RESIDUAL PROPERTIES OF FREE GROUPS

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In this paper the following theorem is proved: if \( \pi \) is an infinite set of primes and \( n \) is an odd integer greater than one, then free groups are residually \( \{\text{PSL}(n, p); p \in \pi\} \). As a by-product of the proof new generators of \( \text{SL}(n, p) \) are obtained for nearly all primes \( p \).

1. The main result. For unexplained notation the reader is referred to [8].

Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be sets of groups. \( \mathcal{A}_1 \) is said to be \textit{residually} \( \mathcal{A}_2 \) iff, for each group \( G \) belonging to \( \mathcal{A}_1 \) and each non-identity element \( g \) of \( G \) there is a homomorphism \( \varphi \) (depending on \( G \) and \( g \)) which maps \( G \) onto some element \( H \) of \( \mathcal{A}_2 \), and is such that \( \varphi(g) \) is not the identity of \( H \). An equivalent formulation is: for each \( G \) in \( \mathcal{A}_1 \) there is a set of normal subgroups \( \{N_i\}_{i \in I} \) of \( G \) such that \( \bigcap_{i \in I} N_i = 1 \) and for each \( i \) in \( I \), \( G/N_i \) is isomorphic to an element of \( \mathcal{A}_2 \). It is obvious that if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are sets of groups and some or all of the members of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are replaced by isomorphic copies, yielding new sets \( \mathcal{A}'_1 \) and \( \mathcal{A}'_2 \) respectively, then \( \mathcal{A}_1 \) is residually \( \mathcal{A}_2 \) iff \( \mathcal{A}'_1 \) is residually \( \mathcal{A}'_2 \). It is also easy to see that if \( \mathcal{A}_1 \) is residually \( \mathcal{A}_2 \), and \( \mathcal{A}_2 \) is residually \( \mathcal{A}_3 \), then \( \mathcal{A}_1 \) is residually \( \mathcal{A}_3 \).

Let \( \{x_1, x_2, x_3, \ldots\} \) be a fixed but arbitrary countably infinite set, and let \( F_n \) be the free group freely generated by \( \{x_1, x_2, \ldots, x_n\} \). Denote by \( \mathcal{F} \) the set \( \{F_n; n \geq 2\} \). In recent years there has been some investigation into which sets, \( \mathcal{A} \) of groups are such that \( \mathcal{F} \) is residually \( \mathcal{A} \). The two-generator groups in \( \mathcal{A} \) must of necessity generate the variety, \( \mathcal{O} \), of all groups. It has been conjectured by S. Meskin that this condition is also sufficient. A rich source of sets of groups which generate \( \mathcal{O} \) is a result of Heineken and Neumann [3] which states that every infinite set of pairwise non-isomorphic known (1967) finite non-abelian simple groups generates the variety of all groups. This theorem has presumably inspired several of the results obtained so far. Thus Katz and Magnus [5] have proved that \( \mathcal{F} \) is residually \( \{A_n; n \in J\} \), where \( A_n \) is the alternating group on \( \{1, 2, \ldots, n\} \) and \( J \) is an infinite set of positive odd integers; and Gorčakov and Levčuk [2] have proved that \( \mathcal{F} \) is residually any infinite subset of the set of simple groups \( \text{PSL}(2, p') \). This latter result generalizes theorems obtained in [6], [5] and [7], which consider the cases \( r = 1 \) and \( p \) variable, \( r > 1 \) and fixed and \( p \) variable, \( p > 11 \) and fixed and \( r \) variable, respectively.

In this paper the following main result is obtained.
THEOREM 1. Let \( n \) be an odd integer greater than one, and let \( \pi \) be an infinite set of primes. Then \( \mathcal{F} \) is residually \( \{ \text{PSL}(n, p) : p \in \pi \} \).

