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Let P be a polynomial with real non-negative coefficients and variables $x_{i,j}, i = 1, \cdots, k, j = 1, \cdots, n_i$. Let $d = \sum_1^k n_i$. Let R_d be the d-dimensional real vector space. Let $\tilde{M}$ be the subset of R_d defined by

$$\tilde{M} = \left\{x \mid x \in R_d, x_{i,j} \geqq 0, \sum_{j=1}^{n_i} x_{i,j} = 1\right\}$$

where the symbols $x_{i,j}$ denote the components of x. If x is a vector in the interior of $\tilde{M}$, define $\tau(x)$ as the vector in $\tilde{M}$ with components $x'_{i,j}$ given by

$$x'_{i,j} = \frac{x_{i,j}\dfrac{\partial P}{\partial x_{i,j}}}{\sum_{h=1}^{n_i} x_{i,h}\dfrac{\partial P}{\partial x_{i,h}}}\,.$$

The expression on the right is evaluated at x. The transformation τ is defined on the boundary of $\tilde{M}$ by the same formula if the denominators do not vanish.

Let $\tilde{F}$ be the set of fixed points of τ in $\tilde{M}$. It is shown that if τ is a homeomorphism of $\tilde{M}$ onto itself, there is a set of $d - k$ functions $f_1, \cdots, f_{d-k}$ defined on $\tilde{M} - \tilde{F}$ such that $f_i(x) = f_i(\tau(x))$ for $x \in \tilde{M} - \tilde{F}$. The functions f_i are continuous and independent on an open dense subset of $\tilde{M} - \tilde{F}$. Explicit expressions for certain invariant functions are also obtained.

1. The transformation τ. The transformation τ defined in the introduction can be used to iteratively find local maxima for the polynomial P. It was shown by L. E. Baum and J. A. Eagon [1] that if P is a homogeneous polynomial with positive coefficients and if x is an element of $\tilde{M}$ such that $\tau(x)$ is defined then either $\tau(x) = (x)$ or $P(\tau(x)) > P(x)$. This result was generalized at the suggestion of 0. Rothaus by L. E. Baum and G. R. Sell [2] to arbitrary polynomials with positive coefficients.

It will be assumed in this paper that the transformation τ is a homeomorphism of $\tilde{M}$ onto itself. According to an unpublished result of L. E. Baum, τ is a homeomorphism of $\tilde{M}$ onto itself if and only if the expression for P as a sum of distinct monomials with positive coefficients contains monomials $c_{i,j}x_{i,j}{}^{w_{i,j}}$ for all $i=1, \cdots, k, j=1, \cdots, n_i$ where $c_{i,j} > 0$ and $w_{i,j}$ is an integer greater than zero. Since this condition is satisfied if and only if τ is defined on all of $\tilde{M}$, a necessary and sufficient condition that τ is a homeomorphism of $\tilde{M}$ onto itself is that τ be defined an all of $\tilde{M}$. We will not prove L. E. Baum's result here, but will give a single example of a polynomial P for which τ is a homeomorphism. Let

$$P = \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{i,j}^m .$$

The τ-transformation associated with P is given by

$$x'_{i,j} = \frac{x_{i,j}^m}{\sum_{h=1}^{n_i} x_{i,j}^m} .$$

The inverse of τ restricted to $\tilde{M}$ is given by

$$x_{i,j} = \frac{x_{i,j}'^{1/m}}{\sum_{h=1}^{n_i} x_{i,h}'^{1/m}}$$

where the real positive mth roots are to be chosen.

2. The existence of invariants.

2.1. *Notation and definitions.* As above, we let $\tilde{M}$ denote the space of real vectors $(x_{1,1}, \cdots, x_{1,n_1}, \cdots, x_{k,1}, \cdots, x_{x,n_k})$ satisfying $x_{i,j} \geqq 0$,
and

$$\sum_{j=1}^{n_i} x_{i,j} = 1 .$$

Let M be set of real vectors

$$(y_{1,1}, \cdots, y_{1,n_{\bar{1}}1}, \cdots, y_{k,1}, \cdots, y_{k,n_{\bar{k}}1})$$

satisfying $y_{i,j} \geqq 0$ and

$$\sum_{j=1}^{n_i - 1} y_{i,j} \leqq 1 .$$

If $y \in M$ let $\psi(y)$ be the point of M with coordinates $x_{i,j} = y_{i,j}$ for $1 \leqq j \leqq n_{i-1}$ and

$$x_{i,n_i} = 1 - \sum_{j=1}^{n_i - 1} y_{i,j} .$$

Clearly ψ is a homeomorphism of M onto $\tilde{M}$.

Let φ be a transformation of a set S onto itself. We inductively define $\varphi^n(x)$ for $n \geqq 0$ and $x \in S$ by $\varphi^0(x) = x$ and $\varphi^n(x) = \psi(\psi^{n-}{}_1(x))$. If φ is a one-to-one transformation of S onto itself, we inductively define $\varphi^n(x)$ for $n < 0$ and $x \in S$ by the rule $\varphi^{n-1}(x) = \varphi^{-1}(\varphi^n(x))$. Also, if φ is a one-to-one transformation of S onto itself, we have $\varphi^{r+s}(x) = \varphi^r(\varphi^s(x))$ for all $x \in S$ and all pairs of integers (r, s).

Let $\{x_n\}$ be a sequence of points of a topological space S. A cluster point of $\{x_n\}$ is a point p of S such that every neighborhood of p contains infinitely many elements of the sequence $\{x_n\}$.

