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**INVARIANT FUNCTIONS OF AN ITERATIVE PROCESS FOR  
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Let  $P$  be a polynomial with real non-negative coefficients and variables  $x_{i,j}$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, n_i$ . Let  $d = \sum_{i=1}^k n_i$ . Let  $R_d$  be the  $d$ -dimensional real vector space. Let  $\tilde{M}$  be the subset of  $R_d$  defined by

$$\tilde{M} = \left\{ x \mid x \in R_d, x_{i,j} \geq 0, \sum_{j=1}^{n_i} x_{i,j} = 1 \right\}$$

where the symbols  $x_{i,j}$  denote the components of  $x$ . If  $x$  is a vector in the interior of  $\tilde{M}$ , define  $\tau(x)$  as the vector in  $\tilde{M}$  with components  $x'_{i,j}$  given by

$$x'_{i,j} = \frac{x_{i,j} \frac{\partial P}{\partial x_{i,j}}}{\sum_{h=1}^{n_i} x_{i,h} \frac{\partial P}{\partial x_{i,h}}}.$$

The expression on the right is evaluated at  $x$ . The transformation  $\tau$  is defined on the boundary of  $\tilde{M}$  by the same formula if the denominators do not vanish.

Let  $\tilde{F}$  be the set of fixed points of  $\tau$  in  $\tilde{M}$ . It is shown that if  $\tau$  is a homeomorphism of  $\tilde{M}$  onto itself, there is a set of  $d - k$  functions  $f_1, \dots, f_{d-k}$  defined on  $\tilde{M} - \tilde{F}$  such that  $f_i(x) = f_i(\tau(x))$  for  $x \in \tilde{M} - \tilde{F}$ . The functions  $f_i$  are continuous and independent on an open dense subset of  $\tilde{M} - \tilde{F}$ . Explicit expressions for certain invariant functions are also obtained.

1. The transformation  $\tau$ . The transformation  $\tau$  defined in the introduction can be used to iteratively find local maxima for the polynomial  $P$ . It was shown by L. E. Baum and J. A. Eagon [1] that if  $P$  is a homogeneous polynomial with positive coefficients and if  $x$  is an element of  $\tilde{M}$  such that  $\tau(x)$  is defined then either  $\tau(x) = (x)$  or  $P(\tau(x)) > P(x)$ . This result was generalized at the suggestion of O. Rothaus by L. E. Baum and G. R. Sell [2] to arbitrary polynomials with positive coefficients.

It will be assumed in this paper that the transformation  $\tau$  is a homeomorphism of  $\tilde{M}$  onto itself. According to an unpublished result of L. E. Baum,  $\tau$  is a homeomorphism of  $\tilde{M}$  onto itself if and only if the expression for  $P$  as a sum of distinct monomials with positive coefficients contains monomials  $c_{i,j} x_{i,j}^{w_{i,j}}$  for all  $i = 1, \dots, k$ ,  $j = 1, \dots, n_i$  where  $c_{i,j} > 0$  and  $w_{i,j}$  is an integer greater than zero. Since this condition is satisfied if and only if  $\tau$  is defined on all of  $\tilde{M}$ , a necessary and sufficient condition that  $\tau$  is a homeomorphism of  $\tilde{M}$  onto itself

is that  $\tau$  be defined on all of  $\tilde{M}$ . We will not prove L. E. Baum's result here, but will give a single example of a polynomial  $P$  for which  $\tau$  is a homeomorphism. Let

$$P = \sum_{i=1}^k \sum_{j=1}^{n_i} x_{i,j}^m .$$

The  $\tau$ -transformation associated with  $P$  is given by

$$x'_{i,j} = \frac{x_{i,j}^m}{\sum_{h=1}^{n_i} x_{i,h}^m} .$$

The inverse of  $\tau$  restricted to  $\tilde{M}$  is given by

$$x_{i,j} = \frac{x_{i,j}^{1/m}}{\sum_{h=1}^{n_i} x_{i,h}^{1/m}}$$

where the real positive  $m$ th roots are to be chosen.

## 2. The existence of invariants.

**2.1. Notation and definitions.** As above, we let  $\tilde{M}$  denote the space of real vectors  $(x_{1,1}, \dots, x_{1,n_1}, \dots, x_{k,1}, \dots, x_{k,n_k})$  satisfying  $x_{i,j} \geq 0$ , and

$$\sum_{j=1}^{n_i} x_{i,j} = 1 .$$

Let  $M$  be set of real vectors

$$(y_{1,1}, \dots, y_{1,n_1-1}, \dots, y_{k,1}, \dots, y_{k,n_k-1})$$

satisfying  $y_{i,j} \geq 0$  and

$$\sum_{j=1}^{n_i-1} y_{i,j} \leq 1 .$$

If  $y \in M$  let  $\psi(y)$  be the point of  $\tilde{M}$  with coordinates  $x_{i,j} = y_{i,j}$  for  $1 \leq j \leq n_{i-1}$  and

$$x_{i,n_i} = 1 - \sum_{j=1}^{n_i-1} y_{i,j} .$$

Clearly  $\psi$  is a homeomorphism of  $M$  onto  $\tilde{M}$ .

Let  $\varphi$  be a transformation of a set  $S$  onto itself. We inductively define  $\varphi^n(x)$  for  $n \geq 0$  and  $x \in S$  by  $\varphi^0(x) = x$  and  $\varphi^n(x) = \psi(\psi^{n-1}(x))$ . If  $\varphi$  is a one-to-one transformation of  $S$  onto itself, we inductively define  $\varphi^n(x)$  for  $n < 0$  and  $x \in S$  by the rule  $\varphi^{n-1}(x) = \varphi^{-1}(\varphi^n(x))$ . Also,

if  $\varphi$  is a one-to-one transformation of  $S$  onto itself, we have  $\varphi^{r+s}(x) = \varphi^r(\varphi^s(x))$  for all  $x \in S$  and all pairs of integers  $(r, s)$ .

Let  $\{x_n\}$  be a sequence of points of a topological space  $S$ . A cluster point of  $\{x_n\}$  is a point  $p$  of  $S$  such that every neighborhood of  $p$  contains infinitely many elements of the sequence  $\{x_n\}$ .

