately consider them as rational vector spaces and subspaces.

Let V be a rational vector space. Define the set $OQ(V)$ of *orderable quotients* of V to be the collection of all *o*-groups of the form $[V/I, P]$ where I is a rational subspace of V (V/I is torsion-free, hence orderable). Partially order $OQ(V)$ by defining $[V/I, P] \geqq [V/J, Q]$ if and only if $I \subseteqq J$ and the natural homomorphism

$$v + I \longmapsto v + J$$

preserves order.

THEOREM 1.2. *Let* G *be a subgroup of each of the o-groups* H *and* K *and let* (F, μ_1, μ_2) *be the free product in* $\mathscr{G}$ *of* H *and* K *with* G *amalgamated. Then* $\mathscr{L}_{\mathscr{O}}(G, H, K)$ *is isomorphic to the subset* $\mathscr{F}$ *of* $OQ(F)$ *consisting of all* $[F/I, P]$ *such that*

(1) $I \cap (H\mu_1 \cup K\mu_2) = 0$, *and*

(2) $H^+\mu_1 + K^+\mu_2 + I \subseteqq P$.

Proof. If $[F/I, P]$ is in $\mathscr{F}$ and $\phi(I)\colon F \to F/I$ is natural then $\mu_1\phi(I)$ and $\mu_2\phi(I)$ are *o*-embeddings. Thus the collection of $([F/I, P], \mu_1\phi(I), \mu_2\phi(I))$, $[F/I, P] \in \mathscr{F}$, is a collection of amalgamations in $\mathscr{O}$ of G in H and K. Furthermore, one easily checks that the partial order on $\mathscr{F}$ is identical with the induced partial order $\succ$. It remains to show that every equivalence class of amalgamations is represented. If $(L, \eta_1, \eta_2) \in \mathscr{L}_{\mathscr{O}}(G, H, K)$ then L is also an amalgamation in $\mathscr{G}$, whence there is a group epimorphism $\phi\colon F \to L$ such that $\mu_i\phi = \eta_i$ $(i = 1, 2)$. If $I = \ker(\phi)$ and $\theta\colon F/I \to L$ is natural then evidently (L, η_1, η_2) and $([F/I, L^+\theta^{-1}], \mu_1\phi, \mu_2\phi)$ are equivalent. This completes the proof.

COROLLARY 1.3. *Let* $\mathscr{L} = \mathscr{L}_{\mathscr{O}}(G, H, K)$. *Then*

(1) $\mathscr{L}$ *is an inverse root system; i.e., the set of all elements exceeded by a given element forms a chain,*

(2) *Every element of* $\mathscr{L}$ *exceeds exactly one minimal element, and is exceeded by at least one maximal element, and*

(3) *Each component of* $\mathscr{L}$ *(i.e., the set of elements exceeding a minimal element) is a lower semilattice.*

Proof. The set of ordered quotients below $[F/I, P]$ in $\mathscr{F}$ is evidently antiisomorphic to the set of all subgroups J, $I \subseteqq J \subseteqq F$, J/I convex in $[F/I, P]$, ordered by inclusion. But this is a totally ordered set. Hence (1) is proven. For $[F/I, P] \in \mathscr{F}$ let Q' be a total order on I and let Q be the total order $Q' \cup \{x \in F: x + I \in P\}$ on F. Then $[F, Q]$ is maximal in $\mathscr{F}$ and exceeds $[F/I, P]$. Going the other way let J be the largest subgroup of F such that $I \subseteqq J$, J/I is convex in $[F/I, P]$ and disjoint from $(H\mu_1 + K\mu_2 + I)/I$. If Q is the order on F/J induced by P, then $[F/J, Q]$ is minimal and exceeded by $[F/I, P]$. By (1) it is unique. Thus (2) is proven. Finally suppose $[F/I, P]$ and $[F/I', P']$ both exceed the minimal element $[F/J', Q']$ of $\mathscr{F}$. Then the collection of subgroups J such that J/I is convex in $[F/I, P]$ and J/I' is convex in $[F/I', P']$ is nonempty and linearly ordered by inclusion. If J^* is their intersection, then evidently $[F/J^*, P + J^*/J^*]$ is in $\mathscr{F}$ and the infimum of the two given ordered quotients. This proves (3) and completes the proof of the corollary.

