THE CONVEX CONE OF \( n \)-MONOTONE FUNCTIONS

ROY MARTIN RAKESTRAW
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R. M. RAKESTRAW

A reformulation of the Krein-Milman Theorem is used to obtain an integral representation of each function in a certain class of real monotonic functions defined on $[0, 1]$.

Let $\{i_1, i_2, i_3, \cdots\}$ denote a fixed sequence all of whose terms are either 0 or 1, and let $M_1$ be the set of real nonnegative functions $f$ on $[0, 1]$ such that

$$(-1)^{i_1} A_h f(x) = (-1)^{i_1} [f(x + h) - f(x)] \geq 0,$$

$h > 0$, for $[x, x + h] \subset [0, 1]$. Let $M_n$, $n > 1$, be the set of functions belonging to $M_{n-1}$ such that

$$(-1)^{i_n} A_h f(x) = (-1)^{i_n} [A_h^{-1} f(x + h) - A_h^{-1} f(x)] \geq 0$$

for $[x, x + nh] \subset [0, 1]$. If $f \in M_n$, then $f$ is said to be an $n$-monotone function. Since the sum of two $n$-monotone functions is in $M_n$ and since a nonnegative real multiple of an $n$-monotone function is an $n$-monotone function, the set $M_n$ is a convex cone. It is the purpose of this paper to give the extremal elements (i.e., the generators of extreme rays) of this cone, and to show that for the $n$-monotone functions an integral representation in terms of extremal elements is possible.

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1. Extremal elements of $M_n$. Let $f$ be a function in $M_i$ which assumes exactly one positive value in $[0, 1]$. If $f = f_1 + f_2$, where $f_1$ and $f_2 \in M_i$, then $f_1$ and $f_2$ are zero where $f$ is zero and $f_1$ and $f_2$ are constant where $f$ is constant. Therefore, $f_1$ and $f_2$ are proportional to $f$ and $f$ is an extremal element of $M_i$. On the other hand, if $f$ assumes at least two positive values in $[0, 1]$, then a nonproportional decomposition can be given by taking

$$f_1(x) = \min \{f(x), (1/2) [f(0) + f(1)]\}$$

and $f_2 = f - f_1$. Therefore, the extremal elements of $M_i$ are precisely the functions in $M_i$ which assume exactly one positive value in $[0, 1]$.

Let $f \in M_n$, $n > 1$, and let $a_0 = 0$ if $i_1 = 0$ and $a_0 = 1$ if $i_1 = 1$. If $f(a_0) > 0$ and $f$ is not constant, then take $f_1 = f(a_0)$ and $f_2 = f - f_1$. 

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In so doing, $f_1$ and $f_2 \in M_n$ and $f_1$ and $f_2$ are not proportional to $f$. Therefore, the only extremal elements $f$ of $M_n$ with $f(a_0) > 0$ are the positive constant functions.

Let $f \in M_n$, $n > 1$, and define $a_0' = 1 - a_0$, if $i_2 = 0$ and $a_0' = a_0$ if $i_2 = 1$, where $a_0$ is defined above. It can be shown that if $f \in M_n$, then $f$ must be continuous on $[0, 1]$ except at $a_0'$ [9, p. 148]. It follows that the only extremal elements of $M_1$ that are in $M_n$ are those which are continuous on $[0, 1]$ except, possibly, at $a_0'$, and these functions are again extremal elements of $M_n$.

If $i_2 = 0, f \in M_n$, $n > 1$, $f$ is not constant on $(0, 1)$ and $f$ is discontinuous at $a_0' = 1 - a_0$, then take $f_1(x) = 0$ for $x \in [0, 1]$ and $x \neq a_0'$,

$$f_1(a_0') = f(a_0') - \lim_{x \to a_0'} f(x) > 0$$

and $f_2 = f - f_1$. In so doing, $f_1$ and $f_2 \in M_n$ and $f_1$ and $f_2$ are not proportional to $f$. Hence, whenever $i_2 = 0$, the only extremal elements of $M_n$ that are discontinuous at $a_0' = 1 - a_0$ are the functions which are positive at $a_0'$ and zero elsewhere on $[0, 1]$.

On the other hand, if $i_2 = 1, f \in M_n$, $n > 1$, $f$ is not constant on $(0, 1)$ and $f$ is discontinuous at $a_0' = a_0$, then let

$$f_1(x) = \lim_{x \to a_0'} f(x) > 0 ,$$

$x \in [0, 1]$ and $x \neq a_0'$, $f_1(a_0') = 0$ and $f_2 = f - f_1$. Then $f_1$ and $f_2$ are in $M_n$ and $f_1$ and $f_2$ are not proportional to $f$. Therefore, whenever $i_2 = 1$, the only extremal elements of $M_n$ that are discontinuous at $a_0' = a_0$ are the functions which are zero at $a_0'$ and equal to a positive constant elsewhere on $[0, 1]$.

Consequently, the extremal elements of $M_n$, $n > 1$, which are not extremal elements of $M_1$ must be zero at $a_0$ and continuous on $[0, 1]$. It will be shown that these extremal elements of $M_n$ are indefinite integrals of the extremal elements of a cone which is similar to $M_1$. This cone is given in Definitions 1 and 2.

**DEFINITION 1.** If $g$ is a real function monotonic on $(0, 1)$ and $n > 1$, then define the (possibly extended real-valued) function $I(g, n - 1; \cdot)$ by the equation

$$I(g, n - 1; x) = \int_{a_0}^{x} \int_{a_1}^{t_1} \cdots \int_{a_{n-2}}^{t_{n-2}} \int_{a_{n-2}}^{t_{n-2}} g(t) \, dt \, dt_{n-2} \cdots \, dt_2 \, dt_1$$

for $x \in (0, 1)$, where $a_0 = (1/2) [1 - (-1)^{i_1}]$ and

$$a_j = (1/2) [1 - (-1)^{i_j+i_{j+1}}], 1 \leq j \leq n - 2.$$
DEFINITION 2. Let $K_n$, $n > 1$, denote the convex cone of real functions $g$ on $(0,1)$ such that

(a) $g$ is right-continuous;
(b) $(-1)^{(1)}(1-n)g(x) \geq 0$, for $x \in (0,1)$;
(c) $(-1)^{(1)}(1-n)\Delta g(x) \geq 0$, for $0 < x < x + h < 1$;
(d) $I(g, n-1; x)$ is finite, for $x \in (0,1)$; and
(e) $\lim_{x \to -1} I(g, n-1; x)$ exists and is finite.

Note. If $g \in K_n$, $n > 1$, then $I(g, n-1; \cdot)$ will denote the function which is the continuous extension to $[0,1]$ of the function given in Definition 1.

DEFINITION 3. Let $a$ and $b$ be two distinct numbers in the interval $[0,1]$ and define the function $\chi_{(a,b)}$ on $(0,1)$ by

$\chi_{(a,b)}(x) = 1$, if $x$ is between $a$ and $b$ or $0 < x = \min \{a, b\}$;
$\chi_{(a,b)}(x) = 0$, otherwise.