2.2. *Proof of the existence theorem.*

LEMMA 2.1. *The transformation $T = \psi^{-1}\tau_\psi$ of M into itself has the following properties:*

(i) *Let $\bar{P}$ be the polynomial defined on M by the formula $\bar{P}(y) = P(\psi(y))$ for $y \in M$. If $y \in M$, either $y = T(y)$ or $\bar{P}(T(y)) > \bar{P}(y)$.*

(ii) *The set of fixed points of T on M is the union of the set of critical points of $\bar{P}$ on M and the sets of critical points of $\bar{P}$ restricted to boundary simplices of M.*

(iii) *The set of fixed points T in M has only finitely many components. Each component of the set of fixed points of T is compact and $\bar{P}$ is constant on each of the components of the set of fixed points of T.*

(iv) *T is a homeomorphism of M onto itself if and only if τ is a homeomorphism of M onto itself.*

(v) *If $x \in M$, every cluster point of a sequence $\{T^n(x)\}$, $n \geqq 0$, is a fixed point of T. If T is a homeomorphism, every cluster point of the sequence $\{T^n(x)\}$ is a fixed point of T.*

Proof. To prove (i), let y be an element of M such that $T(y) \neq y$. Then $\psi^{-1}\tau\psi(y) \neq y$ and $\tau\psi(y) \neq \psi(y)$. Thus $\psi(y)$ is not a fixed point of τ and it follows that

$$\bar{P}(T(y)) = P(\psi\psi^{-1}\tau\psi(y)) = P(\tau\psi(y)) > P(\psi(y)) = \bar{P}(y) .$$

Statement (ii) may be well known but include a proof for the sake of completeness. Note first that ψ maps the set of fixed points T onto the set of fixed points of τ. Let x be a fixed point of τ in $\tilde{M}$ and let x have coordinates $(x_{i,j})$. The equation $\tau(x) = x$ implies the equations

$$x_{i,j}\left(\sum_1^{n_i} x_{i,k}\frac{\partial P}{\partial x_{i,k}} - \frac{\partial P}{\partial x_{i,j}}\right) = 0$$

for all i, j, and since τ is defined at x, these equations imply $\tau(x) = x$. If x is an interior fixed point of M, no $x_{i,j}$ is zero so that $\tau(x) = x$ is equivalent to

$$\frac{\partial P}{\partial x_{i,j}} - \frac{\partial P}{\partial x_{i,n_i}} = 0$$

for all i, j. But this just the condition that $\psi^{-1}(y)$ be a critical point of $\bar{P}$. Thus the fixed points of T interior to M are just the interior critical points of $\bar{P}$.

Now suppose y is a fixed point of T on the boundary of M. Clearly $\psi(y)$ is a fixed point of τ on the boundary of $\tilde{M}$. If $\psi(y) = z = (z_{i,j})$, certain variables $x_{i,i}$ are zero at z. Let $\tilde{M}_y$ be the part of the boundary of $\tilde{M}$ determined by the equations $x_{i,j} = 0$ for all i, j such that $z_{i,j} = 0$. If no z_{i,n_i} is zero, it follows as before that y is a critical point of $\bar{P}$ restricted to $M_y = \psi^{-1}(\tilde{M}_y)$. Note that M_y is a subset of the boundary of M. If some z_{i,n_i} is zero, the variables $u_{i,j}$ describing M_y are subject to the additional constraint $\sum u_{i,j} = 1$, where the sum is over the subscripts i, j such that $z_{i,j} \neq 0$. Since the partial derivatives $\partial P/\partial x_{i,j}(z)$ are equal for i, j such that $z_{i,j} \neq 0$, it follows that y is a critical point of $\bar{P}$ for $\bar{P}$ restricted to M_y. Conversely, if y is a critical point of P restricted to M_y, it follows that y is a fixed point of T.

Let us prove (iii). Let R_d be d-dimensional real space, with coordinates $x_{i,j}$ as described in the introduction. Let P be a polynomial defined on R_d. Let S_1 be the set of points of R_d satisfying the equations:

$$\sum_{j=1}^{n_i} x_{i,j}^2 = 1 \text{ for all } i, \quad \text{and} \quad \frac{\partial P}{\partial x_{i,j}} = \frac{\partial P}{\partial x_{i,n_i}}$$

for all i, j, where the partial derivatives of P are evaluated at $(x_{1,1}^2, \cdots, x_{1,n_1}^2, \cdots, x_{k,n_k}^2)$. According to H. Whitney [5], a real algebraic variety such as S_1 has only finitely many components and each component is a union of finitely many components of differentiable manifolds (of various dimensions). Let $Q = P(x_{1,1}^2, \cdots, x_{k,n_k}^2)$. The partial derivatives of Q with respect to $x_{i,j}$ for $j < n_i$ with the restrictions

$$\sum_{j=1}^{n_i} x_{i,j}^2 = 1, i = 1, \cdots, k$$

are all zero on S_1. Thus Q can have only one value on a component of a differentiable manifold contained in S_1, and thus can have only finitely many values on S_1. Since Q is continuous and the components of S are arcwise connected, Q must be constant on each component of S_1.

Let φ be the mapping of R_d into itself given by $\varphi(x_{1,1}, \cdots, x_{k,n_k}) = (x_{1,1}^2, \cdots, x_{k,n_k}^2)$. The set $S = \varphi(S_1)$ is given by the relations:

(i) $x_{i,j} \geqq 0$ for all i, j,

(ii) $\sum_{j=1}^{n_i} x_{i,j} = 1$ for $i = 1, \cdots, k$, and

(iii) $\partial P/\partial x_{i,j} = \partial P/\partial x_{i,n_i}$ (evaluated at $(x_{1,1}, \cdots, x_{k,n_k})$) for all i, j.

Since φ is continuous, S can have only finitely many components. Since $Q(x) = P(\varphi(x))$ for all $x \in R_d$, the range of P on S is then range of Q on S_1. Hence P assumes only finitely many values on S, and by continuity of P, P is constant on each component of S. Since S is just the ψ image of the set of critical points of $\bar{P}$ on M, S is the ψ image of the subset of fixed points of T corresponding to these critical points.

The same argument applies to the sets of critical points of $\bar{P}$ restricted to the boundary sets of M given by certain $x_{i,j} = 0$. Since the set F of fixed points of T is the union of the set of critical points of $\bar{P}$ on M and the sets of critical points of $\bar{P}$ restricted to each of finitely many subsets of the boundary of M, F has just finitely many components, and $\bar{P}$ assumes only finitely many values on F. By continuity, $\bar{P}$ is constant on each component of F. Since F is compact, each of its finitely many components is also compact.