Before discussing the proof of Theorem 1 some notation and definitions will be introduced. Let \( R \) be a commutative ring with identity \( 1 \). The ring of polynomials in the indeterminant \( x \) with coefficients from \( R \) will be denoted by \( R[x] \). The degree of an element \( f(x) \) of \( R[x] \) will be written as \( \deg(f(x)) \). As is well-known (see [4], page 56) the \( n \times n \) matrices with entries from \( R \) form a ring with identity. The identity will be denoted by \( E \). The \( n \times n \) matrix with 1 in its \( i \)th row and \( j \)th column and zeros elsewhere will be denoted by \( E_{ij} \) \( (i, j = 1, 2, \ldots, n) \), and \( E_{(n+i)(n+j)}, E_{(n+i)(n+j)} \) \( (i, j = 1, 2, \ldots, n) \) will all be defined to be equal to \( E_{ij} \). The multiplicative semigroup of the ring of \( n \times n \) matrices with entries from \( R \) has a sub-semigroup consisting of all matrices which have a single nonzero entry, namely 1, in each row and each column. This sub-semigroup is in fact a group, isomorphic to the symmetric group on \( \{1, 2, \ldots, n\} \). An isomorphism is given by:

\[
\sigma \longrightarrow \sum_{i=1}^{n} E_{\sigma(i)},
\]

where \( \sigma \) is a permutation of \( \{1, 2, \ldots, n\} \). The matrix \( \sum_{i=1}^{n} E_{\sigma(i)} \) will be called the permutation matrix corresponding to \( \sigma \). When no confusion can arise, and if it is convenient to do so, the matrix \( \sum_{i=1}^{n} E_{\sigma(i)} \) will be denoted by the permutation \( \sigma \).

For the rest of this section \( n \) will denote a fixed but arbitrary odd integer greater than one, and \( p \) (possibly subscripted) will stand for a prime number. To simplify the proof of Theorem 1, use is made of the following two results:

(i) \( \mathcal{F} \) is residually \( \{ F_\alpha \} \),

(ii) For each \( k \geq 2 \), \( \{ F_\alpha \} \) is residually \( \{ T_k \} \), where \( T_k = (a, b | a^k) \).

The former result is proved in [6], whilst Lemma 1 of [5] proves (ii) for the case \( k = 2 \), and the proof for \( k > 2 \) is entirely analogous. Using (i) and (ii) reduces the proof of Theorem 1 to showing that \( \{ T_n \} \) is residually \( \{ \text{PSL}(n, p) : p \in \pi \} \).

The first step in proving that \( \{ T_n \} \) is residually \( \{ \text{PSL}(n, p) : p \in \pi \} \) is to find a group of \( n \times n \) matrices which is isomorphic to \( T_n \). Consider the ring of \( n \times n \) matrices with entries from \( Z[x] \). The multiplicative semigroup of this ring has a sub-semigroup consisting of all matrices with determinant \( \pm 1 \). This sub-semigroup is a group, called the group of units. The permutation matrix \( X \) corresponding to the permutation \( (1, 2, 3, \ldots, n) \), and the matrix \( Y = E + x \sum_{j=2}^{n} E_{j1} \) are in the group of units. They therefore generate a subgroup, \( \mathcal{U}_n \),
of this group. Notice that in this group $X$ has order $n$ and $Y$ has infinite order. In §2 the following result is proved.

**Lemma 1.** When a product of the form

$$Y^v X^{m_1} \cdots X^{m_r} Y^{m_{r+1}} X^{m_{r+2}}$$

—where $r \geq 0$, the $d_i$ can have the values $1, 2, \ldots, n - 1$, the $m_i$ can have any integer values except zero, $v$ can have any integer value, $\mu$ can be $0, 1, 2, \ldots, n - 1$, $v$ and $\mu$ cannot be zero simultaneously unless $r \geq 1$—is multiplied out, it has an entry of degree at least one, provided $v$ and $r$ are not both zero.

