Pacific Journal of Mathematics

C_λ-GROUPS AND λ-BASIC SUBGROUPS

KYLE DAVID WALLACE

Vol. 43, No. 3 May 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 3, 1972

C_λ-GROUPS AND λ-BASIC SUBGROUPS

KYLE D. WALLACE

The groups considered in this paper will be abelian primary groups. For λ a fixed but arbitrary countable limit ordinal, C. K. Megibben studied that class C_λ consisting of all p-groups G such that $G/p^\alpha G$ is a direct sum of countable groups for all $\alpha < \lambda$.

Fundamental to the development of C_λ-theory was the introduction of the concept of a λ-basic subgroup, which generalized the familiar concept of a basic subgroup, and the following existence theorem: A primary group G contains a λ-basic subgroup if and only if G is a C_λ-group. This paper extends, in a natural fashion, the concepts of "C_λ-group" and "λ-basic subgroup" to an arbitrary limit ordinal λ, and considers the analogous question of existence. This is used to examine the structure of p^λ-pure subgroups of C_λ-groups for limit ordinals λ such that $\lambda \neq \beta + \omega$ for any ordinal β. For an ordinal λ of this type, if H is a p^λ-pure subgroup of the C_λ-group G then both H and G/H are C_λ-groups.

The classical theory of torsion abelian groups corresponds to Megibben's C_ω-theory, in that the class of all p-groups coincides with C_ω.

1. Preliminaries.

In this section we assemble the basic concepts which are crucial in the following development. For pertinent results related to these concepts, we refer the reader to [2].

A subgroup H of the p-group G is said to be a *p^α-pure subgroup* if $H \rightarrowtail G \twoheadrightarrow G/H$ represents an element of $p^\alpha \mathrm{Ext}\,(G/H, H)$. This notion is due to Nunke and shall assume the same role in our theory as that played by ordinary purity (i.e. p^ω-purity for p-groups) in the classical theory.

The subgroup H is said to be a *p^α-high subgroup* of G if H is maximal among the subgroups of G that intersect $p^\alpha G$ trivially. From [5] or [7], if H is a p^α-pure subgroup of G then $H \cap p^\beta G = p^\beta H$ for all $\beta \leqq \alpha$ and $p^\beta(G/H)[p] = (p^\beta G)[p] + H/H$ for all $\beta < \alpha$. Moreover, if G/H is divisible, where H is a p^α-pure subgroup of G and α is a limit ordinal, then $H + p^\beta G/p^\beta G = G/p^\beta G$ for all $\beta < \alpha$. If H is a p^α-high subgroup of G, then H is a $p^{\alpha+1}$-pure subgroup of G and $H + p^\alpha G/p^\alpha G$ is p^α-pure in $G/p^\alpha G$ (see [3]).

A subgroup H of the p-group G is *neat* if $pG \cap H = pH$. From [3], if H is a neat subgroup of the p-group G and

$$G[p] = H[p] + p^{\beta}G[p]$$

for each $\beta < \alpha$, then H is p^{α}-pure in G. Moreover, if A is a neat subgroup of $p^{\alpha}G$ and if $B \supseteq A$ is maximal in G with respect to $B \cap p^{\alpha}G = A$, then B is $p^{\alpha+1}$-pure in G.

A subgroup H of the p-group G is *nice* in G if each coset $g+H$ contains an element $g + h$ that has maximal height in G. If g has maximal height in the coset $g + H$, we say g is *proper* with respect to H.

Totally projective groups as introduced by Nunke provide a generalization of the concept of a direct sum of countable reduced groups. A p-group G is *p^{α}-projective* if $p^{\alpha}\mathrm{Ext}\,(G, C) = 0$ for all groups C. A reduced p-group G is *totally projective* if $Gp^{\alpha}G$ is p^{α}-projective for every ordinal α. The following characterization of totally projective groups, given and utilized by Hill [4] to show that the Ulm invariants suffice to classify totally projective p-groups, is used extensively.