DEFINITION 4. If $m$ is a nonzero real number, $\xi \in [0,1]$ and $n > 1$, then define the function $e(m, \xi, n-1; \cdot)$ by the equation

$e(m, \xi, n-1; x) = mI(\chi_{(\xi,1-a_{n-1})}, n-1; x)$

for $0 \leq x \leq 1$, where $a_{n-1} = (1/2) [1 + (1/2)^{(1)}]^{(1)}$.

The principal theorem of this section can now be stated and the remainder of the section will be devoted to its proof. The key results are Lemma 3 and Proposition 2.

THEOREM 1. The extremal elements of $M_1$ are the functions in $M_1$ which assume exactly one positive value in $[0,1]$. The positive constant functions and the extremal elements of $M_1$ which are discontinuous at $a_1' = (1/2) [1 + (1/2)^{(1)}]^{(1)}$ are extremal elements of $M_n$, $n > 1$. The functions $e(m, \xi, n-1; \cdot)$, where $(-1)^{(1)}(1-n) m > 0$ and $\xi \in (0,1)$ or $\xi = a_{n-1}$ are extremal elements of $M_n$, $n > 1$. There are no other extremal elements of $M_n$, $n > 2$, are those functions $e(m, a_k, k; \cdot)$, where $(-1)^{(1)}(1-k) m > 0$ and $1 \leq k \leq n - 2$.

In the same manner that the extremal elements of $M_1$ were found, it can be shown that the extremal elements of $K_n$ are precisely those functions in $K_n$ which assume exactly one nonzero value in $(0,1)$. Before determining the extremal elements of $M_n$, it is shown in the following three lemmas how the $n$-monotone functions are related to the functions in $K_n$, where $n > 1$. 
LEMMA 1. If \( f \in M_n \), then \( f^{(n-1)} \in K_n \), where \( n > 1 \).

Proof. Since \( (-1)^{(n)} A_n^* f(x) \geq 0 \) for \( 0 \leq x < x + nh \leq 1 \), then \( f^{(n-2)} \) exists and is continuous on \((0,1)\) and \( (-1)^{(n)} f^{(n-2)} \) is convex [1]. Therefore \( (-1)^{(n)} A_n^* f^{(n-2)} \) has a right-continuous, nondecreasing right-hand derivative [4, p. 10]. It follows that \( (-1)^{(n)} A_n^* f^{(n-1)}(x) \geq 0 \) for \( 0 < x + h < 1 \). If \( f \in M_n \), then \( (-1)^{(n-1)} A_n^{-1} f(x) \geq 0 \) for \( 0 \leq x < x + (n - 1) h \leq 1 \), which implies that

\[
(-1)^{(n-1)} A_{n-1}^* A_{n-2}^* \cdots A_1^* f(x) \geq 0
\]

for \( 0 \leq x < x + \delta_1 + \delta_2 + \cdots + \delta_{n-1} \leq 1 \) [1]. It then follows that \( (-1)^{(n-1)} f^{(n-1)}(x) \geq 0 \) for \( 0 < x < 1 \), since \( f^{(n-1)} \) exists on \((0,1)\). It remains to show that

\[
\lim_{x \to 1^{-}} I(f^{(n-1)}, n - 1; x)
\]

exists and is finite and this proof will be by induction on \( n \).

If \( f \in M_2 \), then

\[
f(x) = \int_{a_0}^{x} f'_+(t) \, dt + \lim_{x \to a_0} f(x),
\]

which implies that

\[
\lim_{x \to a_0^{-}} I(f^{(n-1)}, n - 1; x) = \lim_{x \to a_0^{-}} f(x) - \lim_{x \to a_0^{-}} f(x)
\]

and this latter limit exists and is finite since \( f \) is monotonic on \([0,1]\) [4, Theorem 1.1]. Now assume that \( f \in M_n \) implies that

\[
\lim_{x \to a_0^{-}} I(f^{(n-1)}, n - 1; x)
\]

exists and is finite and let \( f \in M_{n+1} \). Then \( f \in M_n \) and it follows from the first part of the proof that \( (-1)^{(n-1)} f^{(n-1)} \) is nonnegative and monotonic on \((0,1)\) and

\[
(-1)^{(n-1)} f^{(n-1)}(a_{n-1}) = \lim_{x \to a_{n-1}^{-}} (-1)^{(n-1)} f^{(n-1)}(x)
\]

\[
= \inf \{ (-1)^{(n-1)} f^{(n-1)}(x); 0 < x < 1 \}.
\]

Therefore,

\[
\lim_{x \to a_0^{-}} I(f^{(n)}, n; x)
\]

\[
= \lim_{x \to a_0^{-}} I(f^{(n-1)} - f^{(n-1)}(a_{n-1}), n - 1; x)
\]

\[
= \lim_{x \to a_0^{-}} I(f^{(n-1)}, n - 1; x) - f^{(n-1)}(a_{n-1}) I(1, n - 1; x)
\]

exists and is finite by the induction hypothesis.
LEMMA 2. If $g \in K_n$, then $I(g, n - 1; \cdot) \in M_n$, where $n > 1$.

Proof. The proof will be by induction on $n$. If $g \in K_2$, then

$$I(g, 1; x) = \int_{a_0}^{x} g(t) \, dt$$

for $x \in [0, 1]$, and since $(-1)^{(i_1)}g(t) \geq 0$, $t \in (0, 1)$, and

$$a_0 = (1/2) \left[ 1 - (-1)^{(i_1)} \right],$$

then $I(g, 1; x) \geq 0$. If $0 \leq x < x + h \leq 1$, then

$$(-1)^{(i_1)}A_{\chi}^1 I(g, 1; x) = \int_{x}^{x+h} (-1)^{(i_1)}g(t) \, dt \geq 0.$$  

Since $(-1)^{(i_2)}g$ is nondecreasing, then $I((-1)^{(i_2)}g, 1; \cdot)$ is convex [4, p. 13]. It follows that $(-1)^{(i_2)}A_{\chi}^2 I(g, 1; x) \geq 0$ for $0 \leq x < x + 2h \leq 1$, and hence, $I(g, 1; \cdot) \in M_n$. Assume that $I(g, n - 1; \cdot) \in M_n$ for $g \in K_n$ and $n > 1$. If $g \in K_{n+1}$, then let

$$f(x) = \int_{a_{n-1}}^{x} g(t) \, dt,$$  

for $x \in (0, 1)$. Since $(-1)^{(i_n)}g$ is nonnegative and

$$a_{n-1} = (1/2) \left[ 1 - (-1)^{(i_{n-1}+i_n)} \right],$$

it is easily seen that $f \in K_n$ and it follows from the induction hypothesis that $I(g, n; \cdot) = I(f, n - 1; \cdot) \in M_n$. By a repeated application of the mean value theorem for a Riemann integral, it can be shown that

$$A_{\chi}^{n} I(g, n; x) = h^{n-1}f(\xi)$$

for $0 \leq x < \xi < x + (n - 1)h \leq 1$. Since $(-1)^{(i_{n+1})}g$ is nondecreasing, then $(-1)^{(i_{n+1})}f$ is convex on $(0, 1)$ [4, p. 13]. It follows that

$$(-1)^{(i_{n+1})}A_{\chi}^{n+1} I(g, n; x) = (-1)^{(i_{n+1})}A_{\chi}^{n} A_{\chi}^{n} I(g, n; x) = (-1)^{(i_{n+1})}A_{\chi}^{n} f(\xi) \geq 0$$

for $0 \leq x < x + (n + 1)h \leq 1$, and this inequality, together with the fact that $I(g, n; \cdot) \in M_n$ implies that $I(g, n; \cdot) \in M_{n+1}$.