From this lemma follows immediately

**Theorem 2.** $\mathbb{Z}_n$ and $T_n$ are isomorphic.

The problem is now reduced to showing that $\{\mathbb{Z}_n\}$ is residually $\{\text{PSL}(n, p); p \in \pi\}$. There are plenty of homomorphisms from $\mathbb{Z}_n$ into $SL(n, p)$. In fact, let $\alpha$ be a nonzero element of $GF(p)$. Then, by Theorem 4 of Chapter III [4], there is a ring homomorphism of $\mathbb{Z}[x]$ onto $GF(p)$ which maps $x$ to $\alpha$. This homomorphism induces a homomorphism $\varphi_\alpha$ from the multiplicative semigroup of all $n \times n$ matrices with entries from $\mathbb{Z}[x]$ to the multiplicative semigroup of all $n \times n$ matrices with entries from $GF(p)$. The value of $\varphi_\alpha$ at the matrix $M$ is obtained by replacing all appearances of $x$ in $M$ by $\alpha$, and replacing all integers appearing as coefficients in the polynomials in $M$ by their congruence classes modulo the prime $p$. When restricted to $\mathbb{Z}_n$, $\varphi_\alpha$ is a group homomorphism with range contained in $SL(n, p)$. Let $\varphi_\alpha(X) = C$ and $\varphi_\alpha(Y) = D(\alpha)$. It is easy to see that the subgroup of $SL(n, p)$ generated by $C$ and $D(\alpha)$ is the same as that generated by $C$ and $D = D(1)$. For there are integers $t$ and $u$ such that $t \alpha = 1$ and $u \alpha = \alpha$, and so $D(\alpha)^t = D$ and $D^u = D(\alpha)$. In §3 the following result is proved.

**Theorem 3.** Let $p$ be a prime which does not divide $3(n - 1)$. Then $C$ and $D$ generate $SL(n, p)$.

(If $p$ divides $3(n - 1)$, the validity of the theorem remains undecided.)

It follows immediately from Theorem 3 that $\varphi_\alpha$ is a homomorphism of $\mathbb{Z}_n$ onto $SL(n, p)$ for all but a finite number of primes $p$.

Using Lemma 1 and Theorems 2 and 3, it is now possible to prove that $\{\mathbb{Z}_n\}$ is residually $\{\text{PSL}(n, p); p \in \pi\}$. It is well-known (see [8],
that the centre of $SL(n, p)$ consists of all scalar matrices $\lambda E$, where $\lambda^n = 1$. Given a non-identity element $W$ of $\mathbb{F}_n$, it will be shown that there is a prime $p$ in $\pi$, and a homomorphism $\varphi$ of $\mathbb{F}_n$ onto $SL(n, p)$ such that $\varphi(W)$ does not belong to the centre of $SL(n, p)$. Then the composition of $\varphi$ with the natural homomorphism of $SL(n, p)$ onto $PSL(n, p)$ gives a homomorphism of $\mathbb{F}_n$ onto $PSL(n, p)$ which does not map $W$ to the identity.

Thus, let $W$ be a non-identity element of $\mathbb{F}_n$. Then $W$ can be expressed uniquely as a product of the form (*) (see Lemma 1). First suppose that in the product (*) $v = 0$ and $r = 0$, so that $W = X^r$, where $\mu$ is an integer and $0 < \mu < n$. Let $p_0$ be a prime in $\pi$ which does not divide $3(n - 1)$. Then the homomorphism of $\mathbb{F}_n$ onto $SL(n, p_0)$ determined by

$$
X \mapsto C \\
Y \mapsto D
$$

does not map $W$ to the centre of $SL(n, p_0)$.

Suppose now that the product (*) is such that not both of $v$ and $r$ are zero. Then by Lemma 1, $W$ has an entry

$$a_0 + a_1 x + \cdots + a_s x^s$$

with $a_s \neq 0$, $s \geq 1$.

Let $p_0$ be a prime in $\pi$ with the property

$$p_0 - 1 > \max \{|a_s|, s(n + 1)|.$$ 

The congruence class of an integer $k \mod p_0$ will be denoted by $\bar{k}$. Consider the polynomials

$$f(x) = \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_s x^s,$$

$$g(x) = f(x)(f(x))^n - \bar{I},$$

which are elements of $GF(p_0)[x]$. Since $\bar{a}_s \neq \bar{0}$, $\deg(f(x)) = s$, and so $\deg(g(x)) = s(n + 1)$. By the choice of $p_0$ there is a nonzero element $\alpha$ of $GF(p_0)$ which is not a root of $g(x)$.