THEOREM A. *A reduced p-group G is totally projective if and only if G has a collection $\mathscr{C}_G$ of nice subgroups satisfying the following conditions*:

(0) *0 is a member of $\mathscr{C}_G$.*

(1) *$\mathscr{C}_G$ is closed with respect to group-theoretic union.*

(2) *If $A \in \mathscr{C}_G$ and H is a subgroup of G such that $(H + A)/A$ is countable, there exists $B \in \mathscr{C}_G$ such that $B \supseteq H + A$ and B/A is countable.*

In the sequel, we shall refer to these conditions as the third axiom of countability and to condition (2) *as the countable extension property.*

An ordinal λ is said to be *confinal with ω* if λ is the limit of a countable ascending sequence of ordinals. From [7], if α is confinal with ω then every p^{α}-pure subgroup of a p^{α}-projective group is p^{α}-projective.

To extend the concepts of C_{λ}-group and λ-basic subgroup to an arbitrary limit ordinal, we introduce the following definitions. For a fixed but arbitrary limit ordinal λ, C_{λ} shall designate that class of all p-groups G such that $G/p^{\alpha}G$ is totally projective for all $\alpha < \lambda$. Groups in the class C_{λ} will be referred to as C_{λ}-groups. B is said to be a λ-*basic subgroup* of G if

(1) B is totally projective of length at most λ,

(2) B is a p^λ-pure subgroup of G, and

(3) G/B is divisible.

For B a λ-basic subgroup of G and $\alpha < \lambda$, a routine argument yields that the α-th Ulm invariant of B coincides with the α-th Ulm invariant of G. Hence, by Hill's version of Ulm's Theorem, we obtain the following analogue to a well-known property of ordinary basic subgroups

PROPOSITION 1.1. *If B and $\bar{B}$ are λ-basic subgroups of G then $B \cong \bar{B}$.*

We shall require for technical convenience the notion of a λ-high confinal tower. Let λ be an ordinal confinal with ω, and G an abelian p-group. A λ-*high confinal tower* of G is an ascending sequence $\{G_n\}$ of subgroups of G such that:

(1) For each positive integer n, G_n is a $p^{\alpha(n)}$-high subgroup of G;

(2) $\lambda = \sup \{\alpha(n)\}$, $\alpha(n) < \alpha(n+1)$;

(3) If $\lambda = \beta + \omega$ for some limit ordinal β, then $\alpha(n) = \beta + m$ for some positive integer m;

(4) If $\lambda \neq \beta + \omega$ for any ordinal β, then $\alpha(n) = \beta(n) + \omega$ for some limit ordinal $\beta(n)$.

2. The existence theorem.

In this section we determine, for an arbitrary but fixed limit ordinal λ, that class of all abelian p-groups G such that G contains a λ-basic subgroup (see Theorem 2.7).

LEMMA 2.1. *Suppose $G/p^\beta G$ is totally projective and B is a basic subgroup of $p^\beta G$. If H is a subgroup of G such that*

$$G/B = H/B \oplus p^\beta G/B$$

then H is totally projective.

Proof. If H is a subgroup of G such that $G/B = H/B \oplus p^\beta G/B$, then $G = H + p^\beta G$ and H is maximal in G with respect to $H \cap p^\beta G = B$. Thus H is $p^{\beta+1}$-pure in G. Consequently $p^\alpha H = p^\alpha G \cap H$ for all $\alpha \leqq \beta + 1$, and in particular $p^\beta H = p^\beta G \cap H = B$. We now observe that $H/p^\beta H$ is totally projective since

$$H/p^\beta H = H/p^\beta G \cap H \cong (H + p^\beta G)/p^\beta G = G/p^\beta G\ ,$$

and $p^\beta H = B$ is a direct sum of cyclic groups.

LEMMA 2.2. *Let λ be a limit ordinal confinal with ω such that $\lambda \neq \beta + \omega$ for any ordinal β. Suppose $G = \bigcup G_n$ with $\{G_n\}$ a λ-high confinal tower of G. If $A \subseteqq G$ satisfies the conditions:*

(1) A is the union of an ascending sequence of subgroups $A_1 \subseteqq A_2 \subseteqq \cdots$ such that A_n is nice in G_n for each n,

(2) $A \subseteqq p^\alpha G + A_n$ for all $\alpha < \alpha(n)$;

then A is nice in G.