## 2.2. Proof of the existence theorem.

**LEMMA 2.1.** *The transformation  $T = \psi^{-1}\tau\psi$  of  $M$  into itself has the following properties:*

(i) *Let  $\bar{P}$  be the polynomial defined on  $M$  by the formula  $\bar{P}(y) = P(\psi(y))$  for  $y \in M$ . If  $y \in M$ , either  $y = T(y)$  or  $\bar{P}(T(y)) > \bar{P}(y)$ .*

(ii) *The set of fixed points of  $T$  on  $M$  is the union of the set of critical points of  $\bar{P}$  on  $M$  and the sets of critical points of  $\bar{P}$  restricted to boundary simplices of  $M$ .*

(iii) *The set of fixed points  $T$  in  $M$  has only finitely many components. Each component of the set of fixed points of  $T$  is compact and  $\bar{P}$  is constant on each of the components of the set of fixed points of  $T$ .*

(iv)  *$T$  is a homeomorphism of  $M$  onto itself if and only if  $\tau$  is a homeomorphism of  $M$  onto itself.*

(v) *If  $x \in M$ , every cluster point of a sequence  $\{T^n(x)\}$ ,  $n \geq 0$ , is a fixed point of  $T$ . If  $T$  is a homeomorphism, every cluster point of the sequence  $\{T^n(x)\}$  is a fixed point of  $T$ .*

*Proof.* To prove (i), let  $y$  be an element of  $M$  such that  $T(y) \neq y$ . Then  $\psi^{-1}\tau\psi(y) \neq y$  and  $\tau\psi(y) \neq \psi(y)$ . Thus  $\psi(y)$  is not a fixed point of  $\tau$  and it follows that

$$\bar{P}(T(y)) = P(\psi\psi^{-1}\tau\psi(y)) = P(\tau\psi(y)) > P(\psi(y)) = \bar{P}(y).$$

Statement (ii) may be well known but include a proof for the sake of completeness. Note first that  $\psi$  maps the set of fixed points  $T$  onto the set of fixed points of  $\tau$ . Let  $x$  be a fixed point of  $\tau$  in  $\tilde{M}$  and let  $x$  have coordinates  $(x_{i,j})$ . The equation  $\tau(x) = x$  implies the equations

$$x_{i,j} \left( \sum_1^{n_i} x_{i,k} \frac{\partial P}{\partial x_{i,k}} - \frac{\partial P}{\partial x_{i,j}} \right) = 0$$

for all  $i, j$ , and since  $\tau$  is defined at  $x$ , these equations imply  $\tau(x) = x$ . If  $x$  is an interior fixed point of  $M$ , no  $x_{i,j}$  is zero so that  $\tau(x) = x$  is equivalent to

$$\frac{\partial P}{\partial x_{i,j}} - \frac{\partial P}{\partial x_{i,n_i}} = 0$$

for all  $i, j$ . But this just the condition that  $\psi^{-1}(y)$  be a critical point of  $\bar{P}$ . Thus the fixed points of  $T$  interior to  $M$  are just the interior critical points of  $\bar{P}$ .

Now suppose  $y$  is a fixed point of  $T$  on the boundary of  $M$ . Clearly  $\psi(y)$  is a fixed point of  $\tau$  on the boundary of  $\tilde{M}$ . If  $\psi(y) = z = (z_{i,j})$ , certain variables  $x_{i,i}$  are zero at  $z$ . Let  $\tilde{M}_y$  be the part of the boundary of  $\tilde{M}$  determined by the equations  $x_{i,j} = 0$  for all  $i, j$  such that  $z_{i,j} = 0$ . If no  $z_{i,n_i}$  is zero, it follows as before that  $y$  is a critical point of  $\bar{P}$  restricted to  $M_y = \psi^{-1}(\tilde{M}_y)$ . Note that  $M_y$  is a subset of the boundary of  $M$ . If some  $z_{i,n_i}$  is zero, the variables  $u_{i,j}$  describing  $M_y$  are subject to the additional constraint  $\sum u_{i,j} = 1$ , where the sum is over the subscripts  $i, j$  such that  $z_{i,j} \neq 0$ . Since the partial derivatives  $\partial P / \partial x_{i,j}(z)$  are equal for  $i, j$  such that  $z_{i,j} \neq 0$ , it follows that  $y$  is a critical point of  $\bar{P}$  for  $\bar{P}$  restricted to  $M_y$ . Conversely, if  $y$  is a critical point of  $P$  restricted to  $M_y$ , it follows that  $y$  is a fixed point of  $T$ .

Let us prove (iii). Let  $R_d$  be  $d$ -dimensional real space, with coordinates  $x_{i,j}$  as described in the introduction. Let  $P$  be a polynomial defined on  $R_d$ . Let  $S_1$  be the set of points of  $R_d$  satisfying the equations:

$$\sum_{j=1}^{n_i} x_{i,j}^2 = 1 \text{ for all } i, \quad \text{and} \quad \frac{\partial P}{\partial x_{i,j}} = \frac{\partial P}{\partial x_{i,n_i}}$$

for all  $i, j$ , where the partial derivatives of  $P$  are evaluated at  $(x_{1,1}^2, \dots, x_{1,n_1}^2, \dots, x_{k,n_k}^2)$ . According to H. Whitney [5], a real algebraic variety such as  $S_1$  has only finitely many components and each component is a union of finitely many components of differentiable manifolds (of various dimensions). Let  $Q = P(x_{1,1}^2, \dots, x_{k,n_k}^2)$ . The partial derivatives of  $Q$  with respect to  $x_{i,j}$  for  $j < n_i$  with the restrictions

$$\sum_{j=1}^{n_i} x_{i,j}^2 = 1, \quad i = 1, \dots, k$$

are all zero on  $S_1$ . Thus  $Q$  can have only one value on a component of a differentiable manifold contained in  $S_1$ , and thus can have only finitely many values on  $S_1$ . Since  $Q$  is continuous and the components of  $S$  are arcwise connected,  $Q$  must be constant on each component of  $S_1$ .

Let  $\varphi$  be the mapping of  $R_d$  into itself given by  $\varphi(x_{1,1}, \dots, x_{k,n_k}) = (x_{1,1}^2, \dots, x_{k,n_k}^2)$ . The set  $S = \varphi(S_1)$  is given by the relations:

- (i)  $x_{i,j} \geq 0$  for all  $i, j$ ,
- (ii)  $\sum_{j=1}^{n_i} x_{i,j} = 1$  for  $i = 1, \dots, k$ , and
- (iii)  $\partial P / \partial x_{i,j} = \partial P / \partial x_{i,n_i}$  (evaluated at  $(x_{1,1}, \dots, x_{k,n_k})$ ) for all  $i, j$ .

Since  $\varphi$  is continuous,  $S$  can have only finitely many components. Since  $Q(x) = P(\varphi(x))$  for all  $x \in R_d$ , the range of  $P$  on  $S$  is then range

of  $Q$  on  $S_1$ . Hence  $P$  assumes only finitely many values on  $S$ , and by continuity of  $P$ ,  $P$  is constant on each component of  $S$ . Since  $S$  is just the  $\psi$  image of the set of critical points of  $\bar{P}$  on  $M$ ,  $S$  is the  $\psi$  image of the subset of fixed points of  $T$  corresponding to these critical points.

The same argument applies to the sets of critical points of  $\bar{P}$  restricted to the boundary sets of  $M$  given by certain  $x_{i,j} = 0$ . Since the set  $F$  of fixed points of  $T$  is the union of the set of critical points of  $\bar{P}$  on  $M$  and the sets of critical points of  $\bar{P}$  restricted to each of finitely many subsets of the boundary of  $M$ ,  $F$  has just finitely many components, and  $\bar{P}$  assumes only finitely many values on  $F$ . By continuity,  $\bar{P}$  is constant on each component of  $F$ . Since  $F$  is compact, each of its finitely many components is also compact.