Let $\varphi$ be the homomorphism of $\mathbb{F}_n$ onto $SL(n, p_0)$ determined by

$$X \mapsto C \\
Y \mapsto D(\alpha).$$

(Note that $p_0$ does not divide $3(n - 1)$, so Theorem 3 applies.) The entries of $\varphi(W)$ are obtained from those of $W$ by replacing $x$ by $\alpha$ and working mod $p_0$. Hence $\varphi(W)$ has

$$f(\alpha) = \bar{a}_0 + \bar{a}_1 \alpha + \cdots + \bar{a}_s \alpha^s$$

as one of its entries. By the choice of $\alpha$, $f(\alpha) \neq \bar{0}$ and $f(\alpha)^n \neq \bar{I}$, so
clearly \( \varphi(W) \) does not lie in the centre of \( SL(n, p_0) \).

2. Proof of Lemma 1. In this and the next section it will be useful to keep in mind the following rule for calculating with permutation matrices. If \( M \) is a \( u \times u \) matrix and \( P \) is the permutation matrix corresponding to a permutation \( \sigma \) of \( \{1, 2, \cdots, u\} \), then \( PM \) is obtained from \( M \) by replacing row \( i \) by row \( \sigma(i) \), and \( MP \) is obtained from \( M \) by replacing column \( i \) by column \( \sigma^{-1}(i) \) (\( 1 \leq i \leq u \)).

Before proving Lemma 1, it should be pointed out that the result is also valid when \( n \) is even (the proof given below does not depend upon \( n \) being odd), but in this case the permutation matrix corresponding to \( (1, 2, 3, \cdots, n) \) has determinant \(-1\), so that the result is not of any use here.

A product of the form (*) (as in the statement of Lemma 1) in which \( \nu = \mu = 0 \) will be called a product of type-(\( XY \)). When such a product is multiplied out, a matrix with entries \( \xi_{ij}^r \) (\( i, j = 1, 2, \cdots, n \)) from \( \mathbb{Z}[x] \) is obtained. The following assertion will be proved by induction on \( r \).

\[
\deg (\xi_{11}^{(r)}) = r
\]
\[
\deg (\xi_{ij}^{(r)}) < r \text{ for } j = 2, 3, \cdots, n .
\]

For \( r = 1 \) the product is just \( X^{x_1} Y^{m_1} \), which is equal to \( X^{x_1} + m_1x \sum_{j=2}^{n} E_{(n+1-j)} \). Thus

\[
\xi_{11}^{(1)} = \begin{cases} m_1x & i \neq n + 1 - \delta_i \\ 1 & i = n + 1 - \delta_i \end{cases}.
\]

All other entries of \( X^{x_1} Y^{m_1} \) are either zero or one. Since \( 0 < \delta_i < n \), it follows that \( 1 < n + 1 - \delta_i < n + 1 \), so that \( \xi_{11}^{(1)} \) is \( m_1x \). Thus

\((+++) \) holds when \( r = 1 \).

Now assume \((++)\) holds for all \( s \leq r \), where \( r > 1 \). The first row of \( X^{x_1} Y^{m_1} \cdots X^{x_{r-1}} Y^{m_{r-1}} X^{x_r} Y^{m_r} \) is obtained from that of \( X^{x_1} Y^{m_1} \cdots X^{x_{r-1}} Y^{m_{r-1}} \) by right multiplication by \( X^{x_r} Y^{m_r} \). Thus

\[
\xi_{11}^{(r)} = \sum_{|j| \leq n} m_{r}x \xi_{1j}^{(r-1)} + \xi_{11}^{(r-1)} (m_1 + 1 - \delta_r \).
\]

Since \( 1 < n + 1 - \delta_r < n + 1 \), it follows that

\[
\deg (\xi_{11}^{(r)}) = \deg (\xi_{11}^{(r-1)}) + 1 = r .
\]

Now except for column one, every column of \( X^{x_r} Y^{m_r} \) contains only zeros and ones. Hence for \( 2 \leq j \leq n \),

\[
\deg (\xi_{ij}^{(r)}) = \deg (\xi_{ij}^{(r-1)}) + 1 = r .
\]
This shows that (+ +) holds for $r$, and completes the induction proof.