Proof. We show that each coset $x + A$ contains an element $x + a$ that is proper with respect to A.

Let $x \in G - A$, and choose n such that $x \in G_n$. Let

$$\beta = h_G(x) < \alpha(n) .$$

For $k \geqq n$, there exists $a_k \in A_k$ such that $h_G(x + a_k) = h_{G_k}(x + a_k) \geqq h_{G_k}(x + a') = h_G(x + a')$ for any $a' \in A_k$. It suffices to show that the sequence $h_G(x + a_n) \leqq h_G(x + a_{n+1}) \leqq \cdots$ cannot be strictly increasing.

Suppose for some $m \geqq n$ that $h_G(x + a_m) > \beta = h_G(x)$. Then $h_G(a_{m+i}) = h_G(x)$ for $i = 1, 2, \cdots$. Let $\gamma = h_G(x + a_m)$ and observe $\gamma < \alpha(m)$ since $x + a_m \in G_m$. Moreover $\gamma + 1 < \alpha(m)$ since $\alpha(m)$ is a limit ordinal. We shall show that $x + a_m$ is proper with respect to A. Suppose $x + a_m$ is not proper with respect to A. Then for some k, $h_G(x + a_{m+k}) > h_G(x + a_m) = \gamma$ and $x + a_{m+k} \in p^{\gamma+1}G$. Since $A \subseteqq p^{\gamma+1}G + A_m$ we have $a_{m+k} = g_k + a_{m,k}$ with $g_k \in p^{\gamma+1}G$ and $a_{m,k} \in A_m$. Hence

$$x + a_{m,k} = x + g_k + a_{m,k} \in p^{\gamma+1}G$$

and $x + a_{m,k} \in p^{\gamma+1}G$. This however is absurd since

$$h_G(x + a_{m,k}) \leqq h_G(x + a_m) = \gamma .$$

Consequently $x + a_m$ is proper with respect to A and A is nice in G.

With λ and G as in Lemma 2.2, we shall now restrict our attention to the case where G_n is totally projective for each n. Let $\mathscr{C}_n$ denote a collection of nice subgroups of G_n satisfying the third axiom of countability. Let $\mathscr{C}$ be the collection of all subgroups A of G such that

(1) $A = \bigcup A_n$ with $A_1 \subseteqq A_2 \subseteqq \cdots$ and $A_n \in \mathscr{C}_n$ for each n,

(2) $A \subseteqq p^\alpha G + A_n$ for all $\alpha < \alpha(n)$.

The members of $\mathscr{C}$ are nice by Lemma 2.2.

LEMMA 2.3. *$\mathscr{C}$ has the countable extension property.*

Proof. For each n, we have $\alpha(n) = \beta(n) + \omega$ with $\beta(n)$ a limit ordinal. Thus $\lambda = \sup\{\alpha(n)\} = \sup\{\beta(n)\}$. We observe that to show

$B \subseteqq p^{\alpha}G + B_n$ for each ordinal $\alpha < \alpha(n)$, it suffices to show

$$B \subseteqq p^{\beta(n)+k}G + B_n \qquad \text{for each } k < \omega .$$

Let $A \in \mathscr{C}$ and H a subgroup of G such that H/A is countable. Let $S = \{x_i\}_{i<\omega}$ be such that $H = \langle A, S\rangle$ and let $S_n = S \cap G_n$. By induction, we shall construct, for each positive integer n, subgroups $B_1^{(n)} \subseteqq B_2^{(n)} \subseteqq \cdots \subseteqq B_n^{(n)}$ such that

(0) $A_i \subseteqq B_i^{(n)}$ for $i \leqq n$,

(1) $B_j^{(k)} \subseteqq B_i^{(n)}$ for $k \leqq n$,

(2) $B_i^{(n)} \in \mathscr{C}_i$,

(3) $B_{i+1}^{(n)} \subseteqq p^{\beta(i)+k}G + B_i^{(n)}$ for $k < \omega$,

(4) $B_i^{(n)} \supseteqq S_i$,

(5) $B_i^{(n)}/A_i$ is countable for $i \leqq n$.