Part (iv) of the lemma follows from the fact that  $\psi$  is a homeomorphism of  $M$  onto  $\tilde{M}$ . Since  $T = \psi^{-1}\tau\psi$ ,  $T$  is a homeomorphism of  $M$  onto  $M$  if  $\tau$  is a homeomorphism of  $\tilde{M}$  onto  $\tilde{M}$ . Since  $\tau = \psi T \psi^{-1}$ , the converse follows.

The final result, (v), follows directly from the Baum-Eagon inequality (c.f. Section 1 of this paper), and Lemma 2.1 of Bhatia-Szego [3].

In the following, we restrict our attention to those transformations  $\tau$  for which  $\tau$  is a homeomorphism of  $M$  onto itself and  $T$  is a homeomorphism of  $M$  onto itself.

There is an obvious relation between the functions  $f$  defined on  $M$  such that  $f(T(x)) = f(x)$  for all  $x$  in  $M$  and the functions  $g$  defined on  $\tilde{M}$  such that  $g(\tau(y)) = g(y)$  for all  $y \in \tilde{M}$ . If  $f(T(x)) = f(x)$  for all  $x \in M$  then  $g(y) = f(\psi(y))$  is such that

$$g(\tau(y)) = f(\psi\tau\psi^{-1} \cdot \psi(y)) = f(T\psi(y)) = f(\psi(y)) = g(y).$$

Conversely, if  $g(\tau(y)) = g(y)$  it is clear that  $f(x) = g(\psi^{-1}(x))$  is such that  $f(T(x)) = f(x)$ . Thus we can find all invariant functions of  $\tau$  from the invariant functions of  $T$ .

A spherical neighborhood of a point  $x$  of the interior of  $M$  is a  $d - k$  dimensional ball contained in  $M$  with center at  $x$ . If  $x$  is on the boundary of  $M$  in  $d - k$  dimensional real space, a spherical neighborhood of  $x$  in  $M$  is the intersection of  $M$  and an  $d - k$  dimensional ball with center at  $x$ .

**LEMMA 2.2.** *Let  $T$  be a homeomorphism of  $M$  onto itself. If  $x_0$  is a point of  $M$  but not a fixed point of  $T$ , there is a spherical neighborhood  $N$  of  $x_0$  in  $M$  such that the sets  $T^r(N)$  are disjoint for  $-\infty < r < \infty$ .*

*Proof.* Since  $x_0$  is not a fixed point of  $T$ ,  $T(x_0) \neq x_0$ . By Lemma

1, (i)  $\bar{P}(T(x_0)) - \bar{P}(x_0) = \Delta > 0$ . Since  $\bar{P}$  is continuous on  $M$ , there is a neighborhood  $U$  of  $x_0$  such that  $\bar{P}(x) < \bar{P}(x_0) + \Delta/3$  for all  $x \in U$  and a neighborhood  $V$  of  $T(x_0)$  such that  $\bar{P}(y) > \bar{P}(T(x_0)) - \Delta/3$  for all  $y \in V$ . Since  $T$  is a continuous transformation,  $T^{-1}(V) \cap U$  is a neighborhood of  $x_0$ . Let  $N$  be a spherical neighborhood of  $x_0$  contained in  $T^{-1}(V) \cap U$ . Since  $N \subset U$  and  $T(N) \subset V$ , for arbitrary  $x \in N$ ,  $y \in T(N)$  we have

$$\bar{P}(x) < \bar{P}(x_0) + \frac{\Delta}{3} \bar{P}(T(x_0)) - \frac{\Delta}{3} < \bar{P}(y).$$

If  $x \in N$  and  $z \in T^m(N)$  for  $m \geq 1$ ,  $z = T^m(u)$  for some  $u \in N$  and  $\bar{P}(z) \geq \bar{P}(T(u)) > \bar{P}(x)$  since  $T(u) \in T(N)$ . Thus  $T^m(N) \cap N$  is empty for  $m \geq 1$ .

Suppose  $T^r(N) \cap T^s(N)$  is not empty for  $r \neq s$ . We assume  $r > s$  and let  $y \in T^r(N) \cap T^s(N)$ . Then  $T^{-r}(y) \in N$  and  $T^{r-s}(T^{-r}(y)) = T^{-s}(y) \in N$  so that  $N$  and  $T^{r-s}(N)$  intersect. This contradiction shows that  $T^r(N) \cap T^s(N)$  is empty for  $r \neq s$ .

If  $x, y \in M$ , let  $|x - y|$  denote the Euclidean distance between  $x$  and  $y$ .

**LEMMA 2.3.** *Let  $T$  be a homeomorphism of  $M$  onto itself. There is a positive number  $\varepsilon$  such that if  $x$  is a point of  $M$  but not a fixed point of  $T$ , there is at least one element of the sequence  $\{T^n(x)\}$  at distance greater than or equal to  $\varepsilon$  from the set of fixed points of  $T$ .*

It follows from Baum and Sell [2] that the set  $F$  of fixed points of  $T$  is an asymptotically stable set. This Lemma is a consequence of Theorem 4.19 of Bhatia-Szego [3].

A fundamental set  $S$  for  $T$  on  $M$  is a subset of  $M$  defined as follows:  $S$  contains no fixed point of  $T$  but if  $x$  is not a fixed point of  $T$ ,  $T^n(x) \in S$  for a single integer  $n$  depending on  $S$  and  $x$ .

**LEMMA 2.4.** *If  $T$  is a homeomorphism of  $M$  onto itself,  $T$  has a measurable fundamental set.*

*Proof.* Let  $D_\varepsilon$  be the set of points of  $M$  at distance greater than or equal to  $\varepsilon$  from  $F$ , the set of fixed points of  $T$ . According to Lemma 2.3,  $\varepsilon > 0$  may be chosen so that  $D_\varepsilon$  contains at least one element of every sequence  $\{T^n(x)\}$  for  $x \notin F$ . Since  $D_\varepsilon$  does not meet  $F$ , it follows from Lemma 2.2 that about each  $x \in D_\varepsilon$  there is a spherical neighborhood  $N_x$  such that the sets  $T^n(N_x)$  are disjoint (if  $x$  is a boundary point of  $M$ , the set  $N_x$  is the intersection of a ball with  $M$ ). Since  $D_\varepsilon$  is compact, it is compact relative to  $M$  so that there may

be selected a finite covering  $N_1, \dots, N_r$  of  $D_\varepsilon$  from the sets  $N_\alpha$ . Clearly, each sequence  $\{T^n(x)\}$  for  $x \in M - F$  can meet an  $N_i$  in at most one point.

Let

$$L_1 = N_1, L_2 = N_2 - \bigcup_{-\infty}^{+\infty} T^n(N_1), \dots,$$

$$L_r = N_r - \bigcup_{-\infty}^{+\infty} T^n(N_1) - \bigcup_{-\infty}^{+\infty} T^n(N_2) - \dots - \bigcup_{-\infty}^{+\infty} T^n(N_{r-1}).$$

Clearly  $\bigcup_1^r L_i$  is a fundamental set for  $T$  in  $M$ . Since  $T$  is continuous and each  $N_i$  is measurable,  $\bigcup_{-\infty}^{+\infty} T^n(N_i)$  is measurable. Hence each  $L_i$  is measurable and  $\bigcup_1^r L_i$  is measurable.