Now take a product of the general form (*) in which not both of $\nu$ and $r$ are zero, and let $W$ be the matrix obtained when this product is multiplied out. It is required to show that $W$ has an entry of degree at least one.

Case (i). $\nu = \mu = 0$. The product is of type-(XY), so $W$ has an entry of degree $r$, by (+ +).

Case (ii). $\nu \neq 0, \mu \neq 0$. Since

$$W^{-1} = X_{n-\nu} Y^{-m_1} X_{n-\delta_2} \ldots Y^{-m_1} X_{n-\delta_1} Y^{-\nu}$$

and the product on the right is of type-(XY), $W^{-1}$ has an entry of degree at least one by (+ +); consequently $W$ has also.

Case (iii). $\nu \neq 0, \mu = 0$. If $r = 0$, $W$ is just $Y^r$, which has $\nu x$ as one of its entries. Suppose then that $r \geq 1$. $X_{\delta_1} Y^{m_1} \ldots X_{\delta_r} Y^{m_r}$ is a product of type-(XY), so the entries $\xi_{ij}^{(r)}$ ($j = 1, 2, \ldots, n$) in the first row of the matrix $U$ obtained when this product is multiplied out satisfy (+ +). The first row of $W$ is the same as that of $U$, so $W$ has an entry of degree $r$.

Case (iv). $\nu = 0, \mu \neq 0$. If $U$ is the matrix obtained when $X_{\delta_1} Y^{m_1} \ldots X_{\delta_r} Y^{m_r}$ is multiplied out, then $U$ has an entry of degree $r$, and since $W$ is just obtained from $U$ by a permutation of columns, $W$ also has an entry of degree $r$.

This completes the proof of Lemma 1.

3. Proof of Theorem 3. The following definitions are used. A matrix of the form $E + \lambda E_{ij}$, where $\lambda \neq 0$ and $i \neq j$, will be called a transvection. In a group $G$ the commutator $[g_1]$ of $g_1 \in G$ will be defined to be $g_1$, the commutator $[g_1, g_2]$ of $g_1, g_2 \in G$ will be defined to be $g_1 g_2 g_1^{-1} g_2^{-1}$, and for $n \geq 3$, $[g_1, g_2, \ldots, g_n]$ will be defined to be $[g_1, \ldots, g_{n-1}, g_n]$. If $S$ is a nonempty subset of $G$ then $sgpS$ will denote the subgroup of $G$ generated by $S$.

Let $n$ denote a fixed but arbitrary odd integer greater than one, and let $p$ be a fixed but arbitrary prime which does not divide $3n - 3$. It is required to show that the elements

$$C = \sum_{i=1}^{n} E_{i(i+1)}$$
of $SL(n, p)$ generate this group. It will be shown below that the transvection $E + E_{1n}$ belongs to $sgp(C, D)$, and from this the result follows, as is now indicated.

It is well-known (see [8], page 158) that the transvections $E + \lambda E_{ij}$ $(i \neq j; i, j = 1, 2, \ldots, n)$, where $\lambda$ ranges over the nonzero elements of $GF(p)$, generate $SL(n, p)$. In fact, it is enough to choose one value of $\lambda$, say $\lambda_{ij}$, for each pair $(i, j)$. For $\lambda_{ij}$ has order $p$ in the additive group of $GF(p)$, and so as $t$ runs through the integers from 1 to $p - 1$, $t\lambda_{ij}$ assumes every non-zero element of $GF(p)$. Since

$$(E + \lambda_{ij}E_{ij})^t = E + (t\lambda_{ij})E_{ij} \ (i \neq j; i, j = 1, 2, \ldots, n)$$

all transvections can be obtained from the $E + \lambda_{ij}E_{ij}$. Notice that, in particular, the value 1 can be chosen for each $\lambda_{ij}$.

