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**APPROXIMATION BY HOLOMORPHIC FUNCTIONS ON  
CERTAIN PRODUCT SETS IN  $C^n$**

BARNET MORDECAI WEINSTOCK

# APPROXIMATION BY HOLOMORPHIC FUNCTIONS ON CERTAIN PRODUCT SETS IN $C^N$

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**In this paper we prove several theorems concerning approximation by holomorphic functions on product sets in  $C^n$  where each factor is either a compact plane set or the closure of a strongly pseudoconvex domain. In particular we show that every continuous function which is locally approximable by holomorphic functions on such a set is globally approximable. Our results depend on a generalization of a theorem of Andreotti and Stoll on bounded solutions of the inhomogeneous Cauchy-Riemann equations on certain product domains.**

1. Statement of results. If  $K$  is a compact set in  $C^n$  let  $C(K)$  denote the Banach space of continuous complex-valued functions on  $K$  with the uniform norm, and let  $H(K)$  denote the closure in  $C(K)$  of the space of functions which are holomorphic in some neighborhood of  $K$ . When  $n = 1$ , each function in  $H(K)$  is the uniform limit of a sequence of rational functions which are finite on  $K$  and the spaces  $H(K)$  (usually denoted  $R(K)$  in this instance) have been extensively studied. In particular, the following properties of  $H(K)$  are well-known in the case  $n = 1$  (cf. Chapter 3 of [2]):

(1) If  $U$  is a neighborhood of  $K$ ,  $f \in C^1(U)$ , and  $\partial f / \partial \bar{z} \equiv 0$  on  $K$ , then  $f|_K \in H(K)$ .

(2) If  $f \in C(K)$  and if for each  $x \in K$  there is a neighborhood  $U_x$  of  $x$  in  $C$  such that  $f \in H(K \cap \bar{U}_x)$ , then  $f \in H(K)$ .

(3) If  $\mu$  is a complex Borel measure on  $K$ , then  $\mu = \partial \hat{\mu} / \partial \bar{z}$  where

$$\hat{\mu}(z) = -\frac{1}{\pi} \int_K (\zeta - z)^{-1} d\mu(\zeta)$$

is locally integrable on  $C$ . A measure  $\mu$  is an annihilating measure for  $H(K)$  (i.e.,  $\int f d\mu = 0$  for all  $f \in H(K)$ ) if and only if  $\hat{\mu}$  is supported on  $K$ .

Properties (1)-(3) are not valid for arbitrary compact sets of  $C^n$ , even if one restricts one's attention to holomorphically convex, or even polynomially convex sets. A celebrated example of Kallin [6] shows that (2) fails in general for polynomially convex compact sets. Also, Chirka [3], by modifying her example, has shown that for each positive integer  $s$  there is a compact holomorphically convex set  $K^s$  in  $C^3$  and a function  $f_s \in C^\infty(K^s)$  such that  $f_s \notin H(K^s)$  although  $\bar{\partial} f_s$

vanishes on  $K^s$  to order  $s$ .<sup>1</sup>

In this paper we consider compact sets  $K \subset C^n$  of the form  $K = K_1 \times \cdots \times K_r$  where each  $K_i$  is either a compact set in  $C$  or the closure of a strongly pseudo-convex domain in  $C^{n_i}$ . For such  $K$  we prove the following theorems:

**THEOREM 1.1.** *If  $U$  is a neighborhood of  $K$ ,  $f \in C^{r+1}(U)$ , and  $\partial^\alpha f / \partial \bar{z}^\alpha \equiv 0$  on  $K$  for all  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $\sum \alpha_j \leq r$ , then  $f \in H(K)$ .*

**THEOREM 1.2.** *If  $f \in C(K)$  and if for each  $x \in K$  there is a neighborhood  $U_x$  such that  $f \in H(K \cap \bar{U}_x)$ , then  $f \in H(K)$ .*

**THEOREM 1.3.** *A measure  $\mu$  on  $K$  is an annihilating measure for  $H(K)$  if and only if there exist distributions  $\lambda_1, \dots, \lambda_n$  of order  $\leq r - 1$  with support in  $K$  such that  $\mu = \sum \partial \lambda_j / \partial \bar{z}_j$ .*

It is possible that Theorem 1.1 remains valid if  $f$  is merely required to satisfy  $\partial f / \partial \bar{z}_j \equiv 0$  on  $K$ ,  $1 \leq j \leq n$ . We know of no counterexample.