In the proofs that follow, $f^{(k)}(a_k)$ should be interpreted as

$$f^{(k)}(a_k) = \lim_{x \to a_k} f^{(k)}(x),$$

where $f \in M_n$, $n > 2$, and $1 \leq k \leq n - 2$. Since $f^{(k)} \in K_{k+1}$, this limit will always exist and be finite. It is a consequence of Lemmas 1 and
2 that \( f = I(f^{(n-1)}, n-1; \cdot) \) whenever \( f \in \mathcal{M}_n, n > 1 \), and \( f^{(k)}(a_k) = 0 \) for \( 0 \leq k \leq n-2 \). It is shown in the following lemma that extremal elements of \( \mathcal{M}_n \) can be obtained directly from the extremal elements of \( \mathcal{K}_n \).

**Lemma 3.** If \( g \in \mathcal{K}_n \) and \( f = I(g, n-1; \cdot) \), then \( f \) is an extremal element of \( \mathcal{M}_n \) if, and only if, \( g \) is an extremal element of \( \mathcal{K}_n \), where \( n > 1 \).

**Proof.** Suppose that \( f \) is an extremal element of \( \mathcal{M}_n \). If \( g_1 \) and \( g_2 \in \mathcal{K}_n \) such that \( g = g_1 + g_2 \), then

\[
I(g, n-1; \cdot) = I(g_1, n-1; \cdot) + I(g_2, n-1; \cdot).
\]

If \( f_j = I(g_j, n-1; \cdot), j = 1, 2 \), then \( f_1 \) and \( f_2 \in \mathcal{M}_n \) and \( f = f_1 + f_2 \). Since \( f \) is an extremal element of \( \mathcal{M}_n \), there are numbers \( \lambda_j \geq 0 \) such that \( f_j = \lambda_j f_j, j = 1, 2 \), which implies that \( g_j = \lambda_j f_j^{(n-1)} = \lambda_j g, j = 1, 2 \), and \( g \) is therefore an extremal element of \( \mathcal{K}_n \).

Conversely, if \( g \) is an extremal element of \( \mathcal{K}_n \) and \( f_1 \) and \( f_2 \in \mathcal{M}_n \) such that \( f = f_1 + f_2 \), then \( g_1 \) and \( g_2 \in \mathcal{K}_n \) and \( g_1 + g_2 = f_1^{(n-1)} = g \), where \( g_j \) is the \( (n-1) \)th right derivative of \( f_j, j = 1, 2 \). This implies there are constants \( \lambda_j \geq 0, j = 1, 2 \), such that \( g_j = \lambda_j g \). It is evident from the definition of \( f \) that \( f^{(k)}(a_k) = 0 \), where \( 0 \leq k \leq n-2 \). This, together with the fact that \( f_j^{(k)} \in \mathcal{K}_{k+1} \) for \( 1 \leq k \leq n-2 \), implies that \( f_j^{(k)}(a_k) = 0, j = 1, 2 \) and \( 0 \leq k \leq n-2 \).

Hence,

\[
f_j = I(g_j, n-1; \cdot) = I(\lambda_j g, n-1; \cdot) = \lambda_j I(g, n-1; \cdot) = \lambda_j f
\]

for \( j = 1, 2 \), and \( f \) is therefore an extremal element of \( \mathcal{M}_n \).

**Proposition 1.** The function \( e(m, \xi, n-1; \cdot) \) is an extremal element of \( \mathcal{M}_n, n > 1 \), where \( (-1)^{i(n-1)} m > 0 \) and \( \xi \in (0, 1) \) or \( \xi = a_{n-1} \).

**Proof.** Since \( m\chi_{(\xi, a_{n-1})} \) is an extremal element of \( \mathcal{K}_n \) whenever \( (-1)^{(i(n-1)} m > 0 \) and \( \xi \in (0, 1) \) or \( \xi = a_{n-1} \), and

\[
e(m, \xi, n-1; \cdot) = I(m \chi_{(\xi, a_{n-1})}, n-1; \cdot),
\]

the result follows immediately from Lemma 3.

**Proposition 2.** The function \( e(m, a_k, k; \cdot) \) is an extremal element of \( \mathcal{M}_n, n > 2 \), where \( (-1)^{(k)} m > 0 \) and \( 1 \leq k \leq n-2 \).

**Proof.** Since \( \mathcal{M}_n \) is a subcone of \( \mathcal{M}_{k+1} \) and \( e(m, a_k, k; \cdot) \) is an extremal element of \( \mathcal{M}_{k+1} \), it is sufficient to show that
If \( f = e(m, a_k, k; \cdot) \), then \( f = I (f^{(k)}, k; \cdot) \), where

\[
f^{(k)}(x) = m\chi_{(a_k,1)}(x) = m\chi_{(0,1)}(x) = m
\]

for \( 0 < x < 1 \). Since \( f^{(k)} \) is constant on \((0,1)\), it follows from a repeated application of the mean value theorem for a Riemann integral that

\[
\Delta_{k+1}^k f(x) = \Delta_{k}^k \Delta_{k}^1 f(x) = h^k \Delta_{k}^1 f^{(k)}(\xi) = 0
\]

for \( 0 < x < x + (k + 1)h \leq 1 \), where \( x < \xi < x + kh \) and thus, \( \Delta_{k}^1 f(x) = 0 \) for \( 0 < x < x + ph \leq 1 \) and \( p \geq k + 1 \). Hence, \( f \in M_n \), for every \( n \), which implies that \( f \) is an extremal element of \( M_p \), for \( p \geq k + 1 \).

It will follow, as a consequence of the next three lemmas, that no other functions in \( M_n \) are extremal elements of \( M_n \), \( n > 2 \).

**Lemma 4.** Let \( f \in M_n \), \( n > 2 \), such that \( f(a_0) = 0 \), \( f \) is continuous on \([0, 1]\) and \( f \neq e(m, a_k, k; \cdot) \) for \((-1)^{i_k} m > 0 \) and \( 1 \leq k \leq n - 2 \). If there is an integer \( k \) such that \( 1 \leq k \leq n - 2 \) and \( f^{(k)}(a_k) \neq 0 \), then \( f \) is not an extremal element of \( M_n \).