Part (iv) of the lemma follows from the fact that ψ is a homeomorphism of M onto $\tilde{M}$. Since $T = \psi^{-1}\tau\psi$, T is a homeomorphism of M onto M if τ is a homeomorphism of $\tilde{M}$ onto $\tilde{M}$. Since $\tau = \psi T \psi^{-1}$, the converse follows.

The final result, (v), follows directly from the Baum-Eagon inequality (c.f. Section 1 of this paper), and Lemma 2.1 of Bhatia-Szego [3].

In the following, we restrict our attention to those transformations τ for which τ is a homeomorphism of M onto itself and T is a homeomorphism of M onto itself.

There is an obvious relation between the functions f defined on M such that $f(T(x)) = f(x)$ for all x in M and the functions g defined on $\tilde{M}$ such that $g(\tau(y)) = g(y)$ for all $y \in \tilde{M}$. If $f(T(x)) = f(x)$ for all $x \in M$ then $g(y) = f(\psi(y))$ is such that

$$g(\tau(y)) = f(\psi\tau\psi^{-1}\cdot\psi(y)) = f(T\psi(y)) = f(\psi(y)) = g(y) .$$

Conversely, if $g(\tau(y)) = g(y)$ it is clear that $f(x) = g(\psi^{-1}(x))$ is such that $f(T(x)) = f(x)$. Thus we can find all invariant functions of τ from the invariant functions of T.

A spherical neighborhood of a point x of the interior of M is a $d - k$ dimensional ball contained in M with center at x. If x is on the boundary of M in $d-k$ dimensional real space, a spherical neighborhood of x in M is the intersection of M and an $d - k$ dimensional ball with center at x.

LEMMA 2.2. *Let T be a homeomorphism of M onto itself. If x_0 is a point of M but not a fixed point of T, there is a spherical neighborhood N of x_0 in M such that the sets $T^r(N)$ are disjoint for $-\infty < r < \infty$.*

Proof. Since x_0 is not a fixed point of T, $T(x_0) \neq x_0$. By Lemma

1, (i) $\bar{P}(T(x_0)) - \bar{P}(x_0) = \Delta > 0$. Since $\bar{P}$ is continuous on M, there is a neighborhood U of x_0 such that $\bar{P}(x) < \bar{P}(x_0) + \Delta/3$ for all $x \in U$ and a neighborhood V of $\tau(x_0)$ such that $\bar{P}(y) > \bar{P}(T(x_0)) - \Delta/3$ for all $y \in V$. Since T is a continuous transformation, $T^{-1}(V) \cap U$ is a neighborhood of x_0. Let N be a spherical neighborhood of x_0 contained in $T^{-1}(V) \cap U$. Since $N \subset U$ and $T(N) \subset V$, for arbitrary $x \in N$, $y \in T(N)$ we have

$$\bar{P}(x) < \bar{P}(x_0) + \frac{\Delta}{3} \bar{P}(T(x_0)) - \frac{\Delta}{3} < \bar{P}(y) .$$

If $x \in N$ and $z \in T^m(N)$ for $m \geqq 1$, $z = T^m(u)$ for some $u \in N$ and $\bar{P}(z) \geqq \bar{P}(T(u)) > \bar{P}(x)$ since $T(u) \in T(N)$. Thus $T^m(N) \cap N$ is empty for $m \geqq 1$.

Suppose $T^r(N) \cap T^s(N)$ is not empty for $r \neq s$. We assume $r > s$ and let $y \in T^r(N) \cap T^s(N)$. Then $T^{-r}(y) \in N$ and $T^{r-s}(T^{-r}(y)) = T^{-s}(y) \in N$ so that N and $T^{r-s}(N)$ intersect. This contradiction shows that $T^r(N) \cap T^s(N)$ is empty for $r \neq s$.

If $x, y \in M$, let $|x - y|$ denote the Euclidean distance between x and y.

LEMMA 2.3. *Let T be a homeomorphism of M onto itself. There is a positive number ε such that if x is a point of M but not a fixed point of T, there is at least one element of the sequence $\{T^n(x)\}$ at distance greater than or equal to ε from the set of fixed points of T.*

It follows from Baum and Sell [2] that the set F of fixed points of T is an asyptotically stable set. This Lemma is a consequence of Theorem 4.19 of Bhatia-Szego [3].

A fundamental set S for T on M is a subset of M defined as follows: S contains no fixed point of T but if x is not a fixed point of T, $T^n(x) \in S$ for a single integer n depending on S and x.

LEMMA 2.4. *If T is a homeomorphism of M onto itself, T has a measurable fundamental set.*

Proof. Let D_ε be the set of points of M at distance greater than or equal to ε from F, the set of fixed points of T. According to Lemma 2.3, $\varepsilon > 0$ may be chosen so that D_ε contains at least one element of every sequence $\{T^n(x)\}$ for $x \notin F$. Since D_ε does not meet F, it follows from Lemma 2.2 that about each $x \in D_\varepsilon$ there is a spherical neighborhood N_x such that the sets $T^n(N_x)$ are disjoint (if x is a boundary point of M, the set N_y is the intersection of a ball with M). Since D_ε is compact, it is compact relative to M so that there may be selected a finite covering $N_1, \cdots, N_r$ of D_ε from the sets N_x. Clearly, each sequence $\{T^n(x)\}$ for $x \in M - F$ can meet an N_i in at most one point.

Let

$$L_1 = N_1,\ L_2 = N_2 - \bigcup_{-\infty}^{+\infty} T^n(N_1),\ \cdots,$$

$$L_r = N_r - \bigcup_{-\infty}^{+\infty} T^n(N_1) - \bigcup_{-\infty}^{+\infty} T^n(N_2) - \cdots - \bigcup_{-\infty}^{+\infty} T^n(N_{r-1}) .$$

Clearly $\bigcup_1^r L_i$ is a fundamental set for T in M. Since T is continuous and each N_i is measurable, $\bigcup_{-\infty}^{+\infty} T^n(N_i)$ is measurable. Hence each L_i is measurable and $\bigcup_1^r L_i$ is measurable.