We now show that the existence of subgroups $B_i^{(n)}$ satisfying the above conditions (0) $-$ (5) will suffice to establish the lemma. For each $i < \omega$, let $B_i = \bigcup_{n\geqq i} B_i^{(n)}$ and observe that $B_i \in \mathscr{C}_i$ and $B_i \subseteqq B_{i+1}$. Moreover $B_{i+1} = \bigcup_{n\geqq i+1} B_{i+1}^{(n)} \subseteqq \bigcup_{n\geqq i+1}(p^{\beta(i)+k}G + B_i^{(n)}) = p^{\beta(i)+k}G + B_i$ for $k < \omega$, and by induction $B_{i+m} \subseteqq p^{\beta(i)+k}G + B_i$ for all $m < \omega$, $k < \omega$. Let $B = \bigcup_{i<\omega} B_i$. Clearly $B \supseteqq H$ and B/A is countable. Moreover $B \in \mathscr{C}$ since $B \subseteqq p^{\beta(i)+k}G + B_i$ for each i and k.

Suppose we have constructed $B_i^{(n)}$, $1 \leqq i \leqq n$, satisfying (0) $-$ (5) above. We shall now construct $B_i^{(n+1)}$ for $1 \leqq i \leqq n + 1$.

For $1 \leqq i \leqq n$, let $B_{i,0} = A_i$ and $B_{i,1} = B_i^{(n)}$. Set $B_{n+1,0} = A_{n+1}$ and let $B_{n+1,1}$ be a member of $\mathscr{C}_{n+1}$ such that

$$B_{n+1,1} \supseteqq \langle B_n^{(n)} + A_{n+1}, S_{n+1}\rangle$$

and $B_{n+1,1}/A_{n+1}$ is countable. By induction, we shall construct a family of subgroups $B_{i,j}$, with $1 \leqq i \leqq n + 1$ and $j < \omega$, satisfying the conditions

(i) $B_{i,j} \subseteqq B_{i,k}$ for $j \leqq k$,

(ii) $B_{i,j} \in \mathscr{C}_i$,

(iii) $B_{i,j+1}/B_{i,j}$ is countable,

(iv) $B_{i+1,2j} \subseteqq p^{\beta(i)+k}G + B_{i,2j}$ for all $1 \leqq i \leqq n$ and j, $k < \omega$;

(v) $B_{i,2j+1} \subseteqq B_{i+1,2j+1}$ for all $1 \leqq i \leqq n$ and $j < \omega$.

We define $B_i^{(n+1)} = \bigcup_{j<\omega} B_{i,j}$ and observe that

$$\bigcup_{j<\omega} B_{i,2j} = B_i^{(n+1)} = \bigcup_{j<\omega} B_{i,2j+1} .$$

By (iv), we see that

$$B_{i+1}^{(n+1)} = \bigcup_{j<\omega} B_{i+1,2j} \subseteqq \bigcup_{j<\omega} (p^{\beta(i)+k}G + B_{i,2j}) = p^{\beta(i)+k}G + B_i^{(n+1)}$$

for all $k<\omega$, $1 \leqq i \leqq n$. By (v), $B_i^{(n+1)} = \bigcup_{j<\omega} B_{i,2j+1} \subseteqq \bigcup_{j<\omega} B_{i+1,2j+1} = B_{i+1}^{(n+1)}$ for all $1 \leqq i \leqq n$. It is now easy to see that conditions

(0) – (4) are satisfied by the subgroups $B_i^{(j)}$, $1 \leqq i \leqq j \leqq n+1$. Since $B_{i,j+1}/B_{i,j}$ is countable for each

$$j < \omega,\ B_i^{(n+1)}/A_i = B_i^{(n+1)}/B_{i,0} = \bigcup_{j<\omega} B_{i,j+1}/B_{i,0}$$

is countable for all $1 \leqq i \leqq n+1$ and condition (5) is satisfied.