**Proof.** Let \( k \) denote the smallest integer such that \( f^{(k)}(a_k) \neq 0 \). Then \( f \in M_n \subset M_{k+2} \) implies that \( f^{(k+1)}(x) \in K_{k+2} \), and it follows from Lemma 2 that \( I(f^{(k+1)}; k+1; \cdot) \in M_{k+2} \). Since \( f(a_0) = 0 \) and \( f^{(p)}(a_p) = 0 \) for \( 1 \leq p < k \), then

\[
I(f^{(k+1)}; k+1; \cdot) = I(f^{(k)}; k; \cdot) - f^{(k)}(a_k) I(1, k; \cdot) = f - e(m, a_k, k; \cdot)
\]

where \( m = f^{(k)}(a_k) \). Since

\[
\Delta_{k}^p e(m, a_k, k; x) = 0
\]

for \( 0 < x < x + ph \leq 1 \) and \( k + 1 \leq p \leq n \) and \( f \in M_n \), it follows that

\[
(-1)^{i_p} \Delta_{k}^p I(f^{(k+1)}; k+1; x) = (-1)^{i_p} \Delta_{k}^p f(x) \geq 0
\]

for \( 0 < x < x + ph \leq 1 \) and \( k + 1 \leq p \leq n \). Hence,

\[
f - e(m, a_k, k; \cdot) \in M_n ,
\]

where \( m = f^{(k)}(a_k) \), and a nonproportional decomposition of \( f \) can be given by taking \( f_1 = e(m, a_k, k; \cdot) \) and \( f_2 = f - f_1 \). Thus \( f \) is not an extremal element.

**Lemma 5.** Let \( f \in M_n \), \( n > 2 \), such that \( f \neq 0 \), \( f(a_0) = 0 \), \( f \) is
continuous on \([0, 1]\) and \(f \neq e(m, a_k, k; \cdot)\) for \((-1)^{i_k} m > 0\) and 1 \(\leq k \leq n - 2\). If \(f_{+}^{(n-1)} = 0\) on \((0, 1)\), then \(f\) is not an extremal element of \(M_n\).

Proof. If \(f_{+}^{(n-1)} = 0\), then there is a positive integer \(k \leq n - 2\) such that \(f^{(k)} \neq 0\) and \(f^{(k)}\) is constant on \((0, 1)\). Thus, \(f^{(k)}(a_k) \neq 0\) and it follows from Lemma 4 that \(f\) is not an extremal element.

It follows from Lemmas 4 and 5 that if \(f\) is an extremal element of \(M_n\), \(n > 2\) such that \(f(a_0) = 0\), \(f\) is continuous on \([0, 1]\) and either \(f_{+}^{(n-1)} = 0\) or \(f^{(k)}(a_k) \neq 0\) for some \(k\), 1 \(\leq k \leq n - 2\), then \(f = e(m, a_k, k; \cdot)\), where \((-1)^{i_k} m > 0\) and 1 \(\leq k \leq n - 2\).

**Lemma 6.** Let \(f \in M_n\), \(n \geq 2\), such that \(f\) is continuous on \([0, 1]\), \(f_{+}^{(n-1)} \neq 0\) and \(f^{(k)}(a_k) = 0\) for 0 \(\leq k \leq n - 2\). If \(f\) is an extremal element of \(M_n\), then \(f = e(m, \xi, n - 1; \cdot)\), where \((-1)^{i_{n-1}} m > 0\) and \(\xi \in (0, 1)\) or \(\xi = a_{n-1}\).

Proof. Since \(f^{(k)}(a_k) = 0\) for 0 \(\leq k \leq n - 2\), then

\[ f = I(f_{+}^{(n-1)}, n - 1; \cdot) \]

and it follows from Lemma 3 that \(f_{+}^{(n-1)}\) is an extremal element of \(K_n\). Thus, \(f_{+}^{(n-1)} = m\lambda_{\xi, n-1-a_{n-1}}\) for \((-1)^{i_{n-1}} m > 0\) and \(\xi \in (0, 1)\) or \(\xi = a_{n-1}\), which implies that \(f = I(f_{+}^{(n-1)}, n - 1; \cdot) = e(m, \xi, n - 1; \cdot)\). This completes the proof of Theorem 1.

2. Integral representations. The set of functions \(M_n - M_n\), \(n \geq 1\), forms the smallest linear space containing the convex cone \(M_n\). With the topology of simple convergence, \(M_n - M_n\) is a Hausdorff locally convex space such that for each \(x \in [0, 1]\), the linear functional \(L_x\) defined by \(L_x(f) = f(x)\) is continuous.

**Proposition 3.** The set \(M_n\) is closed in \(M_n - M_n\) for \(n \geq 1\).

Proof. The linear functional \(F\) defined on \(M_n - M_n\) by \(F(f) = \Delta^n_x f(x)\), for \([x, x + nh] \subset [0, 1]\), is continuous in the topology of simple convergence. By definition, \(M_n\) is the intersection of a collection of closed half-spaces corresponding to such functionals.

Since \(M_n\) is closed and every \(n\)-monotone function \(f\) is nonnegative and bounded by \(f(1 - a_0)\), Tychonoff's theorem implies that the normalized \(n\)-monotone functions, namely

\[ C_n = \{f \in M_n; f(1 - a_0) = 1\}, \]
form a compact base for $M_n$, $n \geq 1$. Thus, every nonzero $n$-monotone function can be uniquely expressed as a positive multiple of some $f$ in $C_n$ and $f$ is an extreme point of the convex set $C_n$ if, and only if, $f$ is an extremal element of $M_n$ which lies in $C_n$.

**Definition 5.** For $n \geq 2$, let $m_\xi$ denote the number which satisfies the equation $e(m_\xi, \xi, n - 1; 1 - a_0) = 1$, where $\xi \in (0, 1)$ or $\xi = a_{n-1}$. For $n > 2$, let $m_k$ denote the constant which satisfies the equation $e(m_k, a_k, k; 1 - a_0) = 1$, where $1 \leq k \leq n - 2$. Let $\text{ext } C_n$ denote the set of extreme points of $C_n$, $n \geq 1$, and let $e(m_0, a_0, 0; \cdot)$ denote the unique function in $\text{ext } C_n$, $n \geq 2$, which is discontinuous at $a'_0 = (1/2) [1 + (-1)^{(i+1)}]$; that is, $e(m_0, a_0, 0; x) = (1/2)[1 - (-1)^{(i+1)}]$ for $0 < x < 1$, $e(m_0, a_0, 0; a_0) = 0$ and $e(m_0, a_0, 0; 1 - a_0) = 1$.

The principal theorem of this section can now be stated and the remainder of the section will be devoted to its proof.