Let $\tilde{F}$ be the set of fixed points of τ in M.

THEOREM 1. *If T is a homeomorphism of M onto itself, and F is the set of fixed points of T, there exist $d - k$ T-invariant functions of T which are continuous and independent on an open dense subset of $M - F$. Thus there are $d - k\tau$ invariant functions continuous and independent on an open dense subset of $\tilde{M} - \tilde{F}$.*

Proof. Let S be a fundamental set for T on M, as constructed in the proof of Lemma 3.4. Let S^* be the boundary of S and let $B = \bigcup_{-\infty}^{+\infty} T^n(S^*)$. Then $M - F - B$ is dense in $M - F$. For $x \in M - F$ let $\varphi(x)$ be the element of $\{T^n(x)\}$ in S. We will show that φ is continuous on $M - F - B$.

If $x \in M - F - B$, $\varphi(x)$ is the unique intersection of $\{T^n(x)\}$ with S. Hence there is an integer m such that $T^m(x) \in S$. Since $x \notin B$, $T^m(x)$ is an interior point of S. Let U be a neighborhood of $T^m(x)$ in S. Since T^m is continuous, $V = (T^m)^{-1}(U) = T^{-m}(U)$ is a neighborhood of x. If $y \in V$, $T^m(y) \in S$ so that $\varphi(y) = T^m(y)$ for all $y \in V$. Hence φ is continuous in a neighborhood of $x \in M - F - B$, and $M - F - B$ is open. Clearly, $\varphi = T^m$ for some m in a neighborhood of $x \in M - F - B$. If we set $\varphi(x) = (f_{11}(x), \cdots, f_{1,n_1-1}(x), \cdots, f_{k,n_k-1}(x))$ so that the $f_{i,j}(x)$ are the components of $\varphi(x)$, it follows that the $f_{i,j}(x)$ are continuous and independent on $M - F - B$, since $\varphi(x)$ is a local homeomorphism on $M - F - B$. Since $\varphi(T(x)) = \varphi(x), f_{i,j}(T(x)) = f_{i,j}(x)$ so the $f_{i,j}$ are T-invariant.

3. The construction of invariant functions. In order to construct invariant functions, we will use more information about sequences $\{T^n(x)\}$ for x not a fixed point of T in M. As above, we assume that T is a homeomorphism of M onto itself. For $x \in M$, let

L_x be the set of cluster points of $\{T^n(x) \mid n > 0\}$ and let l_x be the set of cluster points of $\{T^n(x) \mid n < 0\}$. Note that L_x and l_x are respectively the ω and α limit sets of x.

LEMMA 3.1. *The set of cluster points of* $\{T^n(x)\}$ *is the union of* l_x *and* L_x. *The value of* $\bar{P}$ *is constant on each of* l_x *and* L_x. *If* $\bar{P}(L_x)$ *denotes the value of* $\bar{P}$ *on* L_x *and* $\bar{P}(l_x)$ *denotes the value of* $\bar{P}$ *on* l_x *we have* $\bar{P}(L_x) > \bar{P}(l_x)$ *whenever* x *is not a fixed point of* T *in* M.

The proof of Lemma 3.1 is straightforward.

LEMMA 3.2. *Let* x_0 *be an element of* M. *Either there is a neighborhood* N *of* x_0 *such that* $\bar{P}(L_x)) = \bar{P}(L_{x_0})$ *for all* $x \in N$ *or in every neighborhood of* x_0 *there is an* x *such that* $\bar{P}(L_x) > \bar{P}(L_{x_0})$.

Proof. Suppose there is a neighborhood N_1 of x_0 in M such that $\bar{P}(L_{x_0}) \geqq \bar{P}(L_x)$ for all $x \in N_1$. Let η be a positive number. Let S^η be the set given by $S_\eta = \{x \mid \bar{P}(L_x) > \bar{P}(L_{x_0}) - \eta\}$. We will show that each S_η is open. If x is an element of S_η, there is an m such that $\bar{P}(T^m(x)) > \bar{P}(L_{x_0}) - \eta$. Since T^m is continuous, there is a neighborhood N_x of x such that $\bar{P}(T^m(y)) > \bar{P}(L_{x_0}) - \eta$ for all y in N_x. But $\bar{P}(L_y) \geqq \bar{P}(T^m(y))$ for all $y \in M$ so that $\bar{P}(L_y) \in S_\eta$ for all y in N_x. Hence S_η is open. Let $N(\eta) = S_\eta \cap N_{x_0}$. Since x_0 is an element of S_η for all positive η, $N(\eta)$ is not empty for $\eta > 0$. Since $N(\eta)$ is contained in N_{x_0} and S_η, $\bar{P}(L_{x_0}) \geqq \bar{P}(L_x) \geqq \bar{P}(L_{x_0}) - \eta$ for all x in $N(\eta)$. Since the points of L_x are in F, the set of fixed points of T, $\bar{P}(L_x)$ can assume only finitely many values. Hence for η sufficiently small

$$\bar{P}(L_{x_0}) \geqq \bar{P}(L_x) \geqq \bar{P}(L_{x_0}) - \eta$$

implies that $\bar{P}(L_x) = \bar{P}(L_{x_0})$, and so for some η, $x \in N(\eta)$ implies that $P(L_x) = \bar{P}(L_{x_0})$.

LEMMA 3.3. *Let* x_0 *be an element of* M. *Either there is a neighborhood* N_{x_0} *of* x_0 *in* M *such that* $\bar{P}(L_{x_0}) = \bar{P}(L_x)$ *for all* x *in* N_{x_0} *or every neighborhood* N *of* x_0 *contains an open subset* Φ_N *such that* $P(L_y) = P(L_z)$ *for all* y *and* z *in* Φ_N.