Let  $\tilde{F}$  be the set of fixed points of  $\tau$  in  $M$ .

**THEOREM 1.** *If  $T$  is a homeomorphism of  $M$  onto itself, and  $F$  is the set of fixed points of  $T$ , there exist  $d - k$   $T$ -invariant functions of  $T$  which are continuous and independent on an open dense subset of  $M - F$ . Thus there are  $d - k\tau$  invariant functions continuous and independent on an open dense subset of  $\tilde{M} - \tilde{F}$ .*

*Proof.* Let  $S$  be a fundamental set for  $T$  on  $M$ , as constructed in the proof of Lemma 3.4. Let  $S^*$  be the boundary of  $S$  and let  $B = \bigcup_{-\infty}^{+\infty} T^n(S^*)$ . Then  $M - F - B$  is dense in  $M - F$ . For  $x \in M - F$  let  $\varphi(x)$  be the element of  $\{T^n(x)\}$  in  $S$ . We will show that  $\varphi$  is continuous on  $M - F - B$ .

If  $x \in M - F - B$ ,  $\varphi(x)$  is the unique intersection of  $\{T^n(x)\}$  with  $S$ . Hence there is an integer  $m$  such that  $T^m(x) \in S$ . Since  $x \notin B$ ,  $T^m(x)$  is an interior point of  $S$ . Let  $U$  be a neighborhood of  $T^m(x)$  in  $S$ . Since  $T^m$  is continuous,  $V = (T^m)^{-1}(U) = T^{-m}(U)$  is a neighborhood of  $x$ . If  $y \in V$ ,  $T^m(y) \in S$  so that  $\varphi(y) = T^m(y)$  for all  $y \in V$ . Hence  $\varphi$  is continuous in a neighborhood of  $x \in M - F - B$ , and  $M - F - B$  is open. Clearly,  $\varphi = T^m$  for some  $m$  in a neighborhood of  $x \in M - F - B$ . If we set  $\varphi(x) = (f_{11}(x), \dots, f_{1, n_1-1}(x), \dots, f_{k, n_{k-1}-1}(x))$  so that the  $f_{i,j}(x)$  are the components of  $\varphi(x)$ , it follows that the  $f_{i,j}(x)$  are continuous and independent on  $M - F - B$ , since  $\varphi(x)$  is a local homeomorphism on  $M - F - B$ . Since  $\varphi(T(x)) = \varphi(x)$ ,  $f_{i,j}(T(x)) = f_{i,j}(x)$  so the  $f_{i,j}$  are  $T$ -invariant.

**3. The construction of invariant functions.** In order to construct invariant functions, we will use more information about sequences  $\{T^n(x)\}$  for  $x$  not a fixed point of  $T$  in  $M$ . As above, we assume that  $T$  is a homeomorphism of  $M$  onto itself. For  $x \in M$ , let



$L_x$  be the set of cluster points of  $\{T^n(x) | n > 0\}$  and let  $l_x$  be the set of cluster points of  $\{T^n(x) | n < 0\}$ . Note that  $L_x$  and  $l_x$  are respectively the  $\omega$  and  $\alpha$  limit sets of  $x$ .

LEMMA 3.1. *The set of cluster points of  $\{T^n(x)\}$  is the union of  $l_x$  and  $L_x$ . The value of  $\bar{P}$  is constant on each of  $l_x$  and  $L_x$ . If  $\bar{P}(L_x)$  denotes the value of  $\bar{P}$  on  $L_x$  and  $\bar{P}(l_x)$  denotes the value of  $\bar{P}$  on  $l_x$  we have  $\bar{P}(L_x) > \bar{P}(l_x)$  whenever  $x$  is not a fixed point of  $T$  in  $M$ .*

The proof of Lemma 3.1 is straightforward.

LEMMA 3.2. *Let  $x_0$  be an element of  $M$ . Either there is a neighborhood  $N$  of  $x_0$  such that  $\bar{P}(L_x) = \bar{P}(L_{x_0})$  for all  $x \in N$  or in every neighborhood of  $x_0$  there is an  $x$  such that  $\bar{P}(L_x) > \bar{P}(L_{x_0})$ .*

*Proof.* Suppose there is a neighborhood  $N_1$  of  $x_0$  in  $M$  such that  $\bar{P}(L_{x_0}) \geq \bar{P}(L_x)$  for all  $x \in N_1$ . Let  $\eta$  be a positive number. Let  $S_\eta$  be the set given by  $S_\eta = \{x | \bar{P}(L_x) > \bar{P}(L_{x_0}) - \eta\}$ . We will show that each  $S_\eta$  is open. If  $x$  is an element of  $S_\eta$ , there is an  $m$  such that  $\bar{P}(T^m(x)) > \bar{P}(L_{x_0}) - \eta$ . Since  $T^m$  is continuous, there is a neighborhood  $N_x$  of  $x$  such that  $\bar{P}(T^m(y)) > \bar{P}(L_{x_0}) - \eta$  for all  $y$  in  $N_x$ . But  $\bar{P}(L_y) \geq \bar{P}(T^m(y))$  for all  $y \in M$  so that  $\bar{P}(L_y) \in S_\eta$  for all  $y$  in  $N_x$ . Hence  $S_\eta$  is open. Let  $N(\eta) = S_\eta \cap N_{x_0}$ . Since  $x_0$  is an element of  $S_\eta$  for all positive  $\eta$ ,  $N(\eta)$  is not empty for  $\eta > 0$ . Since  $N(\eta)$  is contained in  $N_{x_0}$  and  $S_\eta$ ,  $\bar{P}(L_{x_0}) \geq \bar{P}(L_x) \geq \bar{P}(L_{x_0}) - \eta$  for all  $x$  in  $N(\eta)$ . Since the points of  $L_x$  are in  $F$ , the set of fixed points of  $T$ ,  $\bar{P}(L_x)$  can assume only finitely many values. Hence for  $\eta$  sufficiently small

$$\bar{P}(L_{x_0}) \geq \bar{P}(L_x) \geq \bar{P}(L_{x_0}) - \eta$$

implies that  $\bar{P}(L_x) = \bar{P}(L_{x_0})$ , and so for some  $\eta$ ,  $x \in N(\eta)$  implies that  $\bar{P}(L_x) = \bar{P}(L_{x_0})$ .