**Theorem 2.** To each $f \in C_n$, $n \geq 2$, there correspond unique non-negative regular Borel measures $\nu$ and $\mu$ on $[0, 1]$ and

$$\{e(m_k, a_k, k; \cdot): 0 \leq k \leq n - 2\},$$

respectively, such that

$$\nu([0, 1]) + f(a_0) + \sum_{k=0}^{n-2} \mu[e(m_k, a_k, k; \cdot)] = 1$$

and

$$f(x) = \int_0^1 e(m_\xi, \xi, n - 1; x) \, d\nu(\xi) + f(a_0) + \sum_{k=0}^{n-2} \alpha_k e(m, a_k, k; x)$$

for each $x \in [0, 1]$, where $\alpha_k = \mu[e(m_k, a_k, k; \cdot)]$ for each $k$ and

$$e(m_{1-a_{n-1}}, 1 - a_{n-1}, n - 1; \cdot) = e(m_{k_0}, a_{k_0}, k_0; \cdot)$$

denotes the function which is the pointwise limit of the functions $e(m_\xi, \xi, n - 1; \cdot)$ as $\xi$ approaches $1 - a_{n-1}$. Thus, each $n$-monotone function is a scalar multiple of such a representation.

Theorem 2 will be proved by using an integral reformulation of the Krein-Milman theorem. In order to apply this result, it must first be demonstrated that $\text{ext } C_n$ is closed.

**Proposition 4.** The set of extreme points of $C_n$ is closed in $C_n$, $n \geq 2$. 

Proof. Since $C_n$ with the relative topology is a subspace of a first countable space, it will suffice to show that if $\{f_i\}$ is a sequence of functions in $\text{ext} \ C_n$ which converges pointwise to the function $f$, then $f \in \text{ext} \ C_n$ [3, p. 164]. Since all except a finite number of the functions in $\text{ext} \ C_n$ are of the form $e(m_{\xi_i}, \xi_i, n - 1; \cdot)$, where $\xi_i \in (0, 1)$ or $\xi = a_{n-i}$, it can be assumed without loss of generality that $f_i = e(m_{\xi_i}, \xi_i, n - 1; \cdot)$ for each $i$.

If $a_0 = a_1 = \cdots = a_{n-1}$, then the function in $C_n$ are convex and

$$f_i(x) = \left(\frac{x - \xi_i}{1 - a_0 - \xi_i}\right)^{n-1} \chi_{(\xi_i, 1-a_0)}(x)$$

for $x \in (0, 1)$. If the sequence $\{\xi_i\}$ of real numbers converges to $1 - a_0$, then it is easily seen that

$$\lim_{i \to \infty} f_i(x) = 0$$

for $x \in (0, 1)$ or $x = a_0$. Since the topology of simple convergence is a Hausdorff topology, it follows that $f(1 - a_0) = 1$ and $f(x) = 0$, otherwise, which implies that $f = e(m_0, a_0, 0; \cdot)$ and $f \in \text{ext} \ C_n$. On the other hand, if $\{\xi_i\}$ does not converge to $1 - a_0$, then there is a real number $\xi_0 \neq 1 - a_0$ and a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ such that $\{\xi_j\}$ converges to $\xi_0$. Hence,

$$\lim_{j \to \infty} f_j(x) = \lim_{j \to \infty} \left(\frac{x - \xi_j}{1 - a_0 - \xi_j}\right)^{n-1} \chi_{(\xi_j, 1-a_0)}(x)
= \left(\frac{x - \xi_0}{1 - a_0 - \xi_0}\right)^{n-1} \chi_{(\xi_0, 1-a_0)}(x)
= e(m_{\xi_0}, \xi_0, n - 1; x)$$

for each $x \in (0, 1)$. Therefore, since the topology is a Hausdorff topology, $f = e(m_{\xi_0}, \xi_0, n - 1; \cdot)$ and it follows that $f \in \text{ext} \ C_n$.

If $a_1 = a_2 = \cdots = a_{n-1}$ and $a_0 \neq a_{n-1}$, then the functions in $C_n$ are concave and

$$f_i(x) = 1 - \left(\frac{x - \xi_i}{a_0 - \xi_i}\right)^{n-1} \chi_{(\xi_i, a_0)}(x)$$

for $x \in (0, 1)$. If the sequence $\{\xi_i\}$ converges to $a_0$, then

$$\lim_{i \to \infty} f_i(x) = 1$$

for $x \in (0, 1)$ or $x = 1 - a_0$ and $f = e(m_0, a_0, 0; \cdot)$. On the other hand, if there is a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ which converges to $\xi_0 \neq a_0$, then
\[
\lim_{j \to \infty} f_j(x) = \lim_{j \to \infty} \left[ 1 - \left( \frac{x - \xi_j}{a_0 - \xi_j} \right)^{n-1} \chi_{(\xi_j, a_0)}(x) \right] \\
= 1 - \left( \frac{x - \xi_0}{a_0 - \xi_0} \right)^{n-1} \chi_{(\xi_0, a_0)}(x) = e(m_{\xi_0}, \xi_0, n - 1; x)
\]
for each \( x \in (0, 1) \) and \( f = e(m_{\xi_0}, \xi_0, n - 1; \cdot) \). In either case, it follows that \( f \in \text{ext } C_\kappa \).

If there are exactly \( p > 0 \) integers \( k_1, \ldots, k_p \) such that
\[
1 \leq k_1 < k_2 < \cdots < k_p \leq n - 2
\]
and \( a_{k_j} = a_{n-1}, 1 \leq j \leq p, \) and \( a_0 = a_{n-1}, \) then
\[
f_j(x) = m_{\xi_j} \left[ \frac{(x - \xi_j)^{n-1}}{(n - 1)!} \chi_{(\xi_j, a_{n-1})}(x) \right] \\
+ \sum_{r=1}^{p} (-1)^r \sum_{j_r=r}^{j_2-1} (1 - a_0 - \xi_j)^{n-k_{j_r}-1}(1 - 2a_0)^{k_{j_r}-k_{j_1}}(x - a_0)^{k_{j_1}}
\]
for \( x \in (0, 1) \), where
\[
m_{\xi_j} = \frac{(1 - a_0 - \xi_j)^{n-1}}{(n - 1)!} \\
+ \sum_{r=1}^{p} (-1)^r \sum_{j_r=r}^{j_2-1} (1 - a_0 - \xi_j)^{n-k_{j_r}-1}(1 - 2a_0)^{k_{j_r}-k_{j_1}}(x - a_0)^{k_{j_1}}(n - k_{j_r} - 1)!(k_{j_r} - k_{j_r-1})! \cdots (k_{j_2} - k_{j_1})!(k_{j_1})!.
\]
If there is a subsequence \( \{\xi_j\} \) of \( \{\xi_i\} \) which converges to \( \xi_0 \neq 1 - a_0 \), then it is easily seen that
\[
f(x) = \lim_{j \to \infty} f_j(x) = e(m_{\xi_0}, \xi_0, n - 1; x)
\]
for each \( x \in (0, 1) \). On the other hand, if \( \{\xi_i\} \) converges to \( 1 - a_0 \), then
\[
\lim_{j \to \infty} f_j(x) = m_{k_p} \left[ \frac{(x - a_0)^{k_p}}{(k_p)!} \right] \\
+ \sum_{r=1}^{p-1} (-1)^r \sum_{j_r=r}^{j_2-1} \sum_{j_1=1}^{j_r} (1 - 2a_0)^{k_p-k_{j_1}}(x - a_0)^{k_{j_1}}(k_{j_r} - k_{j_r-1})! \cdots (k_{j_2} - k_{j_1})!(k_{j_1})!.
\]
for \( x \in (0, 1) \), where
\[
m_{k_p} = \frac{(1 - 2a_0)^{k_p}}{(k_p)!} \\
+ \sum_{r=1}^{p-1} (-1)^r \sum_{j_r=r}^{j_2-1} \sum_{j_1=1}^{j_r} (1 - 2a_0)^{k_p-k_{j_1}}(k_{j_r} - k_{j_r-1})! \cdots (k_{j_2} - k_{j_1})!(k_{j_1})!.
\]
In either case, it follows that \( f \in \text{ext } C_\kappa \).
Finally if there are exactly \( p > 0 \) integers \( k_1, \ldots, k_p \) such that \( 1 \leq k_1 < k_2 < \cdots < k_p \leq n - 2 \) and \( a_{k_j} \neq a_{n-1} \), \( 1 \leq j \leq p \) and \( a_0 \neq a_{n-1} \), then