Proof. Suppose x_0 is an element of M and there is no neighborhood U of x_0 in M such that $\bar{P}(L_x) = \bar{P}(L_{x_0})$ for all x in U. Let N be a neighborhood of x_0. According to Lemma 3.2, there is an element x of N such that $\bar{P}(L_x) > \bar{P}(L_{x_0})$. Let K be the least upper bound of $\bar{P}(L_x)$ for x in N. Since the range of $\bar{P}(L_x)$ is finite, there is a point y of N such that $\bar{P}(L_y) = K$. Thus $\bar{P}(L_y) \geqq P(L_x)$ for all x in

N, and N is a neighborhood of y. By Lemma 3.2, there is a neighborhood U of y such that $\bar{P}(L_y) = \bar{P}(L_x)$ for all x in U. Let $\Phi_N = N \cap U$.

Using the fact that if T is a homeomorphism of M onto itself, T^{-1} is defined and either $x = T^{-1}(x)$ or $\bar{P}(T^{-1}(x)) < \bar{P}(x)$, we can modify the above arguments to prove a similar lemma about the function $\bar{P}(l_x)$.

LEMMA 3.4. *Let x_0 be an element of M. Either there is a neighborhood N_{x_0} of x_0 in M such that $\bar{P}(l_{x_0}) = \bar{P}(l_x)$ for all x in N_{x_0}, or every neighborhood N of x_0 contains an open subset φ_N such that $\bar{P}(l_y) = \bar{P}(l_z)$ for all y and z in φ_N.*

THEOREM 2. *There is an open dense subset G of $M - F$ such that for any function f continuous on M, the series*

$$\sum_{-\infty}^{+\infty} f(T^n(x))[\bar{P}(T^n(x)) - \bar{P}(T^{n-1}(x))]$$

represents a T invariant function continuous on G.

Proof. Let G_1 be the set of all elements x of M such that $\bar{P}(L_x)$ is constant in a neighborhood of x. Let G_2 be the set of all elements x of M such that $\bar{P}(l_x)$ is constant in a neighborhood of x. Clearly, G_1 and G_2 are open relative to M and by Lemmas 3.3 and 3.4, each of G_1 and G_2 is dense in M. Hence $G = (M - F) \cap G_1 \cap G_2$ is an open dense subset of $M - F$.

For each x in M let $S(x)$ denote the series

$$S(x) = \sum_{-\infty}^{+\infty} \bar{P}(T^n(x)) - \bar{P}(T^{n-1}(x)) \, .$$

Now $S(x)$ converges at each x to $\bar{P}(L_x) - \bar{P}(l_x)$.

Let y be an element of G. There is a neighborhood U of y such that $S(x)$ represents the constant function in U. Since $y \notin F$ and F is compact, there is a neighborhood V of y containing no fixed points of T. Let W be a neighborhood of y such that $\bar{W} \subset U \cap V$. Now $S(x)$ is a series of positive terms converging to a continuous function on $\bar{W}$, and so by E. C. Titchmarsh [4], art. 1.31, $S(x)$ converges uniformly on $\bar{W}$. Let $f(x)$ be any function continuous on M. The series

$$F(x) = \sum_{-\infty}^{+\infty} f(T^n(x))[\bar{P}(T^n(x)) - \bar{P}(T^{n-1}(x)]$$

converges uniformly on $\bar{W}$ since f is bounded on M. Since f, $\bar{P}$ and

T are continuous, $F(x)$ is continuous on $\bar{W}$ and hence at y. Clearly $F(T(x)) = F(x)$, so the function F is a continuous T invariant function on G.

We initiate the study of differentiable T invariant functions by defining certain series of continuous functions on G, the set defined in the proof of Lemma 3.4. Recall that a point x_0 of M is a point of G if and only if x_0 is not a fixed point of T and there is a neighborhood N of x_0 such that the functions $\bar{P}(L_x)$ and $\bar{P}(l_x)$ are constant on N.

LEMMA 3.5. *If a function $h(x)$ is defined on all of $M - F$ by the formula*

$$h(x) = \frac{2(\bar{P}(L_x) - \bar{P}(x)) + \frac{1}{2}(\bar{P}(x) - \bar{P}(l_x))}{\bar{P}(L_x) - \bar{P}(l_x)},$$

then $h(x)$ has the following properties:

(i) *$h(x)$ is defined and nonnegative on $M - F$,*

(ii) *$h(x)$ is continuous at every point of G, and*

(iii) *if x_0 is a point of G, there is a neighborhood V of x_0 such that $\bar{V}$ is contained in G, and an integer $m > 0$ such that*

$$h(T^{-n}(x)) > \frac{7}{4}$$

and

$$0 < h(T^{n}(x)) < \frac{3}{4}$$

for all $n > m$ and $x \in \bar{V}$.

Proof. If x_0 is an element of $M - F$, x_0 is not a fixed point of T and hence $\bar{P}(L_{x_0}) - \bar{P}(l_{x_0}) > 0$. Hence $h(x)$ is defined on $M - F$. Since $\bar{P}((L_x) > \bar{P}(x)$ and $\bar{P}(x) > \bar{P}(l_x)$ for x in $M - F$, $h(x)$ is positive on $M - F$. To prove (ii), let x_0 be a point of G. By the definition of G, there is a neighborhood N_1 of x_0 such that $\bar{P}(L_x)$ and $\bar{P}(l_x)$ are constant on N_1. By the definition of G, x_0 is not a fixed point of T so that $P(L_{x_0}) - \bar{P}(l_{x_0}) > 0$. Hence $\bar{P}(L_x) - \bar{P}(l_x)$ is a nonzero constant on N_1. Since $\bar{P}(x)$, $\bar{P}(L_x)$ and $\bar{P}(l_x)$ are continuous in N_1, $h(x)$ is continuous in N_1 and hence at x_0.