2. Constructing amalgamations using Hahn embeddings.

LEMMA 2.1 [7]. *Every partial order on a set can be extended to a total order.*

LEMMA 2.2. *The class of ordered sets has the amalgamation property. More specifically, if A is a subset of the ordered sets B and Γ, and β^* and γ^* determine the same cut of A—i.e., $\alpha \leqq \beta^*$ if and only if $\alpha \leqq \gamma^* (\alpha \in A)$—then there is an amalgamation $\Delta(\Delta')$ of A in B and Γ in which $\beta^* < \gamma^* (\beta^* = \gamma^*)$.*

Proof. Let $B \cap \Gamma = A$ and extend the orders on B and Γ to a partial order of $B \cup \Gamma$ by letting $\beta \leqq \gamma$ if $\beta \leqq \alpha \leqq \gamma$ for some $\alpha \in A$ or if β, β^*, γ, and γ^* all determine the same cut of A, and letting $\gamma \leqq \beta$ if $\gamma \leqq \alpha \leqq \beta$ for some $\alpha \in A$. Its extension to a total order yields Δ as desired. For Δ', first identify β^* and γ^*, then proceed as above.

LEMMA 2.3. *The class of archimedian o-groups has the amalgamation property. If the amalgamated subgroup is nonzero, then any two amalgamations are equivalent.*

Proof. Let G be a subgroup of each of the archimedian o-groups H and K, and let ν_1; $H \rightarrow R$ and η_2: $K \rightarrow R$ be o-embeddings. Since every o-isomorphism between subgroups of R is realized as multiplication by a positive real number, there exists $0 < r \in R$ such that $(g\nu_1)r = g\eta_1$ for all $g \in G$. Putting $\eta_1 = \nu_1 r$, $L = H\eta_1 + K\eta_2$, we have the desired amalgamation (L, η_1, η_2). If (M, μ_1, μ_2) is also an amalgamation with $M \subseteqq R$, then $\eta_1^{-1}\mu_1 = s$ and $\eta_2^{-1}\mu_2 = t$ for some $0 < s, t \in R$. If $G \neq 0$ we must have $s = t$, and thus L and M are equivalent via the o-isomorphism induced by s.

The following concept is introduced for notational convenience. *A Γ-valuation* of an o-group H is a map $\gamma \mapsto (H^\gamma, H_\gamma)$ from the ordered set Γ to pairs of convex subgroups of H satisfying

(1) $H_\gamma \subseteqq H^\gamma$. If $H_\gamma \subset H^\gamma$, then H^γ covers H_γ,

(2) $\gamma \leqq \delta$ implies $H^\gamma \subseteqq H_\delta$, and

(3) If $h \neq 0$ then $h \in H^\gamma \backslash H_\gamma$ for some $\gamma \in \Gamma$.

H is called a *Γ-group*. γ is a *value* of h if $h \in H^\gamma \backslash H_\gamma$. The *spine* of H is the subset $\Gamma_a = \{\gamma\colon H_\gamma \subset H^\gamma\}$. The valuation is *proper* if $\Gamma = \Gamma_a$. Evidently all proper valuations are identical. For basic results on Γ-valuations see Conrad [2].

An easy consequence of the definition is the

LEMMA 2.4. *For every $\gamma \in \Gamma$,*

$$H^\gamma = \bigcap H_\delta \, (\gamma < \delta, \delta \subset \Gamma_a) \, .$$

A Δ-valuation on H is said to *extend* a Γ-valuation if $\Gamma \subseteqq \Delta$ and $(H^\delta, H_\delta) = (H^\gamma, H_\gamma)$ whenever $\delta = \gamma$ and $\gamma \in \Gamma$.

LEMMA 2.5. *If $\Gamma \subseteqq \Delta$ then every Γ-valuation on H has a unique extension to a Δ-valuation.*

Proof. Evidently defining

$$H^\delta = H_\delta = \bigcap H_\gamma \, (\delta < \gamma, \gamma \in \Gamma_a)$$

for all $\delta \in \Delta \backslash \Gamma$ produces an extension. To prove uniqueness we note that the spine of any extension is Γ_a, so by Lemma 2.4 all extensions are equal to the one just defined.