\[
\begin{align*}
    f_i(x) &= m_{\xi_i} \left[ \frac{(a_0 - \xi_i)^{n-1}}{(n - 1)!} - \frac{(x - \xi_i)^{n-1}}{(n - 1)!} \right] \chi_{(\xi_i, a_0)}(x) \\
    &\quad + \sum_{r=1}^{p} (-1)^r \sum_{j_r=r}^{j_{r-1}} \frac{(a_0 - \xi_i)^{n-k_{j_r}-1}(2a_0 - 1)^{k_{j_r}}}{(n-k_{j_r}-1)! (k_{j_r}-k_{j_{r-1}})! \cdots (k_{j_2}-k_{j_1})! (k_{j_1})!} \\
    &\quad - \sum_{r=1}^{p} (-1)^r \sum_{j_r=r}^{j_{r-1}} \frac{(a_0 - \xi_i)^{n-k_{j_r}-1}(2a_0 - 1)^{k_{j_r}}}{(n-k_{j_r}-1)! (k_{j_r}-k_{j_{r-1}})! \cdots (k_{j_2}-k_{j_1})! (k_{j_1})!} \\
\end{align*}
\]

for \( x \in (0, 1) \), where

\[
    m_{\xi_i} = \frac{(a_0 - \xi_i)^{n-1}}{(n - 1)!} \\
    + \sum_{r=1}^{p} (-1)^r \sum_{j_r=r}^{j_{r-1}} \frac{(a_0 - \xi_i)^{n-k_{j_r}-1}(2a_0 - 1)^{k_{j_r}}}{(n-k_{j_r}-1)! (k_{j_r}-k_{j_{r-1}})! \cdots (k_{j_2}-k_{j_1})! (k_{j_1})!}.
\]

If there is a subsequence \( \{\xi_j\} \) of \( \{\xi_i\} \) which converges to \( \xi_0 \neq a_0 \), then it is evident that

\[
    f(x) = \lim_{j \to \infty} f_j(x) = e(m_{\xi_0}, \xi_0, n - 1; x)
\]

for each \( x \in (0, 1) \). On the other hand, if \( \{\xi_i\} \) converges to \( a_0 \), then

\[
    \lim_{i \to \infty} f_i(x) = m_{k_p} \left[ \frac{(2a_0 - 1)^{(k_p)}}{(k_p)!} - \frac{(x - 1 + a_0)^{(k_p)}}{(k_p)!} \right] \\
    + \sum_{r=1}^{p-1} (-1)^r \sum_{j_r=r}^{j_{r-1}} \frac{(2a_0 - 1)^{k_{j_r}-k_{j_1}}[(2a_0 - 1)^{k_{j_1}} - (x - 1 + a_0)^{k_{j_1}}]}{(k_p-k_{j_r})! (k_{j_r}-k_{j_{r-1}})! \cdots (k_{j_2}-k_{j_1})! (k_{j_1})!} \\
    = e(m_{k_p}, a_{k_p}, k_p; x)
\]

for \( x \in (0, 1) \), where

\[
    m_{k_p}^{-1} = \frac{(2a_0 - 1)^{(k_p)}}{(k_p)!} \\
    + \sum_{r=1}^{p-1} (-1)^r \sum_{j_r=r}^{j_{r-1}} \frac{(2a_0 - 1)^{(k_p)}}{(k_p-k_{j_r})! (k_{j_r}-k_{j_{r-1}})! \cdots (k_{j_2}-k_{j_1})! (k_{j_1})!}.
\]

In either case it follows that \( f \in \text{ext } C_n \) and this completes the proof.
DEFINITION 6. Let $e_0$ denote the function in $\text{ext} C_n$ which is identically one and let $e(m_{1-\sigma_{n-1}}, 1 - a_{n-1}, n - 1; \cdot)$ be the function defined by

$$e(m_{1-\sigma_{n-1}}, 1 - a_{n-1}, n - 1; x) = \lim_{\xi \to 1-} e(m_\xi, \xi, n-1; x)$$

for $0 \leq x \leq 1$ and $n > 1$. Finally, let

$$e(m_{k_0}, a_{k_0}, k_0; \cdot) = e(m_{1-\sigma_{n-1}}, 1 - a_{n-1}, n - 1; \cdot)$$

and notice that $k_0 = 0$ if $a_1 = a_2 = \cdots = a_{n-1}$ or $k_0$ is the largest positive integer such that $a_{k_0} \neq a_{n-1}$.

If the mapping $\phi: [0, 1] \to \text{ext} C_n$, $n \geq 2$, is defined by

$$\phi(\xi) = e(m_\xi, \xi, n-1; \cdot)$$

for $0 \leq \xi \leq 1$, then it follows from the proof of Proposition 4 that $\phi$ is continuous. If $E = \phi([0, 1])$, then $\phi$ is a homeomorphism from $[0, 1]$ onto $E$, since $[0, 1]$ is a compact space and $E$ is a Hausdorff space. By the Krein-Milman representation theorem, to each $f$ in $C_n$ there corresponds a regular Borel probability measure $\mu$ on $\text{ext} C_n$ such that

$$L(f) = \int_{\text{ext} C_n} L d\mu$$

for each continuous linear functional $L$ on $M_n - M_n$, since both $C_n$ and $\text{ext} C_n$ are compact subsets of $M_n - M_n$, $n \geq 2$. For $0 \leq x \leq 1$, the evaluation functional $L_x$ defined by $L_x(f) = f(x)$ is continuous on $M_n - M_n$, so that

$$f(x) = \int_{\text{ext} C_n} L_x d\mu$$

for each $x \in [0, 1]$. Define $\nu$ on each Borel subset $B$ of $[0, 1]$ by

$$\nu(B) = \mu(\phi(B)); \text{ i.e., } \nu = \mu \phi .$$

Since $L_x[\phi(\xi)] = e(m_\xi, \xi, n-1; x)$, then

$$\int_E L_x d\mu = \int_{\phi^{-1}(E)} L_x \phi d(\mu \phi) = \int_0^1 e(m_\xi, \xi, n-1; x) d\nu(\xi)$$

for $0 \leq x \leq 1$. Finally, by observing that $\mu(e_0) = f(a_0)$, since $e_0$ is the only function in $\text{ext} C_n$ which is positive at $a_0$, Equation (1) can be written as
\[ f(x) = \int_0^1 e(m, x, n - 1; \alpha) \, d\nu(\xi) \]

\[ + f(a_0) + \sum_{k=0}^{n-2} e(m_k, a_k, k; x) \mu[e(m_k, a_k, k; \cdot)] . \]