To prove (iii), let x_0 be a point of G and let N_1 be a neighborhood of x_0 such that $\bar{P}(L_x)$ and $\bar{P}(l_x)$ are constant on N_1. Then $G \supset N_1$. Let V be neighborhood of x_0 such that $\bar{V} \subset N_1 \subset G$. As in the proof of (ii), $h(x)$ is continuous on N_1 and hence on $\bar{V}$. Since T is a homeomorphism of M onto itself, T^n is a continuous transformation of M onto itself for arbitrary integral n. Hence $h(T^n(x))$ is continuous on $\bar{V}$ for arbitrary integral n. Let n be an integer. Since $\bar{P}(L_{T(x)}) = \bar{P}(L_x)$ and $\bar{P}(l_{T(x)}) = \bar{P}(l_x)$ the difference between $h(T^{n+1}(x))$ and $h(T^n(x))$ is given by the formula

$$h(T^{n+1}(x)) - h(T^n(x)) = -\frac{\frac{3}{2}[\bar{P}(T^{n+1}(x)) - \bar{P}(T^n(x))]}{\bar{P}(L_x) - \bar{P}(l_x)} .$$

No point of N_1 is a fixed point of T_n since

$$\bar{P}(L_{T^n(x)}) - \bar{P}(l_{T^n(x)}) = \bar{P}(L_x) - \bar{P}(l_x)$$

and

$$\bar{P}(L_x) - \bar{P}(l_x) = \bar{P}(L_{x_0}) - \bar{P}(l_{x_0}) > 0 .$$

Hence

$$h(T^{n+1}(x)) < h(T^n(x))$$

for all x in $\bar{V}$ and all integers n. Hence $h(T^n(x))$ is a monotone decreasing function of n for each x in $\bar{V}$. Since $\lim_{n\to\infty} h(T^n(x)) = 1/2$ and $\lim_{n\to-\infty} h(T^n(x)) = 2$ for all x in $\bar{V}$, it follows from the compactness of $\bar{V}$ that there is an integer m such that

$$2 \geqq h(T^{-n}(x)) > \frac{7}{4}$$

and

$$\frac{3}{4} > h(T^n(x)) \geqq \frac{1}{2}$$

for all integers $n > m$ and all elements x of $\bar{V}$.

LEMMA 3.6. *Let $h(x)$ be the function defined in Lemma 3.5. Let the sequence $p_n(x)$ be inductively defined for integral n by the rules:*

(i) $p_0 = 1$

(ii) $p_{n+1}(x) = h(T^n(x))p_{n-1}(x)$ *for* $n \geqq 1$

(iii) $p_{-n}(x) = p_{-n+1}(x)/h(T^{-n}(x))$ *for* $n \geqq 1$.

If x_0 is an element of G every $p_n(x)$ is continuous at x_0 and there is a neighborhood V of x_0, a constant K and an integer m such that

$$0 < p_n(x) < K\cdot\left(\frac{3}{4}\right)^{|n|-m}$$

for all x in $\bar{V}$ and all n such that $|n| > m$.

The proof of Lemma 3.6 is straightforward and has been omitted.

LEMMA 3.7. *If $q_{n,r}(x)$ is defined by the formula*

$$q_{n,r}(x) = \frac{p_n(x)^r}{\sum_{j=-\infty}^{+\infty} [p_j(x)]^r}$$

then

(i) *each $q_{n,r}(x)$ is defined and continuous for $x \in G$,*

(ii) *if x_0 is an element of G, there is a neighborhood V of x_0 such that $\bar{V} \subset G$, and an integer m such that*

$$0 < q_{n,r}(x) < \left(\left(\frac{3}{4}\right)^r\right)^{|n|-m}$$

for all n such that $|n| > m$ and all positive integers r.

(iii) *for all x in G, $q_{n,r}(T(x)) = q_{n+1,r}(x)$,*

(iv) *if $f(x)$ is a continuous function on M, and r and s are positive integers*

$$\sum_{n=-\infty}^{+\infty} f(T^n(x)) q_{n_1 r}(x)^s$$

defines a continuous T-invariant function on G.

Proof. To prove statement (i), let x_0 be a point of G. According to Lemma 3.6, there is a neighborhood V of x_0 such that $\bar{V} \subset G$ and

$$0 < p_n(x) < K \cdot \left(\frac{3}{4}\right)^{|n|-m}$$

for n sufficiently large. Hence the series

$$\sum_{-\infty}^{+\infty} p_n(x)^r$$

converges uniformly for all x in $\bar{V}$. Since $p_n(x)^r$ is continuous in $\bar{V}$, and

$$\sum_{-\infty}^{+\infty} p_n(x)^r > p_0(x)^r = 1 ,$$

every $q_{n,r}(x)$ is defined and continuous in $\bar{V}$. Since x_0 is an arbitrary point of G, statement (i) is proven.

To prove statement (ii), let x_0 be a point of G. According to Lemma 3.6, there is a neighborhood V of x_0 such that $\bar{V} \subset G$, a constant K and an integer v such that

$$0 < p_n(x) < K\left(\frac{3}{4}\right)^{|n|-v} .$$

Let m be so larger that $K\cdot(3/4)^{m-v} < 1$. Then we have

$$0 < p_n(x) < \left(\frac{3}{4}\right)^{|n|-m},$$

so that

$$0 < p_n(x)^r < \left(\left(\frac{3}{4}\right)^r\right)^{|n|-m}.$$

Since

$$\sum_{-\infty}^{+\infty} p_n(x)^r > p_0(\alpha)^r = 1,$$

we can obtain the inequality of (ii).

Statement (iii) follows directly from the observation that whenever $p_n(x)$ is defined, we have

$$p_n(T(x)) = \frac{p_{n+1}(x)}{h(x)}.$$

To prove statement (iv) note that wherever all $q_{n,r}(x)$ are defined we have

$$f(T_n(T(x)))q_{n,r}(T(x))^s = f(T^{n+1}(x))q_{n+1,r}(x)^s,$$

so that the T invariance of the series of (iv) follows. Since $f(x)$ is continuous on M and M is compact, $|f(x)|$ is bounded on M. By part (iii), the series of part (iv) converges uniformly in a closed neighborhood of each point of G for all positive integers r. Hence if r and s are arbitrary positive integers,

$$\sum_{n=-\infty}^{+\infty} f(T^n(x))[q_{n,r}(x)]^s$$

represents a continuous T-invariant function on G.