LEMMA 3.3. *Let  $x_0$  be an element of  $M$ . Either there is a neighborhood  $N_{x_0}$  of  $x_0$  in  $M$  such that  $\bar{P}(L_{x_0}) = \bar{P}(L_x)$  for all  $x$  in  $N_{x_0}$  or every neighborhood  $N$  of  $x_0$  contains an open subset  $\Phi_N$  such that  $\bar{P}(L_y) = P(L_z)$  for all  $y$  and  $z$  in  $\Phi_N$ .*

*Proof.* Suppose  $x_0$  is an element of  $M$  and there is no neighborhood  $U$  of  $x_0$  in  $M$  such that  $\bar{P}(L_x) = \bar{P}(L_{x_0})$  for all  $x$  in  $U$ . Let  $N$  be a neighborhood of  $x_0$ . According to Lemma 3.2, there is an element  $x$  of  $N$  such that  $\bar{P}(L_x) > \bar{P}(L_{x_0})$ . Let  $K$  be the least upper bound of  $\bar{P}(L_x)$  for  $x$  in  $N$ . Since the range of  $\bar{P}(L_x)$  is finite, there is a point  $y$  of  $N$  such that  $\bar{P}(L_y) = K$ . Thus  $\bar{P}(L_y) \geq P(L_x)$  for all  $x$  in

$N$ , and  $N$  is a neighborhood of  $y$ . By Lemma 3.2, there is a neighborhood  $U$  of  $y$  such that  $\bar{P}(L_y) = \bar{P}(L_x)$  for all  $x$  in  $U$ . Let  $\Phi_N = N \cap U$ .

Using the fact that if  $T$  is a homeomorphism of  $M$  onto itself,  $T^{-1}$  is defined and either  $x = T^{-1}(x)$  or  $\bar{P}(T^{-1}(x)) < \bar{P}(x)$ , we can modify the above arguments to prove a similar lemma about the function  $\bar{P}(l_x)$ .

**LEMMA 3.4.** *Let  $x_0$  be an element of  $M$ . Either there is a neighborhood  $N_{x_0}$  of  $x_0$  in  $M$  such that  $\bar{P}(l_{x_0}) = \bar{P}(l_x)$  for all  $x$  in  $N_{x_0}$ , or every neighborhood  $N$  of  $x_0$  contains an open subset  $\varphi_N$  such that  $\bar{P}(l_y) = \bar{P}(l_z)$  for all  $y$  and  $z$  in  $\varphi_N$ .*

**THEOREM 2.** *There is an open dense subset  $G$  of  $M - F$  such that for any function  $f$  continuous on  $M$ , the series*

$$\sum_{-\infty}^{+\infty} f(T^n(x)) [\bar{P}(T^n(x)) - \bar{P}(T^{n-1}(x))]$$

*represents a  $T$  invariant function continuous on  $G$ .*

*Proof.* Let  $G_1$  be the set of all elements  $x$  of  $M$  such that  $\bar{P}(L_x)$  is constant in a neighborhood of  $x$ . Let  $G_2$  be the set of all elements  $x$  of  $M$  such that  $\bar{P}(l_x)$  is constant in a neighborhood of  $x$ . Clearly,  $G_1$  and  $G_2$  are open relative to  $M$  and by Lemmas 3.3 and 3.4, each of  $G_1$  and  $G_2$  is dense in  $M$ . Hence  $G = (M - F) \cap G_1 \cap G_2$  is an open dense subset of  $M - F$ .

For each  $x$  in  $M$  let  $S(x)$  denote the series

$$S(x) = \sum_{-\infty}^{+\infty} \bar{P}(T^n(x)) - \bar{P}(T^{n-1}(x)).$$

Now  $S(x)$  converges at each  $x$  to  $\bar{P}(L_x) - \bar{P}(l_x)$ .

Let  $y$  be an element of  $G$ . There is a neighborhood  $U$  of  $y$  such that  $S(x)$  represents the constant function in  $U$ . Since  $y \notin F$  and  $F$  is compact, there is a neighborhood  $V$  of  $y$  containing no fixed points of  $T$ . Let  $W$  be a neighborhood of  $y$  such that  $\bar{W} \subset U \cap V$ . Now  $S(x)$  is a series of positive terms converging to a continuous function on  $\bar{W}$ , and so by E. C. Titchmarsh [4], art. 1.31,  $S(x)$  converges uniformly on  $\bar{W}$ . Let  $f(x)$  be any function continuous on  $M$ . The series

$$F(x) = \sum_{-\infty}^{+\infty} f(T^n(x)) [\bar{P}(T^n(x)) - \bar{P}(T^{n-1}(x))]$$

converges uniformly on  $\bar{W}$  since  $f$  is bounded on  $M$ . Since  $f$ ,  $\bar{P}$  and

$T$  are continuous,  $F(x)$  is continuous on  $\bar{W}$  and hence at  $y$ . Clearly  $F(T(x)) = F(x)$ , so the function  $F$  is a continuous  $T$  invariant function on  $G$ .

We initiate the study of differentiable  $T$  invariant functions by defining certain series of continuous functions on  $G$ , the set defined in the proof of Lemma 3.4. Recall that a point  $x_0$  of  $M$  is a point of  $G$  if and only if  $x_0$  is not a fixed point of  $T$  and there is a neighborhood  $N$  of  $x_0$  such that the functions  $\bar{P}(L_x)$  and  $\bar{P}(l_x)$  are constant on  $N$ .

LEMMA 3.5. *If a function  $h(x)$  is defined on all of  $M - F$  by the formula*

$$h(x) = \frac{2(\bar{P}(L_x) - \bar{P}(x)) + \frac{1}{2}(\bar{P}(x) - \bar{P}(l_x))}{\bar{P}(L_x) - \bar{P}(l_x)},$$

*then  $h(x)$  has the following properties:*

- (i)  $h(x)$  is defined and nonnegative on  $M - F$ ,
- (ii)  $h(x)$  is continuous at every point of  $G$ , and
- (iii) if  $x_0$  is a point of  $G$ , there is a neighborhood  $V$  of  $x_0$  such that  $\bar{V}$  is contained in  $G$ , and an integer  $m > 0$  such that

$$h(T^{-n}(x)) > \frac{7}{4}$$

*and*

$$0 < h(T^n(x)) < \frac{3}{4}$$

*for all  $n > m$  and  $x \in \bar{V}$ .*

*Proof.* If  $x_0$  is an element of  $M - F$ ,  $x_0$  is not a fixed point of  $T$  and hence  $\bar{P}(L_{x_0}) - \bar{P}(l_{x_0}) > 0$ . Hence  $h(x)$  is defined on  $M - F$ . Since  $\bar{P}(L_x) > \bar{P}(x)$  and  $\bar{P}(x) > \bar{P}(l_x)$  for  $x$  in  $M - F$ ,  $h(x)$  is positive on  $M - F$ . To prove (ii), let  $x_0$  be a point of  $G$ . By the definition of  $G$ , there is a neighborhood  $N_1$  of  $x_0$  such that  $\bar{P}(L_x)$  and  $\bar{P}(l_x)$  are constant on  $N_1$ . By the definition of  $G$ ,  $x_0$  is not a fixed point of  $T$  so that  $\bar{P}(L_{x_0}) - \bar{P}(l_{x_0}) > 0$ . Hence  $\bar{P}(L_x) - \bar{P}(l_x)$  is a nonzero constant on  $N_1$ . Since  $\bar{P}(x)$ ,  $\bar{P}(L_x)$  and  $\bar{P}(l_x)$  are continuous in  $N_1$ ,  $h(x)$  is continuous in  $N_1$  and hence at  $x_0$ .