Suppose we have constructed $B_{i,j}$ satisfying (i) – (v), for all $1 \leqq i \leqq n+1$ and all $j \leqq 2m+1$. We shall now construct $B_{i,2m+2}$ for $1 \leqq i \leqq n+1$. Define $B_{n+1,2m+2} = B_{n+1,2m+1}$. Assuming, for some positive integer $l \leqq n$, that $B_{i,2m+2}$ has been constructed for each $l+1 \leqq i \leqq n+1$, we let $\{x_j\}_{j<\omega} \subseteqq B_{l+1,2m+2}$ be such that $B_{l+1,2m+2} = \langle B_{l+1,2m}, \{x_j\}_{j<\omega}\rangle$. Since $G \subseteqq p^{\beta(l)+k}G + G_l$ we obtain decompositions $x_j = g_{j,k} + x_{j,k}$, with $g_{j,k} \in p^{\beta(l)+k}G$ and $x_{j,k} \in G_l$, for each $j, k < \omega$. Let $T_{l,2m+2} = \{x_{j,k}\}_{j,k<\omega} \subseteqq G_l$. Let $B_{l,2m+2}$ be a member of $\mathscr{C}_l$ such that $B_{l,2m+2} \supseteqq \langle B_{l,2m+1}, T_{l,2m+2}\rangle$ and $B_{l,2m+2}/B_{l,2m+1}$ is countable. Observe, for each $k < \omega$, $B_{l+1,2m+2} \subseteqq p^{\beta(l)+k}G + B_{l,2m+2}$ since

$$B_{l+1,2m} \subseteqq p^{\beta(l)+k}G + B_{l,2m} \subseteqq p^{\beta(l)+k}G + B_{l,2m+1} \subseteqq p^{\beta(l)+k}G + B_{l,2m+2}$$

and

$$\{x_j\}_{j<\omega} \subseteqq p^{\beta(l)+k}G + \langle T_{n,2m+2}\rangle \subseteqq p^{\beta(l)+k}G + B_{l,2m+2}\,.$$

To conclude the proof, it suffices to construct $B_{i,2m+3}$ for $1 \leqq i \leqq n+1$, having been given a collection $B_{i,j}$ satisfying (i) – (v), for all $1 \leqq i \leqq n+1$ and all $j \leqq 2m+2$. Define $B_{1,2m+3} = B_{1,2m+2}$ and assume, for some positive integer $l \leqq n$, that $B_{i,2m+3}$ has been constructed for each $1 \leqq i \leqq l$. Since $B_{l,2m+3}/B_{l,2m+1}$ is countable and $B_{l,2m+1} \subseteqq B_{l+1,2m+1} \subseteqq B_{l+1,2m2}$, $(B_{l,2m+3} + B_{l+1,2m+2})/B_{l+1,2m+2}$ is countable. Thus there exists $B_{l+1,2m+3} \in \mathscr{C}_{l+1}$ such that $B_{l+1,2m+3} \supseteqq B_{l+1,2m+3} + B_{l+1,2m+2}$ and $B_{l+1,2m+3}/B_{l+1,2m+2}$ is countable. The collection of subgroups $B_{i,j}$, for $1 \leqq i \leqq n+1$ and $0 \leqq j \leqq 2m+3$, clearly satisfies conditions (i) – (v).

LEMMA 2.4. *If α is confinal with ω and $G/p^\alpha G$ is totally projective, then every p^α-high subgroup of G is totally projective.*

Proof. Let α be an ordinal confinal with ω, and H a p^α-high subgroup of G. Since $H \cong (H + p^\alpha G)/p^\alpha G$ and $(H + p^\alpha G)/p^\alpha G$ is p^α-pure in the p^α-projective group $G/p^\alpha G$, H is p^α-projective. Since α is a limit ordinal, $H/p^\beta H \cong G/p^\beta G$ is p^β-projective for all $\beta < \alpha$. Consequently H is totally projective.

PROPOSITION 2.5. *Let λ be a limit ordinal confinal with ω, and*

$\{G_n\}$ a λ-high confinal tower of G. If G is a C_λ-group then $\bigcup G_n$ is totally projective of length at most λ.