An o-embedding $\sigma\colon G \to H$ between Γ-groups is a *Γ-embedding* if $G_\gamma\sigma = H_\gamma \cap G\sigma$ and $G^\gamma\sigma = H^\gamma \cap G\sigma$. G is a *Γ-subgroup* of H if the inclusion map is a Γ-embedding. Every subgroup of a Γ-group admits a unique Γ-valuation making it a Γ-subgroup.

For each $\gamma \in \Gamma$ let R_γ be an archimedian o-group. The *Hahn group* $V(\Gamma, R_\gamma)$ is the subgroup of $\Pi R_\gamma (\gamma \in \Gamma)$ consisting of all functions with inversely well-ordered support, ordered by letting f be positive if its greatest nonzero component is positive. $V(\Gamma, R_\gamma)$ admits a natural Γ-valuation where $V^\gamma(V_\gamma) = \{f;\, f(\delta) = 0 \text{ for all } \delta > \gamma(\delta \geqq \gamma)\}$. If H is a Γ-group we define $V(H) = V(\Gamma, H^\gamma/H_\gamma)$. Banaschewski's proof [1] that a divisible Γ-group H is Γ-embeddable in $V(H)$, although elegant, is too restrictive (§ 4), so we will use Conrad's decomposition proof from [2], outlined below. A collection T of subgroups T_γ of a Γ-group H is a *Γ-decomposition* of H if

(1) $H_\gamma = H^\gamma \cap T_\gamma$,

(2) $H = H^\gamma + T_\gamma$, and

(3) For every h, $h \in T_\gamma$ for all but at inversely well-ordered subset of Γ.

The map $\bar{T}$: $H \rightarrow V(H)$, $h\bar{T}(\gamma) = (h + T_\gamma) \cap H^\gamma$, is a Γ-embedding. If S is a Γ-decomposition of the Γ-subgroup G, then there is a Γ-decomposition T of H such that $S = T \cap G$ (i.e., $S_\gamma = T_\gamma \cap G$ for all $\gamma \in \Gamma$).

A *natural embedding* σ: $V(\Gamma, P_\gamma) \rightarrow V(\Gamma, R_\gamma)$ is one which is induced by o-embeddings σ_γ: $P_\gamma \rightarrow R_\gamma$, where $f\sigma(\gamma) = f(\gamma)\sigma_\gamma$. In particular, if G is a Γ-subgroup of H, then the natural o-embeddings

$$\sigma_\gamma\colon G^\gamma/G_\gamma \longrightarrow H^\gamma/H_\gamma$$

induce σ: $V(G) \rightarrow V(H)$. One more fact from [2]: if S and T are Γ-decomposition on G and H respectively and $S = T \cap G$, then $\bar{S}\sigma = \bar{T} \mid G$.

The standard amalgamation. For the remainder of the paper let G be an ordered subgroup of the o-groups H and K, and let

$$\alpha \longmapsto (G^\alpha, G_\alpha)\ (\alpha \in A),\ \ \beta \longmapsto (H^\beta, H_\beta)\ (\beta \in B)\ ,$$

and $\gamma \rightarrow (K^\gamma, K_\gamma)$ $(\gamma \in \Gamma)$ be proper valuations of the respective groups. There are unique embeddings of A in B and Γ such that $G^\alpha = H^\alpha \cap G = K^\alpha \cap G$ and $G_\alpha = H_\alpha \cap G = K_\alpha \cap G$. By Lemma 2.2 let Δ be an amalgamation of A in B and Γ. By Lemma 2.5 extend to Δ-valuations on H and K. Evidently the maps