To prove (iii), let  $x_0$  be a point of  $G$  and let  $N_1$  be a neighborhood of  $x_0$  such that  $\bar{P}(L_x)$  and  $\bar{P}(l_x)$  are constant on  $N_1$ . Then  $G \supset N_1$ . Let  $V$  be neighborhood of  $x_0$  such that  $\bar{V} \subset N_1 \subset G$ . As in the

proof of (ii),  $h(x)$  is continuous on  $N_1$  and hence on  $\bar{V}$ . Since  $T$  is a homeomorphism of  $M$  onto itself,  $T^n$  is a continuous transformation of  $M$  onto itself for arbitrary integral  $n$ . Hence  $h(T^n(x))$  is continuous on  $\bar{V}$  for arbitrary integral  $n$ . Let  $n$  be an integer. Since  $\bar{P}(L_{T^n(x)}) = \bar{P}(L_x)$  and  $\bar{P}(l_{T^n(x)}) = \bar{P}(l_x)$  the difference between  $h(T^{n+1}(x))$  and  $h(T^n(x))$  is given by the formula

$$h(T^{n+1}(x)) - h(T^n(x)) = -\frac{\frac{3}{2}[\bar{P}(T^{n+1}(x)) - \bar{P}(T^n(x))]}{\bar{P}(L_x) - \bar{P}(l_x)}.$$

No point of  $N_1$  is a fixed point of  $T_n$  since

$$\bar{P}(L_{T^n(x)}) - \bar{P}(l_{T^n(x)}) = \bar{P}(L_x) - \bar{P}(l_x)$$

and

$$\bar{P}(L_x) - \bar{P}(l_x) = \bar{P}(L_{x_0}) - \bar{P}(l_{x_0}) > 0.$$

Hence

$$h(T^{n+1}(x)) < h(T^n(x))$$

for all  $x$  in  $\bar{V}$  and all integers  $n$ . Hence  $h(T^n(x))$  is a monotone decreasing function of  $n$  for each  $x$  in  $\bar{V}$ . Since  $\lim_{n \rightarrow \infty} h(T^n(x)) = 1/2$  and  $\lim_{n \rightarrow -\infty} h(T^n(x)) = 2$  for all  $x$  in  $\bar{V}$ , it follows from the compactness of  $\bar{V}$  that there is an integer  $m$  such that

$$2 \geq h(T^{-n}(x)) > \frac{7}{4}$$

and

$$\frac{3}{4} > h(T^n(x)) \geq \frac{1}{2}$$

for all integers  $n > m$  and all elements  $x$  of  $\bar{V}$ .

**LEMMA 3.6.** *Let  $h(x)$  be the function defined in Lemma 3.5. Let the sequence  $p_n(x)$  be inductively defined for integral  $n$  by the rules:*

- (i)  $p_0 = 1$
- (ii)  $p_{n+1}(x) = h(T^n(x))p_{n-1}(x)$  for  $n \geq 1$
- (iii)  $p_{-n}(x) = p_{-n+1}(x)/h(T^{-n}(x))$  for  $n \geq 1$ .

*If  $x_0$  is an element of  $G$  every  $p_n(x)$  is continuous at  $x_0$  and there is a neighborhood  $V$  of  $x_0$ , a constant  $K$  and an integer  $m$  such that*

$$0 < p_n(x) < K \cdot \left(\frac{3}{4}\right)^{|n|-m}$$

*for all  $x$  in  $\bar{V}$  and all  $n$  such that  $|n| > m$ .*

The proof of Lemma 3.6 is straightforward and has been omitted.

LEMMA 3.7. *If  $q_{n,r}(x)$  is defined by the formula*

$$q_{n,r}(x) = \frac{p_n(x)^r}{\sum_{j=-\infty}^{+\infty} [p_j(x)]^r}$$

*then*

- (i) *each  $q_{n,r}(x)$  is defined and continuous for  $x \in G$ ,*
- (ii) *if  $x_0$  is an element of  $G$ , there is a neighborhood  $V$  of  $x_0$  such that  $\bar{V} \subset G$ , and an integer  $m$  such that*

$$0 < q_{n,r}(x) < \left( \left( \frac{3}{4} \right)^r \right)^{|n|-m}$$

*for all  $n$  such that  $|n| > m$  and all positive integers  $r$ .*

- (iii) *for all  $x$  in  $G$ ,  $q_{n,r}(T(x)) = q_{n+1,r}(x)$ ,*
- (iv) *if  $f(x)$  is a continuous function on  $M$ , and  $r$  and  $s$  are positive integers*

$$\sum_{n=-\infty}^{+\infty} f(T^n(x)) q_{n+1,r}(x)^s$$

*defines a continuous  $T$ -invariant function on  $G$ .*

*Proof.* To prove statement (i), let  $x_0$  be a point of  $G$ . According to Lemma 3.6, there is a neighborhood  $V$  of  $x_0$  such that  $\bar{V} \subset G$  and

$$0 < p_n(x) < K \cdot \left( \frac{3}{4} \right)^{|n|-m}$$

for  $n$  sufficiently large. Hence the series

$$\sum_{n=-\infty}^{+\infty} p_n(x)^r$$

converges uniformly for all  $x$  in  $\bar{V}$ . Since  $p_n(x)^r$  is continuous in  $\bar{V}$ , and

$$\sum_{n=-\infty}^{+\infty} p_n(x)^r > p_0(x)^r = 1,$$

every  $q_{n,r}(x)$  is defined and continuous in  $\bar{V}$ . Since  $x_0$  is an arbitrary point of  $G$ , statement (i) is proven.

To prove statement (ii), let  $x_0$  be a point of  $G$ . According to Lemma 3.6, there is a neighborhood  $V$  of  $x_0$  such that  $\bar{V} \subset G$ , a constant  $K$  and an integer  $v$  such that

$$0 < p_n(x) < K \left( \frac{3}{4} \right)^{|n|-v}.$$

Let  $m$  be so larger that  $K \cdot (3/4)^{m-v} < 1$ . Then we have

$$0 < p_n(x) < \left(\frac{3}{4}\right)^{|n|-m},$$

so that

$$0 < p_n(x)^r < \left(\left(\frac{3}{4}\right)^r\right)^{|n|-m}.$$

Since

$$\sum_{-\infty}^{+\infty} p_n(x)^r > p_0(\alpha)^r = 1,$$

we can obtain the inequality of (ii).

Statement (iii) follows directly from the observation that whenever  $p_n(x)$  is defined, we have

$$p_n(T(x)) = \frac{p_{n+1}(x)}{h(x)}.$$

To prove statement (iv) note that wherever all  $q_{n,r}(x)$  are defined we have

$$f(T_n(T(x)))q_{n,r}(T(x))^s = f(T^{n+1}(x))q_{n+1,r}(x)^s,$$

so that the  $T$  invariance of the series of (iv) follows. Since  $f(x)$  is continuous on  $M$  and  $M$  is compact,  $|f(x)|$  is bounded on  $M$ . By part (iii), the series of part (iv) converges uniformly in a closed neighborhood of each point of  $G$  for all positive integers  $r$ . Hence if  $r$  and  $s$  are arbitrary positive integers,

$$\sum_{n=-\infty}^{+\infty} f(T^n(x))[q_{n,r}(x)]^s$$

represents a continuous  $T$ -invariant function on  $G$ .