Proof. Clearly $\bigcup G_n$ is an isotype subgroup of G and hence has length at most λ. The proof that $\bigcup G_n$ is totally projective shall consist of two cases.

Case 1. $\lambda = \beta + \omega$.

Consider the subgroup $(\bigcup G_n) \cap p^\beta G$ of $p^\beta C$, and observe that

$$G_n \cap p^{\beta+\omega} G = 0$$

for each n. Consequently $p^\omega((\bigcup G_n) \cap p^\beta G) = \bigcup (G_n \cap p^{\beta+\omega} G) = 0$ and thus $(\bigcup G_n) \cap p^\beta G$ is a p-group without elements of infinite height. Since

$$(\bigcup G_n) \cap p^\beta G = \bigcup (G_n \cap p^\beta G) = \bigcup p^\beta G_n$$

is the union of an ascending sequence of bounded subgroups, it follows, by the Kulikov criterion, that $(\bigcup G_n) \cap p^\beta G$ is a direct sum of cyclic groups. It is easy to see that $(\bigcup G_n) \cap p^\beta G$ is a pure subgroup of $p^\beta G$ and that $\bigcup G_n + p^\beta G = G$. Consequently $(\bigcup G_n) \cap p^\beta G$ is a basic subgroup of $p^\beta G$. Since G is a C_λ-group, $G/p^\beta G$ is totally projective and, by Lemma 2.1, it follows that $\bigcup G_n$ is totally projective.

Case 2. $\lambda \neq \beta + \omega$ for any ordinal β.

By Lemma 2.4, it follows that in this case G_n is totally projective for each n. To show that $\bigcup G_n$ contains a collection of nice subgroups satisfying the third axiom of countability, let $\mathscr{C}$ denote the collection of nice subgroups of $\bigcup G_n$ as defined preceding Lemma 2.3. Clearly $0 \in \mathscr{C}$. By Lemma 2.3, $\mathscr{C}$ has the countable extension property. Thus it suffices to show that $\mathscr{C}$ is closed with respect to group-theoretic union. Suppose $\{A_\gamma\}_{\gamma\in I} \subseteqq \mathscr{C}$ with $A_\gamma = \bigcup_{n<\omega} A_{n,\gamma}$ where

(1) $A_{n,\gamma} \subseteqq A_{k,\gamma}$ for $n \leqq k$,

(2) $A_{n,\gamma} \in \mathscr{C}_n$ for each n.

(3) For each n and $\alpha < \alpha(n)$, $A_\gamma \subseteqq p^\alpha G + A_{\gamma,n}$.

Then $\sum_{\gamma\in I} A_\gamma = \sum_{\gamma\in I}(\bigcup_{n<\omega} A_{n,\gamma}) = \bigcup_n(\sum_{\gamma\in I} A_{n,\gamma})$ with

$$\sum_{\gamma\in I} A_{n,\gamma} \subseteqq \sum_{\gamma\in I} A_{k,\gamma} \qquad \text{for } n \leqq k\,,$$

and $\sum_{\gamma\in I} A_{n,\gamma} \in \mathscr{C}_n$. Moreover, for each n and $\alpha < \alpha(n)$, we have

$$\sum_{\gamma\in I} A_\gamma \subseteqq \sum_{\gamma\in I} (p^\alpha G + A_{n,\gamma}) = p^\alpha G + (\sum_{\gamma\in I} A_{n,\gamma})\,.$$

Consequently $\sum_{\gamma \in I} A_\gamma \in \mathscr{C}$.

LEMMA 2.6. *Let $\{G_n\}$ be a λ-high confinal tower of G. If $H = \bigcup G_n$ then H is p^λ-pure in G.*

Proof. Let $\alpha < \lambda$, and recall that H is an isotype, and hence a neat, subgroup of G. There exists a positive integer n such that $\alpha < \alpha(n)$, and $G[p] = G_n[p] + (p^\alpha G)[p] = H[p] + (p^\alpha G)[p]$.