Let $\mathcal{H} = sgp(E + E_{1n}, C)$. Now for $i, j = 1, \ldots, n$

$$CE_{ij}C^{-1} = E_{(n+i-1)(n+j-1)}^{(m+i-1)(n+j-1)}$$

Therefore

$$C^r(E + E_{1n})C^{-r} = E + E_{(n+1-r)(n-r)}^{(m+1-r)(n-r)}$$

$$= \tau_r, \text{ say } (0 \leq r \leq n - 1) .$$

It is easily shown that

$$[\tau_0, \tau_1, \ldots, \tau_s] = E + E_{(n-s)}^{(m-s)} \ (0 \leq s \leq n - 2) .$$

Thus $\mathcal{H}$ contains all the transvections

$$E + E_{ih} \ h = 2, 3, \ldots, n .$$

Finally, using (***) $k$ times $(0 \leq k \leq n - 1)$ gives

$$C^k(E + E_{1h})C^{-k} = E + E_{(n+1-k)(n+h-k)}^{(m+1-k)(n+h-k)}, \ h = 2, 3, \ldots, n ,$$

and so $\mathcal{H}$ contains all the transvections

$$E + E_{ij} \ (i \neq j; i, j = 1, 2, \ldots, n) .$$

Therefore $\mathcal{H} = SL(n, p)$.

It will now be shown that $E + E_{1n}$ belongs to $sgp(C, D)$. Straight-forward computations show
\[ [D^{-1}, C^{-1}]D = E + E_{11} + E_{12} - E_{21} - E_{22} = P, \text{ say} \]

\[ [D^{-1}, C^{-2}]D = E + E_{11} + E_{13} - E_{31} - E_{33} = Q, \text{ say} \]

\[ C^{-1}([D^{-1}, C^{-1}]D)C = E + E_{22} + E_{23} - E_{32} - E_{33} = R, \text{ say.} \]

Let \( t \) be an integer such that \( 6t \equiv 1 \mod p \) (such a \( t \) exists since \( p \) is not 2 or 3). Then

\[ (QP^{-1}R^{-i})^{2t} = E - E_{13} + E_{23}. \]

This element will be denoted by \( T \). It turns out to be extremely useful.

Another useful element is

\[ T^2RP = \sum_{i=1}^{n} E_{ii} + E_{12} + E_{23} + E_{31}. \]

This is just the permutation matrix corresponding to the permutation (123). Since, for \( m \geq 3 \) and odd, the permutations (123) and (123 \( \cdots m \)) generate the alternating group \( A_m \) ([1], page 67), it follows that \( sgp\{C, D\} \) contains all even permutation matrices.

Suppose that \( n \) is greater than 3. It is easy to see that

\[
\begin{align*}
(1) & \quad (34 \cdots n)T^{-1}(34 \cdots n)^{-1} = E + E_{1n} - E_{2n} \\
& \quad (1s)(2, s + 1)(E + E_{1n} - E_{2n})(1s)(2, s + 1) = E + E_{sn} - E_{(s+1)n} \quad (3 \leq s \leq n - 2)
\end{align*}
\]

and

\[
\begin{align*}
(2) & \quad (123)^{-i}E_{1n} - E_{2n}(123) = E + E_{2n} - E_{3n}.
\end{align*}
\]

From (1), (2) and (3) it follows that \( sgp\{C, D\} \) contains all the matrices

\[ A_i = E + E_{2n} - E_{(i+1)n} \quad 1 \leq i \leq n - 2. \]

This is also obviously true if \( n \) equals 3.

Now take the matrix

\[ CDC^{-1} = E + \sum_{i=1}^{n-1} E_{ii}. \]

Multiplying by \( A_{n-2} \) (on either side, since each \( A_i \) commutes with \( CDC^{-1} \)) gives \( E + \sum_{i=1}^{n-3} E_{ii} + 2E_{(n-2)n} \). Then multiplying by \( A_{n-3}^2 \) gives \( E + \sum_{i=1}^{n-1} E_{ii} + 3E_{(n-3)n} \). Continuing in this manner finally gives the matrix \( E + (n - 1)E_{1n} \). Formally,
Since \( p \) does not divide \( n - 1 \), there is an integer \( t \) such that \( t(n - 1) \equiv 1 \mod p \). Then

\[
(E + (n - 1)E_{1n})^t = E + E_{1n}.
\]

This shows that \( sgp(C, D) \) contains the transvection \( E + E_{1n} \), and completes the proof of Theorem 3.

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