It remains to prove that \( \mu \) is unique. Since \( \mu \) is supported by \( \text{ext } C_n \), then \( \mu \) is a maximal measure in Choquet's ordering \([6, \text{pp. 24, 70}]\). Thus, by the Choquet-Meyer uniqueness theorem, it suffices to prove that \( C_n \) is a simplex \([6, \text{p. 66}]\).

**Lemma 7.** Suppose \( f \in M_n - M_n \) and \( n \geq 2 \). Then there is a function \( g \in K_n \) such that \( g - f_{+}^{(n-1)} \in K_n \) and if \( h \) is any function in \( K_n \) such that \( h - f_{+}^{(n-1)} \in K_n \), then it must follow that \( h - g \in K_n \).

**Proof.** First assume that \( i_{n-1} = i_n = 0 \). Since \( f_{+}^{(n-1)} \in K_n - K_n \), then \( f_{+}^{(n-1)} \) is of bounded variation on every interval \([0, x]\), where \( 0 < x < 1 \). Define \( g(x) = f_{+}^{(n-1)}(0) + P_0^x(f_{+}^{(n-1)}) \), where \( P_0^x(f_{+}^{(n-1)}) \) denotes the positive variation of \( f_{+}^{(n-1)} \) over \([0, x]\), \( 0 \leq x < 1 \) \([8, \text{p. 85}]\). Then both \( g \) and \( g - f_{+}^{(n-1)} \) are nonnegative, nondecreasing and right-continuous on \([0, 1]\). If \( h \in K_n \) such that \( h - f_{+}^{(n-1)} \in K_n \), then it follows that \( h - g \) is nonnegative, nondecreasing and right-continuous on \([0, 1]\). Therefore,

\[ 0 \leq \lim_{x \to 1-a_0} I(h - g, n - 1; x) \leq \lim_{x \to 1-a_0} I(h, n - 1; x) , \]

which implies that both \( g \) and \( h - g \) are in \( K_n \).

If \( i_{n-1} \) and \( i_n \) are not both zero, then define

\[ y = (1/2) \left[ 1 - (-1)^{i_{n-1}+i_n}(1 - 2x) \right] \]

and

\[ F(x) = (-1)^{(i_{n-1})} f_{+}^{(n-1)}(y) \quad \text{for } 0 \leq x < 1 . \]

Let \( G(x) = F(0) + P_0^x(F) \) for \( 0 \leq x < 1 \) and define \( g(x) = (-1)^{(i_{n-1})} G(y) \). Then \( g \) and \( g - f_{+}^{(n-1)} \in K_n \) and it follows from the first part of the proof that if \( h \) and \( h - f_{+}^{(n-1)} \in K_n \), then \( h - g \in K_n \).

**Definition 7.** If \( u \) is a function in \( M_n - M_n \), \( n \geq 2 \), then define the functions \( u_k, 0 \leq k \leq n - 2 \), by

\[ u_0(x) = u(a_0) \quad \text{and} \quad u_k(x) = I(u^{(k)}(a_k), k; x) \quad \text{for } 1 \leq k \leq n - 2 \]

where \( x \in [0, 1] \).

**Lemma 8.** Suppose \( f \in M_n - M_n \) and \( n \geq 2 \). Then there is a
function \( g \in M_n \) such that \( g - f \in M_n \) and if \( h \) is any \( n \)-monotone function such that \( h - f \in M_n \), then it must follow that \( h - g \in M_n \).

**Proof.** First assume that \( f^{(k)}(a_k) = 0 \) for \( 0 \leq k \leq n - 2 \) and let \( g^{(n-1)}_+ \) denote the function in \( K_n \) guaranteed by Lemma 7. Define \( g = I(g^{(n-1)}_+, n - 1; \cdot) \); then \( g \in M_n \) and

\[
g - f = I(g^{(n-1)}_+ - f^{(n-1)}_+, n - 1; \cdot) \in M_n.
\]

If \( h \) is an \( n \)-monotone function such that \( h - f \in M_n \), then \( h^{(n-1)}_+ \) and \( h^{(n-1)}_+ - f^{(n-1)}_+ \in K_n \) and it follows that \( h^{(n-1)}_+ - g^{(n-1)}_+ \in K_n \). If \( h^{(k)}(a_k) = 0 \) for \( 0 \leq k \leq n - 2 \), then

\[
h - g = I(h^{(n-1)}_+ - g^{(n-1)}_+, n - 1; \cdot) \in M_n.
\]

If there is some integer \( p \) such that \( 0 \leq p \leq n - 2 \) and \( h^{(p)}(a_p) \neq 0 \), then let

\[
h = h - \sum_{k=0}^{n-2} h_k,
\]

where \( h_0 = h(a_0) \) and \( h_k = I(h^{(k)}(a_k), k; \cdot) \) for \( 1 \leq k \leq n - 2 \). Then \( h^{(k)}(a_k) = 0 \) for \( 0 \leq k \leq n - 2 \) and \( h \) and \( h - f \in M_n \), since \( h \) and \( h - f \in M_n \) (cf. proof of Lemma 4). It follows that \( h - g \in M_n \) which implies that

\[
h - g = \tilde{h} - g + \sum_{k=0}^{n-2} h_k \in M_n
\]

since \( h_k \) is an \( n \)-monotone function for \( 0 \leq k \leq n - 2 \).