THEOREM 2.7. (a) *If G is a C_λ-group with λ confinal with ω then G contains a λ-basic subgroup.*

(b) *If G is a reduced p-group which contains a proper λ-basic subgroup then G is a C_λ-group and λ is confinal with ω.*

Proof. Part (a) follows from Proposition 2.5 and Lemma 2.6.

Conversely, suppose H is a proper λ-basic subgroup of the reduced p-group G. For $\alpha < \lambda$,

$$G/p^\alpha G = (H + p^\alpha G)/p^\alpha G \cong H/(H \cap p^\alpha G) = H/p^\alpha H$$

is totally projective. Thus G is a C_λ-group. That λ must be confinal with ω is immediate from (3.7) of [1] and (3.10) of [7].

3. C_λ-Groups for $\lambda \neq \beta + \omega$. The purpose of this section is to examine the structure of p^λ-pure subgroups of C_λ-groups. We shall restrict our attention to ordinals that cannot be expressed in the form $\beta + \omega$ for any ordinal β. The techniques utilized are essentially those of Megibben in [6] and rely upon the existence of λ-basic subgroups as established in § 2.

The proofs given for Lemma 3 in [6] can, with the aid of § 2, be reproduced to yield the following lemmas.

LEMMA 3.1. *Let λ be an ordinal confinal with ω. Suppose H is a p^λ-pure subgroup of G and that $\{H_n\}$ is a λ-high confinal tower of H. Then there exists a λ-high confinal tower $\{G_n\}$ of G such that, for each n, $H_n \subseteqq G_n$ and $H_n = H \cap G_n$.*

LEMMA 3.2. *Let λ be an ordinal confinal with ω such that $\lambda \neq \beta + \omega$ for any β. Suppose G is totally projective and that $G = \bigcup G_n$ where $\{G_n\}$ is a λ-high confinal tower. If H is a p^λ-pure subgroup of G such that, for each n, $H \cap G_n$ is a $p^{\alpha(n)}$-high subgroup of H, then H is a direct summand of G.*

THEOREM 3.3. *Let λ be any limit ordinal such that $\lambda \neq \beta + \omega$ for any β, and let G be a C_λ-group. If H is a p^λ-pure subgroup of G then H is a C_λ-group.*

Proof. It suffices to establish the proposition for ordinals λ such that λ is confinal with ω and $\lambda \neq \beta + \omega$ for and ordinal β. For such an ordinal λ, let $\{H_n\}$ be a λ-high confinal tower of H. By Lemma 3.1, there exists a λ-high confinal tower $\{G_n\}$ of G such that $H_n = H \cap G_n$ for each n. Since G is a C_λ-group, $\bigcup G_n$ is totally projective, by Proposition 2.5. By Lemma 3.2, it follows that $\bigcup H_n$ is a λ-basic subgroup of H and consequently, by Theorem 2.7, H is a C_λ-group.

LEMMA 3.4. *Let λ be confinal with ω, $\lambda \neq \beta + \omega$ for any β. Let A be a totally projective group of length at most λ and suppose A is a p^λ-pure subgroup of the C_λ-group G. Then there exists a subgroup C of G such that $A \oplus C$ is a λ-basic subgroup of G.*

Proof. Since A is a totally projective group of length at most λ, it follows from Proposition 1.1 that A is the union of a λ-high confinal tower $\{A_n\}$ of itself. By Lemma 3.1, there exists a λ-high confinal tower $\{G_n\}$ of G such that $A_n = A \cap G_n$ for each n. Let $B = \bigcup G_n$. By the proof of Theorem 2.7, B is a λ-basic subgroup of G. But $\{G_n\}$ is also a λ-high confinal tower of B and, by Lemma 3.2, we have the desired decomposition $B = A \oplus C$.