Theorem 1.2 implies an approximation theorem of the Keldysh-Mergelyan type if, in addition to the above hypotheses,  $K$  has the "segment property", i.e., if there is an open cover  $\{U_i\}$  of  $\partial K$  and corresponding vectors  $\{w_i\}$  such that for  $0 < t < 1$   $z + tw_i$  lies in the interior of  $K$  whenever  $z \in K \cap U_i$ . In this case every function which is continuous on  $K$  and holomorphic in the interior of  $K$  satisfies the hypotheses of Theorem 1.2 so lies in  $H(K)$ . In particular, if  $K$  is a product of smoothly bounded domains, then  $K$  has the segment property. The case  $r = 1$  when  $K$  is the closure of a strongly pseudo-convex domain in  $C^n$  has been treated by Lieb [8] and Kerzman [7]. We use their method to prove Theorem 1.2.

If we consider Theorem 1.3 in the case  $r = 1$  we conclude that each annihilating measure for  $H(K)$  where  $K$  is the closure of a strongly pseudo-convex domain is the  $\bar{z}$ -divergence of an  $n$ -tuple of measures supported on  $K$  (distributions of order 0). This implies the following localization theorem for annihilating measures which is well-known in case  $n = 1$  [2, Lemma 3.2.11] and which is a sort of dual version of Theorem 1.2, which it clearly implies.

**THEOREM 1.4.** *Let  $K$  be the closure of a strongly pseudoconvex domain in  $C^n$ . Let  $\mu$  be an annihilating measure for  $H(K)$ . If  $\{U_i\}$  is a finite open covering of  $K$  there exist annihilating measures*

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<sup>1</sup> The referee has pointed out that Kallin, in an unpublished remark and without knowledge of Chirka's paper, observed that her counterexample could be made to yield this additional property.

$\mu_i$  for  $H(K \cap \bar{U}_i)$  (in particular each  $\mu_i$  is supported in  $U_i$ ) such that  $\mu = \sum \mu_i$ .

All of our results are derived from an estimate (Theorem 2.2 below) for Cauchy-Riemann operator in certain product domains. This theorem is a generalization of a theorem of Andreotti and Stoll [1] for the case of a polycylinder. Our proof, like theirs, follows the induction procedure used in the proof of the familiar Dolbeaut-Grothendieck lemma, but we make essential use of the representation theorem of Grauert and Lieb [5] for bounded solutions of the Cauchy-Riemann equations in strongly pseudo-convex domains.

## 2. The basic estimate.

**DEFINITION 2.1.** *An open set  $G$  in  $C^n$  is called admissible if (a)  $n = 1$  and  $G$  is a bounded open set or (b)  $n > 1$  and  $G$  is a strongly pseudo-convex domain with  $C^\infty$  boundary.*

The following theorem is due to Grauert and Lieb [5] in the case  $n > 1$ , and is simply a restatement of known properties of the Cauchy kernel when  $n = 1$ .

**THEOREM 2.1.** *Let  $G$  be an admissible open set in  $C^n$ . Then there exists a differential form  $\Omega(\zeta, z)$  of type  $(n, n-1)$  in  $\zeta$ , of type  $(0, 0)$  in  $z$ , defined in a neighborhood of  $\bar{G} \times \bar{G}$  such that*

- (i)  $\Omega$  is of class  $C^\infty$  off the diagonal of  $G \times G$ ;
- (ii) there is a neighborhood of  $\partial G \times G$  in  $\bar{G} \times G$  in which  $\bar{\partial}_z \Omega = 0$ ;
- (iii) if  $g$  is a bounded  $(0, 1)$  form of class  $C^\infty$  such that  $\bar{\partial} g = 0$  on  $G$ , and if

$$f(z) = - \int_G g(\zeta) \wedge \Omega(\zeta, z)$$

then  $f \in C^\infty(G)$  and  $\bar{\partial} f = g$  in  $G$ ;

- (iv) there is a constant  $\Delta(G)$ , independent of  $z$  such that

$$\int_G |a(\zeta, z)| dm(\zeta) \leq \frac{\Delta(G)}{n}$$

where  $a(\zeta, z)$  is any coefficient of  $\Omega$  and  $dm$  is Lebesgue measure on  $G$ ;

- (v)  $\bar{G}$  has a sequence  $\{G_\nu\}$  of admissible neighborhoods whose intersection is  $\bar{G}$  such that  $\{\Delta(G_\nu)\}$  is a constant sequence.