Let J be the Jacobian of the transformation T and let $|J|$ be the determinant of J. If $|J|$ is bounded away from zero on M, we can construct T invariant functions which are differentiable on an open dense subset of $M - F$. We can show that the hypothesis that $|J|$ is bounded away from zero on M and T is a homeomorphism are reasonable by an example. Let P be any polynomial with positive coefficients defined on $\tilde{M}$. Let R be the polynomial given by the formula

$$R = \sum_{i=1}^{k}\left(\sum_{j=1}^{n_i} x_{i,j}\right)$$

and let $Q_\varepsilon = R + \varepsilon P$. For $\varepsilon > 0$, Q_ε has positive coefficients and by the unpublished result of L. E. Baum stated above, the T transformation T_ε associated with Q_ε is a homeomorphism of M onto itself. The T transformation associated with $R = Q_0$ is the identity transformation so that the determinant of the Jacobian of T_0 is 1. If we let J_ε be the Jacobian of T_ε, $|J_\varepsilon|$ is a continuous function of ε, and $|J_\varepsilon| \to 1$ as $\varepsilon \to 0$ at each point of M. Since M is compact, there is an ε such that $|J_\varepsilon| > 1/2$ at every point of M.

In the following we will assume that $|J|$ is bounded away from zero on M, but we note that local results can be obtained by restricting our attention to elements x of M such that $|J|$ is bounded away from zero in some neighborhood of the sequence $\{T^n(x)\}$.

LEMMA 3.8. *If T is a homeomorphism of M onto itself, the Jacobian determinant $|J|$ of T is bounded away from zero on M and $t_{n,u,v}(x)$ denotes the (u, v) component of $T^n(x)$, then:*

(i) *for every n and subscript pair i, j, $\partial/\partial x_{i,j}(t_{n,u,v}(x))$ is continuous on M;*

(ii) *there is a constant B such that*

$$\left|\frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x)\right| < B^{|n|}$$

for all (i, j) and all x in M;

(iii) *if C is a compact subset of G there is a positive integer r such that the first partial derivatives*

$$\frac{\partial}{\partial x_{i,j}} q_{n,r}(x)$$

(see Lemma 3.7 for the definition of the functions $q_{n,r}(x)$) are continuous in C and there are constants L_1 and L_2 such that

$$\left|\frac{\partial}{\partial x_{i,j}} q_{n,r}(x)\right| < L_1 L_2^{|n|}$$

and $0 < L_2 < 1$, for all x in C.

Proof. Since T^n is a rational transformation of M, with nonzero denominators, $\partial/\partial x_{i,j}(t_{n,u,v}(x))$ is continuous on M for all $n \geqq 0$. Since the Jacobian determinant of T is bounded away from zero on M, the same result holds for $\partial/\partial x_{i,j}(t_{-n,u,v}(x))$.

To prove (ii), note that

$$\frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x) = \sum_{r,s} \frac{\partial t_{1,u,v}}{\partial x_{r,s}}(T^{n-1}(x)) \frac{\partial t_{n-1,u,v}(x)}{\partial x_{i,j}}$$

for all n and every x in M. Since

$$\frac{\partial t_{1,u,v}}{\partial x_{r,s}}(x)$$

is bounded on M for all (r, s), it follows inductively that there are bounds L_1 and R_1 such that

$$\left|\frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x)\right| < L_1 \cdot R_1^n$$

for all $n > 0$. Since the determinant of

$$J = \left(\frac{\partial}{\partial x_{i,j}} t_{1,u,v}(x)\right)$$

is bounded away from zero on M_1 the elements of the matrix J^{-1} are bounded on M. It follows that there are constants L_2 and R_2 such that

$$\left|\frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x)\right| < L_2 \cdot R_2^n$$

for all $n \leqq 0$. Clearly there is a constant B such that $B^{|n|} > L_4 \cdot R_1^{|n}$ and $B^{|n|} > L_2 \cdot R_2^{|n|}$, so that

$$\left|\frac{\partial}{\partial x_{i,j}} t_{n,u,v}(x)\right| < B^{|n|}$$

for all n, u, v and all $x \in M$.

To prove (iii) we will show first that for a given $x_0 \in G$ there is a closed neighborhood V_{x_0} of x_0 and an integer i such that

$$\sum_{-\infty}^{+\infty} \frac{\partial}{\partial x_{i,j}} [p_n(x)]^r$$

converges uniformly in $\bar{V}_{x_0}$ for all $r \geqq i$. By Lemma 3.6, there is a neighborhood V of x_0 such that $G \supset \bar{V}$, a bound K and an integer m such that

$$0 < p_n(x) < K \cdot \left(\frac{3}{4}\right)^{n-m}$$

for all $x \in \bar{V}$.

For $n > 0$ we will inductively find a bound S such that

$$\left|\frac{\partial}{\partial x_{i,j}} p_n(x)\right| < S^{n+1} .$$

We have

$$\begin{aligned}\frac{\partial}{\partial x_{i,j}} p_n(x) &= \frac{\partial}{\partial x_{i,j}} h(T^n(x)) p_{n-1}(x) \\ &= h(T^n(x)) \frac{\partial}{\partial x_{i,j}} p_{n-1}(x) + p_{n-1}(x) \sum_{u,v} \frac{\partial}{\partial x_{u,v}} \cdot \frac{\partial t_{n,u,v}}{\partial x_{i,j}} .\end{aligned}$$

Now $0 < h(T^n(x)) < 2$ for $x \in \bar{V}$, and there is a bound B_1 such that $|p_{n-1}(x)| < B_1$ for $x \in \bar{V}$. For every subset of G,

$$\frac{\partial h}{\partial x_{u,v}} = -\frac{3}{2} \frac{\frac{\partial}{\partial x_{u,v}} (\bar{P})}{\bar{P}(L_x) - \bar{P}(l_x)}$$

is bounded on G since $\bar{P}$ is a polynomial and $\bar{P}(L_x) - \bar{P}(l_x)$ ranges over a finite set not including zero for all $x \in G$. Since G is closed under the transformation T, there is a constant B_2 such that $|\partial h / \partial x_{u,v}| < B_2$ at $T^n(x)$ for every element x of $\bar{V}$. Thus

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| < 2 \left| \frac{\partial}{\partial x_{i,j}} p_{n-1}(x) \right| + d B_1 B_2 B^n .$$

If K_1 is maximum of $2, dB_1B_2$ and B we have

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| < K_1 \left(\left| \frac{\partial}{\partial x_{i,j}} p_{n-1}(x) \right| + K_1^n \right) .$$

Since $p_0(x) = 1$, we have

$$\left| \frac{\partial}{\partial x_{i,j}} p_1(x) \right| < K_1^2$$

$$\left| \frac{\partial}{\partial x_{i,j}} p_2(x) \right| < 2K_1^3$$

and

$$\left| \frac{\partial}{\partial x_{i,j}} p_n(x) \right| < n K_1^{n+1} < S^{n+1}$$

for some bound S and all x in $\bar{V}$.