Let  $J$  be the Jacobian of the transformation  $T$  and let  $|J|$  be the determinant of  $J$ . If  $|J|$  is bounded away from zero on  $M$ , we can construct  $T$  invariant functions which are differentiable on an open dense subset of  $M - F$ . We can show that the hypothesis that  $|J|$  is bounded away from zero on  $M$  and  $T$  is a homeomorphism are reasonable by an example. Let  $P$  be any polynomial with positive coefficients defined on  $\bar{M}$ . Let  $R$  be the polynomial given by the formula

$$R = \sum_{i=1}^k \left( \sum_{j=1}^{n_i} x_{i,j} \right)$$

and let  $Q_\varepsilon = R + \varepsilon P$ . For  $\varepsilon > 0$ ,  $Q_\varepsilon$  has positive coefficients and by the unpublished result of L. E. Baum stated above, the  $T$  transformation  $T_\varepsilon$  associated with  $Q_\varepsilon$  is a homeomorphism of  $M$  onto itself. The  $T$  transformation associated with  $R = Q_0$  is the identity transformation so that the determinant of the Jacobian of  $T_0$  is 1. If we let  $J_\varepsilon$  be the Jacobian of  $T_\varepsilon$ ,  $|J_\varepsilon|$  is a continuous function of  $\varepsilon$ , and  $|J_\varepsilon| \rightarrow 1$  as  $\varepsilon \rightarrow 0$  at each point of  $M$ . Since  $M$  is compact, there is an  $\varepsilon$  such that  $|J_\varepsilon| > 1/2$  at every point of  $M$ .

In the following we will assume that  $|J|$  is bounded away from zero on  $M$ , but we note that local results can be obtained by restricting our attention to elements  $x$  of  $M$  such that  $|J|$  is bounded away from zero in some neighborhood of the sequence  $\{T^n(x)\}$ .

LEMMA 3.8. *If  $T$  is a homeomorphism of  $M$  onto itself, the Jacobian determinant  $|J|$  of  $T$  is bounded away from zero on  $M$  and  $t_{n,u,v}(x)$  denotes the  $(u, v)$  component of  $T^n(x)$ , then:*

(i) *for every  $n$  and subscript pair  $i, j$ ,  $\partial/\partial x_{i,j}(t_{n,u,v}(x))$  is continuous on  $M$ ;*

(ii) *there is a constant  $B$  such that*

$$\left| \frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x) \right| < B^{|n|}$$

*for all  $(i, j)$  and all  $x$  in  $M$ ;*

(iii) *if  $C$  is a compact subset of  $G$  there is a positive integer  $r$  such that the first partial derivatives*

$$\frac{\partial}{\partial x_{i,j}} q_{n,r}(x)$$

*(see Lemma 3.7 for the definition of the functions  $q_{n,r}(x)$ ) are continuous in  $C$  and there are constants  $L_1$  and  $L_2$  such that*

$$\left| \frac{\partial}{\partial x_{i,j}} q_{n,r}(x) \right| < L_1 L_2^{|n|}$$

*and  $0 < L_2 < 1$ , for all  $x$  in  $C$ .*

*Proof.* Since  $T^n$  is a rational transformation of  $M$ , with nonzero denominators,  $\partial/\partial x_{i,j}(t_{n,u,v}(x))$  is continuous on  $M$  for all  $n \geq 0$ . Since the Jacobian determinant of  $T$  is bounded away from zero on  $M$ , the same result holds for  $\partial/\partial x_{i,j}(t_{-n,u,v}(x))$ .

To prove (ii), note that

$$\frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x) = \sum_{r,s} \frac{\partial t_{1,u,v}}{\partial x_{r,s}}(T^{n-1}(x)) \frac{\partial t_{n-1,u,v}(x)}{\partial x_{i,j}}$$

for all  $n$  and every  $x$  in  $M$ . Since

$$\frac{\partial t_{1,u,v}(x)}{\partial x_{r,s}}$$

is bounded on  $M$  for all  $(r, s)$ , it follows inductively that there are bounds  $L_1$  and  $R_1$  such that

$$\left| \frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x) \right| < L_1 \cdot R_1^n$$

for all  $n > 0$ . Since the determinant of

$$J = \left( \frac{\partial}{\partial x_{i,j}} t_{1,u,v}(x) \right)$$

is bounded away from zero on  $M_1$  the elements of the matrix  $J^{-1}$  are bounded on  $M$ . It follows that there are constants  $L_2$  and  $R_2$  such that

$$\left| \frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x) \right| < L_2 \cdot R_2^n$$

for all  $n \leq 0$ . Clearly there is a constant  $B$  such that  $B^{|n|} > L_4 \cdot R_1^n$  and  $B^{|n|} > L_2 \cdot R_2^n$ , so that

$$\left| \frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x) \right| < B^{|n|}$$

for all  $n, u, v$  and all  $x \in M$ .

To prove (iii) we will show first that for a given  $x_0 \in G$  there is a closed neighborhood  $V_{x_0}$  of  $x_0$  and an integer  $i$  such that

$$\sum_{-\infty}^{+\infty} \frac{\partial}{\partial x_{i,j}} [p_n(x)]^r$$

converges uniformly in  $\bar{V}_{x_0}$  for all  $r \geq i$ . By Lemma 3.6, there is a neighborhood  $V$  of  $x_0$  such that  $G \supset \bar{V}$ , a bound  $K$  and an integer  $m$  such that

$$0 < p_n(x) < K \cdot \left( \frac{3}{4} \right)^{n-m}$$

for all  $x \in \bar{V}$ .

For  $n > 0$  we will inductively find a bound  $S$  such that

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| < S^{n+1}.$$

We have



$$\begin{aligned}\frac{\partial}{\partial x_{i,j}} p_n(x) &= \frac{\partial}{\partial x_{i,j}} h(T^n(x)) p_{n-1}(x) \\ &= h(T^n(x)) \frac{\partial}{\partial x_{i,j}} p_{n-1}(x) + p_{n-1}(x) \sum_{u,v} \frac{\partial}{\partial x_{u,v}} \cdot \frac{\partial t_{n,u,v}}{\partial x_{i,j}}.\end{aligned}$$

Now  $0 < h(T^n(x)) < 2$  for  $x \in \bar{V}$ , and there is a bound  $B_1$  such that  $|p_{n-1}(x)| < B_1$  for  $x \in \bar{V}$ . For every subset of  $G$ ,

$$\frac{\partial h}{\partial x_{u,v}} = -\frac{3}{2} \frac{\frac{\partial}{\partial x_{u,v}}(\bar{P})}{\bar{P}(L_x) - \bar{P}(l_x)}$$

is bounded on  $G$  since  $\bar{P}$  is a polynomial and  $\bar{P}(L_x) - \bar{P}(l_x)$  ranges over a finite set not including zero for all  $x \in G$ . Since  $G$  is closed under the transformation  $T$ , there is a constant  $B_2$  such that  $|\partial h / \partial x_{u,v}| < B_2$  at  $T^n(x)$  for every element  $x$  of  $\bar{V}$ . Thus