Also,  $G$  is the union of an increasing sequence of admissible open sets  $\{G_\nu\}$  for which  $\{\Delta(G_\nu)\}$  is a constant sequence.

If  $G$  is an open set in  $C^n$  we denote by  $BC^\infty(G)$  the space of functions on  $G$  whose derivatives of all orders are bounded and continuous on  $G$ . We will need the following corollary of Theorem 2.1.

**COROLLARY.** *Let  $G$  be an admissible open set in  $C^n$ . Let  $U_k$  be open in  $C^{n_k}$ ,  $k = 1, 2$ . Suppose that*

$$g_1(z, x, w), \dots, g_n(z, x, w) \in BC^\infty(G \times U_1 \times U_2)$$

*and that  $g = \sum g_j d\bar{z}_j$  satisfies  $\bar{\partial}_z g = 0$  in  $G \times U_1 \times U_2$ . Then there exists  $f \in BC^\infty(G \times U_1 \times U_2)$  such that*

- (i)  $\bar{\partial}_z f = g$  in  $G \times U_1 \times U_2$ ;
- (ii)  $\|f\| \leq \Delta(G) \max_{1 \leq j \leq n} \|g_j\|$ ;
- (iii) *if  $D$  is a differential operator on  $C^{n_1} \times C^{n_2}$  with constant coefficients, then*

$$\|Df\| \leq \Delta(G) \max_{1 \leq j \leq n} \|Dg_j\|.$$

*(In particular, if each  $g_j$  is holomorphic in  $U_2$  for fixed  $(z, x) \in G \times U_1$ , then  $f$  has the same property.)*

Here  $\|f\| = \sup_{G \times U_1 \times U_2} |f|$ .

*Proof.* Let  $f(z, x, w) = - \int_G g(\zeta, x, w) \wedge \Omega(\zeta, z)$ , where  $\Omega$  is as in Theorem 2.1. Since all the derivatives of  $g$  are bounded we may differentiate under the integral sign as often as we wish. The corollary then follows immediately from Theorem 2.1.

Let  $G = G_1 \times \dots \times G_r$  be an open set in  $C^n$  where each  $G_i$  is an open set in  $C^{n_i}$ . If  $f$  is a function on  $G$  and  $g = \sum g_j d\bar{z}_j$  is a  $(0, 1)$  form on  $G$  we will use the following notation:

$$\begin{aligned} \|f\|_G &= \sup_G |f| \\ \|f\|_G^{(k,t)} &= \max_{\alpha \in A_{k,t}} \|\partial^\alpha f / \partial \bar{z}^\alpha\|_G \end{aligned}$$

(where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is an  $n$ -tuple of nonnegative integers,  $|\alpha| = \sum \alpha_j$ ,

$$\partial^\alpha f / \partial \bar{z}^\alpha = \partial^{|\alpha|} f / \partial \bar{z}_1^{\alpha_1} \dots \partial \bar{z}_n^{\alpha_n},$$

and

$$\begin{aligned} A_{k,t} &= \{\alpha = (\alpha_1, \dots, \alpha_n) : |\alpha| \leq k \text{ and } \alpha_j = 0 \\ &\quad \text{if } j > n_1 + \dots + n_t\} \end{aligned}$$

$$\begin{aligned} \|f\|_G^{(k)} &= \|f\|_G^{(k,r)} \\ \|g\|_G^{(k,t)} &= \max_{1 \leq j \leq n} \|g_j\|_G^{(k,t)} \\ \|g\|_G^{(k)} &= \|g\|_G^{(k,r)}. \end{aligned}$$

We can now state our basic result.

**THEOREM 2.2.** *Let  $G = G_1 \times \cdots \times G_r$  be an open set in  $C^n$  where each  $G_i$  is an admissible open set in  $C^{n_i}$ . Let  $g$  be a  $C^\infty(0, 1)$  form in  $G$  such that  $\bar{\partial}g = 0$  in  $G$  and  $\|g\|_G^{(r-1)} < \infty$ . Then there exists  $f \in C^\infty(G)$  such that*

- (i)  $\bar{\partial}f = g$  in  $G$
- (ii)  $\|f\|_G \leq (3\Delta)^r \|g\|_G^{(r-1)}$ .