$$\delta \longmapsto (H^\delta \cap G, H_\delta \cap G)$$

and

$$\delta \longmapsto (K^\delta \cap G, K_\delta \cap G)$$

are Δ-valuations on G which extend the given A-valuation. Thus by 2.5 they are the same, and we can therefore consider G as a Δ-subgroup of each of H and K. Let S, T, and U be Δ-decompositions of G, H, and K respectively such that $S = T \cap G = U \cap G$, and let σ_1: $H \rightarrow V(H)$ and σ_2: $K \rightarrow V(K)$ be natural Δ-embeddings. Finally, by 2.3 let $(R_\delta, \nu_{1\delta}, \nu_{2\delta})$ be an archimedian amalgamation of G^δ/G_δ in H^δ/H_δ and K^δ/K_δ, and let ν_1: $V(H) \rightarrow V(\Delta, R_\delta)$ and ν_2: $V(K) \rightarrow V(\Delta, R_\delta)$ be natural Δ-embeddings. $\bar{T}\nu_1$ and $\bar{U}\nu_1$ agree on G, whence $(H\bar{T}\nu_1 = K\bar{U}\nu_2, \bar{T}\nu_1, \bar{U}\nu_2)$ is an amalgamation of G in H and K. We will call it a *standard amalgamation.*

Evidently the spine of a standard amalgamation is the amalgamation Δ of the spines. Thus every standard amalgamation is minimal. The converse is false, as this example shows:

EXAMPLE 2.6. Let $H = K = R \oplus {}^*R$ and let G be all elements with second component zero. Embed H and K in $R \oplus {}^*R \oplus {}^*R$ by letting

$$(x, y)\eta_1 = (x, o, y) ,$$

and

$$(x, y)\eta_2 = (x, y, y) .$$

The induced amalgamation is not standard since its spine properly contains an amalgamation of the spine of G in those of H and K. If π is the projection on the middle factor then replacing η_2 by $\eta_2 - \eta_2\pi$ yields a standard amalgamation. This motivates the following construction.

Let (L, η_1, η_2) be a minimal amalgamation and suppose that L is an o-subgroup of the o-group N. If $\pi\colon H \to N$ is a group homomorphism satisfying

(1) $G \subseteqq \ker(\pi)$, and

(2) $|h\pi| \ll |h\eta_1|$ for all $h \in H$ (that is $|h\eta_1|$ exceeds every multiple of $|h\pi|$), then we can form an amalgamation (M, μ_1, μ_2) by defining $\mu_1 = \eta_1 + \pi$, $\mu_2 = \eta_2$, and $M = H\mu_1 + K\mu_2$. We call M an *expansion* of L.

A Δ-subgroup H of L is a *c-subgroup* if each of the natural embeddings $H^\delta/H_\delta \to L^\delta/L_\delta$ is surjective.

LEMMA 2.7 [2]. *If H is a c-subgroup of L and T is a Δ-decomposition of L then $T \cap H$ is a Δ-decomposition of H.*

REMARK. This lemma, and hence also the proof of the following theorem, fails if we use Banaschewski functions instead of decompositions. For example, let Γ be the set of strictly negative integers, $L = V(\Gamma, R_\delta)$ $(R_\delta = R)$ and let H be the divisible subgroup generated $\Sigma R_\delta (\gamma \leqq -2)$ and the element $(\cdots, 1, 1, 1)$. If π_L is the natural Banaschewski function on L, then $H \cap L^{-2}\pi_L = \phi$, and hence π_L is not the extension of any Banaschewski function on H.

THEOREM 2.8. *Every amalgamation is equivalent to an expansion of a standard amalgamation.*

Proof. Suppose that (M, μ_1, μ_2) is an amalgamation and Δ' is its spine. When B and Γ are naturally embedded in Δ', the embeddings agree on A, so some amalgamation Δ of A in B and Γ is embedded in Δ'. We consider G, H, and K as Δ'-groups and μ_1 and μ_2 as Δ'-embeddings. Let S and U be Δ'-decompositions of G and K respectively such that $S = U \cap G$. By choosing an appropriate Δ'-decomposition of M we can assume that M is a Δ'-subgroup of $V(M)$, and if T' is the natural Δ'-decomposition of $V(M)$, then