THEOREM 3.5. *Let λ be a limit ordinal such that $\lambda \neq \beta + \omega$ for any ordinal β, and let G be a C_λ-group. If H is a p^λ-pure subgroup of G then G/H is a C_λ-group.*

Proof. It suffices to establish the result for an arbitrary but fixed ordinal λ satisfying the conditions that λ is confinal with ω and $\lambda \neq \beta + \omega$ for any ordinal β. Let λ be such an ordinal and H a p^λ-pure subgroup of the C_λ-group G. By Theorem 3.3, H is a C_λ-group and thus, by Theorem 2.7, contains a λ-basic subgroup. Let A be a λ-basic subgroup of H and choose C, by Lemma 3.4, such that $A \oplus C$ is a λ-basic subgroup of G. If $x \in (H \cap C)[p]$, we can write, for each $\alpha < \lambda$, $x = a_\alpha + z_\alpha$ where $a_\alpha \in A[p]$ and $z_\alpha \in p^\alpha H$. Thus $-a_\alpha + x \in p^\alpha(A \oplus C) = p^\alpha A \oplus p^\alpha C$ and $x \in \bigcap p^\alpha C = p^\lambda C = 0$. We then have a direct decomposition $H \oplus C$. If $pg \in H \oplus C$, then

$$pg = a + ph + c$$

where $a \in A, h \in H$ and $c \in C$. Since $pG \cap (A \oplus C) = p(A \oplus C)$, we conclude that $pG \cap (H \oplus C) = p(H \oplus C)$ and $H \oplus C$ is neat in G. Moreover, $G[p] \subseteq (A \oplus C)[p] + p^{\alpha}G \subseteq (H \oplus C)[p] + p^{\alpha}G$ for all $\alpha < \lambda$ and therefore $H \oplus C$ is a p^{λ}-pure subgroup of G. Consequently, $(H \oplus C)/H$ is p^{λ}-pure in G/H. Also $(H \oplus C)/H \cong C$ is totally projective of length at most λ, and

$$(G/H)/(H \oplus C/H) \cong (G/A \oplus C)/(H \oplus C/A \oplus C)$$

is divisible. We have constructed a λ-basic subgroup of G/H and we conclude that G/H is indeed a C_{λ}-group.

As easy consequences of Theorem 3.5, we have the following analogues of familiar properties of pure subgroups.

COROLLARY 3.6. *Suppose λ is a limit ordinal such that*

$$\lambda \neq \beta + \omega$$

for any ordinal β. A subgroup H of a C_{λ}-group G is a p^{λ}-pure subgroup if and only if $(H + p^{\alpha}G)/p^{\alpha}G$ is a direct summand of $G/p^{\alpha}G$ for all $\alpha < \lambda$.

COROLLARY 3.7. *Suppose λ is a limit ordinal such that*

$$\lambda \neq \beta + \omega$$

for any ordinal β. If H is a p^{λ}-pure subgroup of the C_{λ}-group G and if $p^{\alpha}H = 0$ for some $\alpha < \lambda$, then H is a direct summand of G.

4. Remark. As noted above, we have not dealt with the problems of p^{λ}-pure subgroups of C_{λ}-groups where λ is a limit ordinal which may be expressed in the form $\lambda = \beta + \omega$. It would not be surprising, however, if the results of § 3 fail to hold for certain of such ordinals.

REFERENCES

1. P. Crawley and A. W. Hales, *The structure of abelian p-groups given by certain presentations*, J. of Algebra, **12** (1969), 10-23.
2. P. Griffith, *Infinite Abelian Group Theory*, Chicago, The University of Chicago Press, 1970.
3. P. Hill, *Isotype subgroups of direct sums of countable groups*, Illinois J. Math., **13** (1969), 281-290.
4. ———, *On the classification of abelian groups*, (to appear).
5. J. Irwin, C. Walker and E. Walker, *On p^{α}-pure Sequences of Abelian Groups*, Topics in Abelian Groups, Chicago, (1963), 69-119.
6. C. Megibben, *A generalization of the classical theory of primary groups*, Tôhoku Math. J., **22** (1970), 347-356.

7. R. Nunke, *Homology and direct sums of countable groups*, Math. Z., **101** (1967), 182–212.

Received July 13, 1971.

VANDERBILT UNIVERSITY
AND
WESTERN KENTUCKY UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 43, No. 3 May, 1972