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| < 2 \left| \frac{\partial}{\partial x_{i,j}} p_{n-1}(x) \right| + dB_1 B_2 B^n.$$

If  $K_1$  is maximum of  $2, dB_1 B_2$  and  $B$  we have

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| < K_1 \left( \left| \frac{\partial}{\partial x_{i,j}} p_{n-1}(x) \right| + K_1^n \right).$$

Since  $p_0(x) = 1$ , we have

$$\begin{aligned}\left| \frac{\partial}{\partial x_{i,j}} p_1(x) \right| &< K_1^2 \\ \left| \frac{\partial}{\partial x_{i,j}} p_2(x) \right| &< 2K_1^3\end{aligned}$$

and

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| < nK_1^{n+1} < S^{n+1}$$

for some bound  $S$  and all  $x$  in  $\bar{V}$ .

Since  $h(T^{-n}(x)) > 1/2$  for all  $x \in \bar{V}$ , a similar argument yields a constant  $S_1$  such that

$$\left| \frac{\partial p_n}{\partial x_{i,j}}(x) \right| < S_1^{|n|+1}$$

for negative  $n$  all  $x \in \bar{V}$ . Hence there is a single constant  $S$  so that

$$\left| \frac{\partial P_n(x)}{\partial x_{i,j}} \right| < S^{|n|+1}$$

for all  $x \in \bar{V}$ . Now  $\bar{V}$  was selected so that

$$0 < p_n(x) < K \left( \frac{3}{4} \right)^{|n|-m}$$

for  $n$  such that  $|n| > m$ . For  $p$  sufficiently large,

$$0 < p_n(x) < \left( \frac{3}{4} \right)^{|n|-p}$$

for all  $n$  such that  $|n| > p > m$ , and so

$$0 < p_n(x)^{r-1} < \left( \left( \frac{3}{4} \right)^{r-1} \right)^{|n|-p}$$

and

$$\begin{aligned} \left| \frac{\partial}{\partial x_{i,j}} p_n(x)^r \right| &= \left| r p_n(x)^{r-1} \right| \left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| \\ &\leq r s \left( S \left( \frac{3}{4} \right)^{r-1} \right)^p \left( \left( \frac{3}{4} \right)^{r-1} S \right)^{|n|-p}. \end{aligned}$$

Now  $r$  can be chosen so large that  $S(3/4)^{r-1} < 1$ . Thus there are constants  $C$  and  $D$  so that  $0 < D < 1$  and

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x)^r \right| \leq CD^{|n|-p}$$

for all  $x$  in  $\bar{V}$ . Hence the series

$$\sum_{-\infty}^{+\infty} p_n(x)^r$$

has continuous first partial derivatives for  $x \in \bar{V}$ . Since

$$\sum_{-\infty}^{+\infty} p_n(x)^r > p_0(x)^r = 1,$$

we have that  $q_{0,r}(x)$  has continuous first partial derivatives for all  $x$  in  $\bar{V}$ . But

$$q_{n,r}(x) = p_n(x)^r \cdot q_{0,r}(x)$$

so that  $q_{n,r}(x)$  has continuous first partial derivatives for all  $x$  in  $\bar{V}$ . Since  $\bar{V}$  is compact, there is a bound  $U$  on the partial derivatives of  $q_{0,r}(x)$  in  $\bar{V}$ . Thus

$$\frac{\partial}{\partial x_{i,j}} q_{n,r}(x) = q_{0,r}(x) \frac{\partial}{\partial x_{i,j}} p_n(x)^r + p_n(x)^r \frac{\partial}{\partial x_{i,j}} q_{0,r}(x)$$

and

$$\begin{aligned} \left| \frac{\partial}{\partial x_{i,j}} q_{n,r}(x) \right| &\leq WCD^{|n|-p} + \left( \frac{3}{4} \right)^{r(|n|-p)} \cdot U \\ &\leq R_1 \cdot R_2^{|n|} \end{aligned}$$

with  $0 < R_2 < 1$ .

Since  $C$  is compact, it can be covered with a finite set of neighborhoods as  $\bar{V}$ , so part (iii) of the lemma follows immediately.

**THEOREM 3.** *If  $C$  is a compact subset of  $G$ , there are integers  $r$  and  $s$  such that for every function  $f(x)$ , defined and with continuous first partial derivatives on  $M$ , the function*

$$F(x) = \sum_{-\infty}^{+\infty} f(T^n(x)) [q_{n,r}(x)]^s$$

*is continuous and has continuous first partial derivatives for all  $x$  in  $\bigcup_{-\infty}^{+\infty} T^n(C)$  and*

$$F(x) = F(T(x))$$

*wherever  $F(x)$  is defined.*

*Proof.* Clearly  $F(x) = F(T(x))$  wherever  $F(x)$  is defined. Also, if all first partial derivatives  $\partial/\partial x_{i,j} F(x)$  are defined and continuous at  $x = x_0$ , it follows by elementary methods from the fact that  $|J|$  is bounded away from zero on  $M$  and  $J$  is continuous on  $M$  that  $\partial/\partial x_{i,j} F(x)$  is defined and continuous at  $T^n(x_0)$  for all  $n$ . Hence we need only show that  $F(x)$  has continuous first partial derivatives for all elements  $x$  of  $C$ . We choose to show that the series of partial derivatives

$$\sum_{-\infty}^{+\infty} \frac{\partial}{\partial x_{i,j}} f(T^n(x)) [q_{n,r}(x)]^s$$

converges uniformly in  $C$ .

Note that

$$\begin{aligned} \frac{\partial}{\partial x_{i,j}} f(T^n(x)) [q_{n,r}(x)]^s &= q_{n,r}(x)^s \sum_{u,v} \frac{\partial f}{\partial x_{u,v}} \bigg|_{T^n(x)} \frac{\partial t_{n,u,v}}{\partial x_{i,j}} \\ &\quad + s [q_{n,r}(x)]^{s-1} \frac{\partial q_{n,r}(x)}{\partial x_{i,j}} f(T^n(x)). \end{aligned}$$

Since  $f$  and its first partial derivatives are bounded on  $M$  it follows directly from Lemma 3.8 that  $s$  may be chosen so large that the series

$$\sum_{-\infty}^{+\infty} \frac{\partial}{\partial x_{i,j}} f(T^n(x)) [q_{n,r}(x)]^s$$

is majorized by a geometric series, and the choice of  $s$  is independent of  $f$ .

The analysis of this section can be extended to obtain a local analogue of Theorem 1, i.e., a set of  $d - k$  functions can be found which are continuous on  $G$ , have nonzero Jacobian at a point  $x_0$  of  $G$  and are invariant with respect to  $T$ .

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