$$U = T' \cap K \text{(i.e., } U_\delta = (T'_\delta \cap K\mu_2)\mu_2^{-1} :$$

we will abuse the notation similarly throughout the proof). Let ν_{1_δ} and ν_{2_δ} be the natural embeddings of H^δ/H_δ and K^δ/K_δ respectively into M^δ/M_δ, inducing ν_1; $V(H) \rightarrow V(M)$ and ν_2: $V(K) \rightarrow V(M)$. Note that ν_{1_δ} and ν_{2_δ} agree on G^δ/G_δ and the choice of decompositions implies $\mu_2 = \bar{U}\nu_2$. Let ρ_δ be a projection of M^δ/M_δ upon $(H^\delta/H_\delta)\nu_{1_\delta}$ and define η_1: $H \rightarrow V(M)$ by $h\eta_1(\delta) = h\mu_1(\delta)\rho_\delta$. If $h \in H^\delta \backslash H_\delta$ then $h\mu_1(\delta) = (h + H_\delta)\nu_{1\delta}$, $h\eta_1(\delta) = h\mu_1(\delta)$, and $h\mu_1(\delta') = h\eta_1(\delta') = 0$ for all $\delta' > \delta$. It follows that η_1 is an o-embedding. By the choice of decompositions again,

$$g\mu_1(\delta) \in (H^\delta/H_\delta)\nu_{1\delta}$$

for all $\delta \in \Delta'$ and all $g \in G$, whence μ_1 and η_1 agree on G. Therefore, if we put $\eta_2 = \mu_2$ and $L = H\eta_1 + K\eta_2$, then (L, η_1, η_2) is an amalgamation. Furthermore M is an expansion of L by the homomorphism $\pi = \mu_1 - \eta_1$. It remains to show that L is a standard amalgamation. Evidently the spine of L is precisely Δ, so we can pare down to Δ-groups and Δ-decompositions. $T'' = T' \cap V(H)$ is evidently the natural Δ-decomposition of $V(H)$. Since $\eta_1\nu_1^{-1}$ embeds H as a c-subgroup of $V(H)$, then by Lemma 2.7, $T = T'' \cap H = T' \cap H$ is a decomposition of H. Now $T \cap G = S$: since $g\mu_1(\delta) \in (H^\delta/H_\delta)\nu_{1_\delta}$ for all $g \in G$ and all $\delta \in \Delta$, then we have the following chain of equivalences: $g \in T_\delta \Leftrightarrow g\eta_1\nu_1^{-1} \in T''_\delta \Leftrightarrow g\eta_1 \in T'_\delta \Leftrightarrow g\mu_1 \in T'_\delta \Leftrightarrow g\mu_2 \in T'_\delta \Leftrightarrow g \in S_\delta$. Finally, we show that $\eta_1 = \bar{T}\nu_1$. Let $h \in H$, $\delta \in \Delta$. There exists $h^* \in H^\delta$ such that

$$h\eta_1(\delta) = (h^* + H_\delta)\nu_{1\delta} = h^*\eta_1(\delta) .$$

Then $(h - h^*)\eta_1 \in T'_\delta$, whence $h - h^* \in T_\delta$. Thus

$$h\eta_1(\delta)\nu_{1\delta}^{-1} = h^* + H_\delta = (h^* + T_\delta) \cap H^\delta = (h + T_\delta) \cap H^\delta = h\bar{T}(\delta) .$$

Thus $\eta_1 = \bar{T}\nu_1$, and it follows that

$$(L, \eta_1, \eta_2) = (H\bar{T}\nu_1 + K\bar{U}\nu_2, \bar{T}\nu_1, \bar{U}\nu_2)$$

is standard.

3. The structure of components of $\mathscr{L}$. Let (L, η_1, η_2) be an amalgamation and let

$$H_L = (H\eta_1 \cap K\eta_2)\eta_1^{-1}$$

and

$$K_L = (H\eta_1 \cap K\eta_2)\eta_2^{-1} .$$

Evidently there is a unique o-isomorphism $h \mapsto h^*$ from H_L onto K_L such that $h\eta_1 = h^*\eta_2$. If $G = H_L$ — that is, $H\eta_1 \cap K\eta_2 = G\eta_1$ — we say that the amalgamation has the *strong intersection property.*

We form next a particular kind of expansion. Let $H = H_L \oplus H_L'$ and let ρ be the projection of H on H_L. Let C be an o-group, π: $H_L \to C$ a group epimorphism such that $G \subseteqq \ker(\pi)$. Embed H and K in $C \oplus^* L$ by defining $\mu_1 = \eta_1 + \rho\pi$ and $\mu_2 = \eta_2$. Evidently $(H\mu_1 + K\mu_2, \mu_1, \mu_2)$ is an amalgamation which is an expansion of L by the homomorphism $\rho\pi$, and which is freer than L via the natural projection map. We will call such a construction a *vertical expansion* of L.