On the other hand, if there is a nonnegative integer \( p \leq n - 2 \) such that \( f^{(p)}(a_p) \neq 0 \), then let

\[
\tilde{f} = f - \sum_{k=0}^{n-2} f_k
\]

where \( f_k \) is given by Definition 7. Since \( \tilde{f} \in M_n - M_n \) and \( \tilde{f}^{(k)}(a_k) = 0 \) for \( 0 \leq k \leq n - 2 \), it follows from the first part of the proof that there is an \( n \)-monotone function \( \tilde{g} \) such that \( \tilde{g} - \tilde{f} \in M_n \) and if \( h \) is an \( n \)-monotone function such that \( h - \tilde{f} \in M_n \), then \( h - \tilde{g} \in M_n \). Let \( k_j, 0 \leq j \leq p < n - 1 \), denote those integers for which

\[
(-1)^k f^{(k)}(a_k) > 0
\]

and define

\[
g = \tilde{g} + \sum_{j=0}^{p} f_{k_j}.
\]

Then \( g \in M_n \) since
Let \( f_{k_j} = I(f^{(k_j)}(a_{k_j}), k_j; \cdot) = e(f^{(k_j)}(a_{k_j}), a_{k_j}, k_j; \cdot) \in M_n \)
for \( 0 \leq j \leq p \), and
\[
g - f = \bar{g} + \sum_{j=0}^{p} f_{k_j} - f = \bar{g} - \bar{f} - \sum_{k \neq k_j} f_k \in M_n
\]
since \(-f_k \in M_n\) if \( k \neq k_j \). Suppose that \( h \) is an \( n \)-monotone function such that \( h - f \in M_n \). Then
\[
h - f - \sum_{k=0}^{n-2} (h - f)_k \in M_n
\]
which implies that
\[
h - f - \sum_{k \neq k_j} (h - f)_k = h - f - \sum_{k=0}^{n-2} (h - f)_k + \sum_{j=0}^{p} (h - f)_{k_j} \in M_n
\]
since \((h - f)_{k_j} \in M_n\) (cf. proof of Lemma 4). Since \( h_k \) is an \( n \)-monotone function for \( 0 \leq k \leq n - 2 \), then
\[
h - f + \sum_{k \neq k_j} f_k = h - f - \sum_{k \neq k_j} (h_k - f_k) + \sum_{k \neq k_j} h_k
\]
\[
= h - f - \sum_{k \neq k_j} (h - f)_k + \sum_{k \neq k_j} h_k \in M_n.
\]
Therefore,
\[
h - \sum_{j=0}^{p} f_{k_j} - \bar{f} = h - f + \sum_{k \neq k_j} f_k \in M_n
\]
and \( h - \sum_{j=0}^{p} f_{k_j} \in M_n \) since \( h - \sum_{j=0}^{p} h_{k_j} \in M_n \) and
\[
h - \sum_{j=0}^{p} f_{k_j} = h - \sum_{j=0}^{p} h_{k_j} + \sum_{j=0}^{p} (h_{k_j} - f_{k_j}) = h - \sum_{j=0}^{p} h_{k_j} + \sum_{j=0}^{p} (h - f)_{k_j}.
\]
It follows that \( h - \sum_{j=0}^{p} f_{k_j} - \bar{g} \in M_n \), which implies that \( h - g \in M_n \).

If the function \( g \) of Lemma 8 is denoted by \( f \lor 0 \), then the least upper bound of two functions \( f_1 \) and \( f_2 \in M_n - M_n \) can be given by \( f_1 + (f_2 - f_1) \lor 0 \) and therefore \( M_n - M_n \) is a vector lattice. Thus, \( C_n \) is a simplex and the proof of Theorem 2 is complete.

3. Remarks. If \( i_2 = 0 \), then \( C_2 \) is the set of functions \( f \) which are monotonic and convex on \([0, 1] \) such that \( \max \{f(x) : 0 \leq x \leq 1\} = 1 \). If \( i_1 = 0 \), then the \( C_2 \) functions are nondecreasing and \( e(m_\xi, \xi, 1; x) = 0 \), \( x \in [0, \xi] \) and \((x - \xi)/(1 - \xi)\) for \( x \in [\xi, 1] \), where \( 0 \leq \xi < 1 \). Thus, to each \( f \in C_2 \) there corresponds a unique nonnegative regular Borel measure \( \nu \) on \([0, 1] \) such that
\[
f(x) = f(0) + \int_0^x \frac{x - \xi}{1 - \xi} d\nu(\xi)
\]
for $0 < x < 1$. On the other hand, if $i_k = 1$, then these functions are nonincreasing and $e(m_{i_k}, \xi, 1; x) = 1 - (x/\xi)$, $x \in [0, \xi]$ and 0 for $x \in [\xi, 1]$, where $0 < \xi \leq 1$. It follows from Theorem 2 that to each $f$ in $C_0$ there corresponds a unique nonnegative regular Borel measure $\nu$ on $[0, 1]$ such that

$$f(x) = f(1) + \int_x^1 [1 - (x/\xi)] \, d\nu(\xi)$$

for $0 < x < 1$.

If $i_k = 0$ for every $k \leq n$, then $e(m_{i_k}, \xi, n - 1; x) = 0$, $x \in [0, \xi]$ and $[(x - \xi)/(1 - \xi)]^{n-1}$ for $x \in [\xi, 1]$, where $0 \leq \xi < 1$, and

$$e(m_{i_k}, 0, k; x) = x^k$$

for $x [0, 1]$, where $1 \leq k \leq n - 2$. Thus, for each function $f$ in $C_n$, there exist unique nonnegative real numbers $\alpha_1, \ldots, \alpha_{n-2}$ and a unique nonnegative regular Borel measure $\nu$ on $[0, 1]$ such that

$$f(x) = f(0) + \sum_{k=1}^{n-2} \alpha_k x^k + \int_0^x \left( \frac{x - \xi}{1 - \xi} \right)^{n-1} \, d\nu(\xi)$$

for $0 < x < 1$. In this case, the intersection of the $M_n$ cones is the class of absolutely monotonic functions on $[0, 1]$. It is well known that if $f \in C_n$ for every $n$, then

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \left( x^n/n! \right)$$

for $0 \leq x < 1$. For a discussion of these cones see [5].

Lastly, if $i_k = (1/2) [1 + (-1)^k]$ for $1 \leq k \leq n$, then

$$e(m_{i_k}, \xi, n - 1; x) = 1 - [1 - (x/\xi)]^{n-1},$$

$x \in [0, \xi]$ and 1 for $x \in [\xi, 1]$, where $0 < \xi \leq 1$, and

$$e(m_{i_k}, 1, k; x) = 1 - (1 - x)^k$$

for $x \in [0, 1]$, where $1 \leq k \leq n - 2$. It follows from Theorem 2 that for each function $f$ in $C_n$, there exist unique nonnegative real numbers $\alpha_1, \ldots, \alpha_{n-2}$ and a unique nonnegative regular Borel measure $\nu$ on $[0, 1]$ such that

$$f(x) = 1 - \sum_{k=1}^{n-2} \alpha_k (1 - x)^k - \int_x^1 [1 - (x/\xi)]^{n-1} \, d\nu(\xi)$$

for $0 < x < 1$. In this case, the $C_n$ functions were called alternating of order $n$ by Choquet [2, p. 170]. It can be shown that if $f \in C_n$ for every $n$, then

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(1) \left[ (x - 1)^n/n! \right]$$
for $0 < x \leq 1$. For a proof of this fact together with a discussion of these cones see [7].

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