Since $h(T^{-n}(x)) > 1/2$ for all $x \in \bar{V}$, a similar argument yields a constant S_1 such that

$$\left| \frac{\partial p_n}{\partial x_{i,j}} (x) \right| < S_1^{|n|+1}$$

for negative n all $x \in \bar{V}$. Hence there is a single constant S so that

$$\left| \frac{\partial P_n(x)}{\partial x_{i,j}} \right| < S^{|n|+1}$$

for all $x \in \bar{V}$. Now $\bar{V}$ was selected so that

$$0 < p_n(x) < K\left(\frac{3}{4}\right)^{|n|-m}$$

for n such that $|n| > m$. For p sufficiently large,

$$0 < p_n(x) < \left(\frac{3}{4}\right)^{|n|-p}$$

for all n such that $|n| > p > m$, and so

$$0 < p_n(x)^{r-1} < \left(\left(\frac{3}{4}\right)^{r-1}\right)^{|n|-p}$$

and

$$\left|\frac{\partial}{\partial x_{i,j}} p_n(x)^r\right| = \left|r p_n(x)^{r-1}\right|\left|\frac{\partial}{\partial x_{i,j}} p_n(x)\right|$$
$$\leqq rs\left(S\left(\frac{3}{4}\right)^{r-1}\right)^p\left(\left(\frac{3}{4}\right)^{r-1} S\right)^{|n|-p} .$$

Now r can be chosen so large that $S(3/4)^{r-1} < 1$. Thus there are constants C and D so that $0 < D < 1\tau$ and

$$\left|\frac{\partial}{\partial x_{i,j}} p_n(x)^r\right| \leqq C D^{|n|-p}$$

for all x in $\bar{V}$. Hence the series

$$\sum_{-\infty}^{+\infty} p_n(x)^r$$

has continuous first parial derivatives for $x \in \bar{V}$. Since

$$\sum_{-\infty}^{+\infty} p_n(x)^r > p_0(x)^r = 1 ,$$

we have that $q_{0,r}(x)$ has continuous first partial derivatives for all x in $\bar{V}$. But

$$q_{n,r}(x) = p_n(x)^r \cdot q_{0,r}(x)$$

so that $q_{n,r}(x)$ has continuous first partial derivatives for all x in $\bar{V}$. Since $\bar{V}$ is compact, there is a bound U on the partial derivatives of $q_{0,r}(x)$ in $\bar{V}$. Thus

$$\frac{\partial}{\partial x_{i,j}} q_{n,r}(x) = q_{0,r}(x)\frac{\partial}{\partial x_{i,j}} p_n(x)^r + p_n(x)^r \frac{\partial}{\partial x_{i,j}} q_{0,r}(x)$$

and

$$\left|\frac{\partial}{\partial x_{i,j}} q_{n,r}(x)\right| \leqq WCD^{|n|-p} + \left(\frac{3}{4}\right)^{r(|n|-p)} \cdot U$$
$$\leqq R_1 \cdot R_2^{|n|}$$

with $0 < R_2 < 1$.

Since C is compact, it can be covered with a finite set of neighborhoods as $\bar{V}$, so part (iii) of the lemma follows immediately.

THEOREM 3. *If C is a compact subset of G, there are integers r and s such that for every function $f(x)$, defined and with continuous first partial derivatives on M, the function*

$$F(x) = \sum_{-\infty}^{+\infty} f(T^n(x))[q_{n,r}(x)]^s$$

is continuous and has continuous first partial derivatives for all x in $\bigcup_{-\infty}^{+\infty} T^n(C)$ and

$$F(x) = F(T(x))$$

wherever $F(x)$ is defined.

Proof. Clearly $F(x) = F(T(x))$ wherever $F(x)$ is defined. Also, if all first prtial derivatives $\partial/\partial_{xi,j}F(x)$ are defined and continuous at $x = x_0$, it follows by elementary methods from the fact that $|J|$ is bounded away from zero on M and J is continuous on M that $\partial/\partial x_{i,j}F(x)$ is defined and continuous at $T^n(x_0)$ for all n. Hence we need only show that $F(x)$ has continuous first partial derivatives for all elements x of C. We choose to show that the series of partial derivatives

$$\sum_{-\infty}^{+\infty} \frac{\partial}{\partial x_{i,j}} f(T^n(x))[q_{n,r}(x)]^s$$

converges uniformly in C.

Note that

$$\frac{\partial}{\partial x_{i,j}} f(T^n(x))[q_{n,r}(x)]^s = q_{n,r}(x)^s \sum_{u,v} \frac{\partial f}{\partial x_{u,v}}\bigg|_{T^n(x)} \frac{\partial t_{n,u,v}}{\partial x_{i,j}}$$
$$+ s[q_{n,r}(x)]^{s-1} \frac{\partial q_{n,r}(x)}{\partial x_{i,j}} f(T^n(x)) .$$

Since f and its first partial derivatives are bounded on M it follows directly from Lemma 3.8 that s may be chosen so large that the series

$$\sum_{-\infty}^{+\infty} \frac{\partial}{\partial x_{i,j}} f(T^n(x))[q_{n,r}(x)]^s$$

is majorized by a geometric series, and the choice of s is independent of f.

The analysis of this section can be extended to obtain a local analogue of Theorem 1, i.e., a set of $d - k$ functions can be found which are continuous on G, have nonzero Jacobian at a point x_0 of G and are invariant with respect to T.

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