LEMMA 3.1. *Every amalgamation freer than L is equivalent to a vertical expansion of L.*

Proof. Let $(M, \mu_1, \mu_2) \succ (L, \eta_1, \eta_2)$ via θ and let $C = \ker(\theta)$. Note that $c \in C$ if and only if $c = h\mu_1 - h^*\mu_2$ for some $h \in H_L$. This implies that $C \cap (H_L'\mu_1 + K\mu_2) = 0$. Thus $M = C \oplus^* D$ for some subgroup D which contains $H_L'\mu_1 + K\mu_2$. Let ρ project H on H_L, and τ_1 and τ_2 project M on C and D recpectively. By the above representation of elements of C, τ_1 maps $H_L\mu_1$ onto C. Let $\pi = \rho\mu_1\tau_1$. Then $\mu_2 = \mu_2\rho$ since $K\mu_2 \subseteqq D$, and

$$h\mu_1 = h\rho\mu_1 + (h - h\rho)\mu_1 = h\rho\mu_1\tau_1 + h\rho\mu_1\tau_2 + (h - h\rho)\mu_1 = h\pi + h\mu_1\tau_2 .$$

Thus M is a vertical expansion of the amalgamation $(D, \mu_1\tau_2, \mu_2\tau_2)$. But D is equivalent to L via $\tau_2^{-1}\theta$.

THEOREM 3.2. *Let $\mathscr{M}$ be a component of $\mathscr{L}_{\mathscr{O}}(G, H, K)$ with least member (L, η_1, η_2). Then $\mathscr{M}$ is order isomorphic to $OQ(H_L/G)$.*

Proof. Let $M \subseteqq C \oplus^* L$ and $M' \subseteqq C' \oplus^* L$ be vertical expansions on L induced by π and π' respectively. One easily checks that $M \succ M'$ if and only if $\ker(\pi) \subseteqq \ker(\pi')$ and the induced homomorphism from C to C' preserves order; that is, $M \succ M'$ if and only if $H_L/\ker(\pi) \geqq H_L/\ker(\pi')$ in $OQ(H_L/G)$. This provides the desired isomorphism.

COROLLARY 3.3. *Let δ be the dimension of H_L/G as a rational vector space.*

(1) *If* $\delta = 0$ (*i.e.*, *L has the strong intersection property*) *then* $\mathscr{M} = \{L\}$.

(2) *If* $\delta = 1$ *then* $\mathscr{M}$ *has* 3 *members-two incomparable members exceeding* L.

(3) *If* $\delta > 1$ *then* $\mathscr{M}$ *has max* $\{c, 2^\delta\}$ *members, and the same number of maximal members. If* δ *is finite then every maximal chain is finite, and* $\delta > c$ *if and only if every maximal chain has cardinality at least* c.

Proof. (1) is obvious, and (2) is an immediate consequence of there being exactly two orders on Q, both archimedian. As for (3), Teh [8] has shown that a rational vector space of rank greater than 1 admits at least c orders. If δ is infinite, there are 2^δ orderings of a basis of $V = H_L/G$, and each of these yields a distinct lexicographic direct sum ordering of V. Thus there are at least $c \cdot 2^\delta$ orderings. But since V has $\aleph_0, \delta$ elements, there are at most $c \cdot 2^\delta$ subsemigroups-let alone that many partial orders — and thus there are exactly $c \cdot 2^\delta$ orders on V. Each of these yields a distinct maximal member of $OQ(V)$. Now maximal chains in $OQ(V)$ correspond to the collection of all o-homomorphic images of a maximal member $[V, P]$ of $OQ(V)$. But $[V, P]$ has at most $c \cdot 2^\delta$ convex subgroups and hence admits at most that many o-homomorphic images. Thus $|OQ(V)| = c \cdot 2^\delta$. If δ is finite then again from [8], any order on V has finite archimedian rank of at most δ. Using similar arguments, one can show that if $\delta > c$ then the cardinality of the archimedian rank of any order on V is greater than c. Thus every maximal chain in $OQ(V)$ also has cardinality greater than c. (See [4] for a discourse on archimedian ranks of abelian groups.)

4. Applications to the study of $\mathscr{L}$.

LEMMA 4.1. *Let* (L, η_1, η_2) *and* (M, μ_1, μ_2) *be amalgamations. The following are equivalent*:

(1) $L \succ M$.

(2) *For all* $h \in H$ *and* $k \in k$, $h\eta_1 \leqq k\eta_2$ *implies* $h\mu_1 \leqq k\mu_2$.

Proof. If $L \succ M$ via θ then $h\eta_1 \leqq k\eta_2$ implies

$$h\mu_1 = h\eta_1\theta \leqq k\eta_2 = k\mu_2 .$$

Conversely, if (2) is true then the map $\theta\colon h\eta_1 + k\eta_2 \mapsto h\mu_1 + k\mu_2$ is a well-defined o-epimorphism by which $L \succ M$.

Let $h \in H$ and $k \in K$. We say that G *separates* h *from* k if there exists $g \in G$ such that either $h \leqq g \leqq k$ or $k \leqq g \leqq h$, where at least one of the pair of inequalities is strict.

G has a *basic archimedean value* $\alpha \in A$ if α is the least element of B and of Γ. Then G^α, H^α, and K^α are nonzero archimedean subgroups.

LEMMA 4.2. *If G, H, and K are archimedean, $G \neq 0$, and G does not separate h from k, then $h\eta_1 = k\eta_2$ in any archimedean amalgamation (L, η_1, η_2).*

Proof. Without loss of generality, $L \subseteqq R$, η_1 and η_2 are inclusion maps, and $1 \in G$. Since G is divisible, h and k determine the same cut of $Q \subseteqq G$, whence they must be equal.

THEOREM 4.3. *The following are equivalent:*

(1) $\mathscr{L} = \mathscr{L}_{\mathscr{O}}(G, H, K)$ *has one component.*

(2) *For every $h \in H$ and $k \in K$, either G separates h from k, or G has a basic archimedean value α with $h \in H^\alpha$ and $k \in K^\alpha$.*

Proof. Suppose (2) is true, and let (L, η_1, η_2) and (M, μ_1, μ_2) be minimal. We will show that $L \approx M$. Let $h\eta_1 \leqq k\eta_2$. If G separates h from k then we must have $h\mu_1 \leqq k\mu_2$. If G doesn't separate h from k, then the second part of (2) holds. Inspecting the proof of Theorem 1.2, we see that an amalgamation is minimal if and only if it has no proper convex subgroup disjoint from the images of H and K. Thus L and M have convex subgroups F and F' which cover zero and contain the respective images of H^α and K^α. In fact, $(F, \eta_1 | D, \eta_2 | E)$ and $(F', \mu_1 | D, \mu_2 | E)$ are archimedean amalgamations of G^α in H^α and K^α. By Lemma 2.3, they are equivalent, and hence $h\mu_1 \leqq k\mu_2$. Thus $L \succ M$ by Lemma 4.1. But since L is minimal, then $L \approx M$.

Conversely, suppose that (2) fails for some $h \in H$ and $k \in K$, let h have value $\beta \in B$, and let k have value $\gamma \in \Gamma$. We consider two cases, depending on whether or not β and γ are in A. If neither is in A then the hypotheses on h and k imply that for all $\alpha \in A$, either α is greater than both β and γ or less than both. By the proof of Lemma 2.2 we can find amalgamations Δ and Δ' of A in B and Δ such that $\beta < \gamma$ in Δ, but $\gamma < \beta$ in Δ'. These then lead to two standard amalgamations, in one of which the image of h is strictly less than that of k, and in the other the order is reversed. Thus by Lemma 4.1 they are incomparable. Since standard amalgamations are minimal, $\mathscr{L}$ has at least two components. On the other hand, suppose that $\beta = \gamma = \alpha \in A$, and let (L, η_1, η_2) be any amalgamation. As usual, embed B and Γ in Δ, the spine of L, and without loss of generality consider $L \subseteqq V(L)$. Let $V = V(\Delta', R_\delta)$ where Δ' arises from Δ by adjoining ε, and the order is extended by defining $\varepsilon < \delta$ for all $\delta \geqq \alpha$, and $\delta < \varepsilon$ for all $\delta < \alpha$. Let $R_\delta = L^\delta/L_\delta$ and let $R_\varepsilon = Q$. $V(L)$ can be naturally embedded in V. Since $G \cap Qh = 0$, one can find a group homomorphism $\tau: H \to R_\varepsilon$ such that $G + H_\alpha \subseteqq \ker(\tau)$ and $h\tau = \pm 1$, depending on whether $h\eta_1 \leqq k\eta_2$ or $k\eta_2 \leqq k\eta_1$. Finally then, using τ we can expand to the amalgamation (M, μ_1, μ_2); i.e., $\mu_2 = \eta_2$, and for $x \in H$, $x\mu_1(\delta) = x\eta_1(\delta)$, and $x\mu_1(\varepsilon) = x\tau$. By hypothesis α is not the least element of both B and Γ, so M is also minimal. We claim that L and M are incomparable. G/G_α, as embedded in H/H_α and K/K_α, does not separate $h + H_\alpha$ from $k + K_\alpha$. Thus by Lemma 4.2, $h\eta_1(\alpha) = k\eta_2(\alpha)$. But $(h\mu_1 - k\mu_2)(\varepsilon) = \pm 1$, the sign chosen so that $h\mu_1 - k\mu_2$ has sign opposite from $h\eta_1 - k\eta_2$. Thus by Lemma 4.1 the amalgamations are incomparable, so again $\mathscr{L}$ has at least two components.

THEOREM 4.4. *The following are equivalent*:
(1) $|\mathscr{L}| = 1$.
(2) *For every* $h \in H$ *and* $k \in K$, G *separates* h *from* k.

Proof. Suppose (2) is true. By Theorem 4.3, $\mathscr{L}$ has exactly one component with minimal element L. But a consequence of the hypothesis is that L has the strong intersection property. Thus by Corollary 3.3, $|\mathscr{L}| = 1$. Conversely, if $\mathscr{L} = \{(L, \eta_1, \eta_2)\}$, then in particular condition (2) of Theorem 4.3 holds. But if the second part of that condition is true then by Lemma 4.2 L fails to have the strong intersection property, whence again by Corollary 3.3 there exists an amalgamation properly exceeding L. But this is impossible.

THEOREM 4.5. $\mathscr{L}$ *has either one, or at least c, components*.

Proof. Suppose $\mathscr{L}$ has more than one component. By Theorem 4.3 there exist $0 < h \in H$ and $0 < k \in K$, not having the basic archimedian value if one exists, such that G does not separate h from k. Let h have value β, and k have value γ. We distinguish two cases. Suppose β is the least element of B and γ is the least element of Γ. Since neither is in A, there is an amalgamation Δ of A in B and Γ such that $\beta = \gamma$. Since $G_\beta = G^\beta$, the choice of ν_{1_β}: $H^\beta/H_\beta \to R_\beta$ and ν_{2_β}: $K^\beta/K_\beta \to R_\beta$ in the construction of a standard amalgamation can be arbitrary. We let $(h + H_\beta)\nu_{1_\beta} = 1$ and for some positive real number r let $(k + K_\beta)\nu_{1_\beta} = r$. Evidently the induced standard amalgamations, one for each r, are mutually nonequivalent. On the other hand, suppose not both H_β and K_β are zero. Let (L, η_1, η_2) be any standard amalgamation. We imitate the construction in the latter part of the proof of Theorem 4.3, except that we also choose a group homomorphism $\pi\colon K \to R_\varepsilon$ such that $G + K_\beta \subseteqq \ker(\pi)$ and $k\pi = r$. Then we define μ_1 as before, and μ_2 as $x\mu_2(\delta) = x\eta_2(\delta)$, and $x\mu_2(\varepsilon) = x\pi$. Then the collection of amalgamations, one for each r, again form a mutually nonequivalent set of minimal amalgamations.

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