Here  $\Delta$  is any number  $\geq 1$  such that  $\Delta \geq \Delta(G_i)$  as defined in Theorem 2.1.

*Proof.* We first prove the theorem in the case when each coefficient of  $g$  is in  $BC^\infty(G)$ .

If  $g$  is a  $(0, 1)$  form on  $G$ , then  $g = \sum g^i$  where each  $g^i$  is a  $(0, 1)$  form on  $G_i$  with coefficients depending only on the other variables as parameters. For each  $k, 1 \leq k \leq r$  we consider the following Assertion  $k$ :

Let  $G = G_1 \times \cdots \times G_r$  be as above. Let  $g$  be a  $(0, 1)$  form on  $G$  whose coefficients lie in  $BC^\infty(G)$  and such that  $\bar{\partial}g = 0$  in  $G$ . Suppose that

$$g = \sum_{i=1}^k g^i$$

where each  $g^i$  is a  $(0, 1)$  form on  $G_i$ , (with coefficients depending also on the other variables). Then there exists  $f \in BC^\infty(G)$  such that

- (i)  $\bar{\partial}f = g$
- (ii)  $\|f\|_G \leq (3\Delta)^k \|g\|^{(k-1, k-1)}$ .

We shall prove Assertion 1 and then show that for  $k = 1, 2, \dots, r-1$ , Assertion  $k$  implies Assertion  $k+1$ . (Of course Assertion  $r$  implies Theorem 2.2 in the case  $g$  is of class  $BC^\infty$ .)

If  $g$  satisfies the hypotheses of Assertion 1, then  $g$  is a  $\bar{\partial}$ -closed  $(0, 1)$  form in  $G_1$  whose coefficients are holomorphic functions in  $G_2 \times \cdots \times G_r$  for fixed  $z$  in  $G_1$ . Assertion 1 thus follows directly from the Corollary to Theorem 2.1.

Suppose now that Assertion  $k$  is true and that  $g$  satisfies the hypotheses of Assertion  $(k+1)$ . Then

$$g = \sum_{i=1}^{k+1} g^i.$$

Notice that  $\bar{\partial}g = 0$  implies  $\bar{\partial}_{k+1}g^{k+1} = 0$  (where  $\bar{\partial}_{k+1}$  differentiates only with respect to the variables from  $G_{k+1}$ ) and that the coefficients of  $g$  are holomorphic in  $G_{k+2} \times \cdots \times G_r$ . Applying the corollary to Theorem 2.1 we conclude the existence of  $u \in BC^\infty(G)$  such that  $\bar{\partial}_{k+1}u = g^{k+1}$  and such that  $u$  is holomorphic in  $G_{k+2} \times \cdots \times G_r$ . Let  $s = g - \bar{\partial}u$ .

Then  $\bar{\partial}s = \bar{\partial}g = 0$  and  $s$  is clearly a sum of  $(0, 1)$ -forms involving only differentials in the variables from  $G_1, \dots, G_k$ . By Assertion  $k$  there exists  $t \in BC^\infty(G)$  such that  $\bar{\partial}t = s$ .

Let  $f = u + t$ . Then  $\bar{\partial}f = g$ . Also,

$$\begin{aligned} \|f\| &\leq \|u\| + \|t\| \\ &\leq \mathcal{A}\|g\| + (3\mathcal{A})^k \|s\|^{(k-1, k-1)}. \end{aligned}$$

But

$$\begin{aligned} \|s\|^{(k-1, k-1)} &\leq \left\| \sum_{i=1}^k g^i \right\|^{(k-1, k-1)} \\ &\quad + \max_{1 \leq j \leq n_k} \|\partial u / \partial \bar{z}_j\|^{(k-1, k-1)} \\ &\leq \|g\|^{(k-1, k-1)} + \|u\|^{(k, k)} \\ &\leq \|g\|^{(k-1, k-1)} + \mathcal{A}\|g\|^{(k, k)} \\ &\leq (2\mathcal{A})\|g\|^{(k, k)}. \end{aligned}$$

Hence

$$\begin{aligned} \|f\| &\leq (\mathcal{A} + (2\mathcal{A})(3\mathcal{A})^k)\|g\|^{(k, k)} \\ &\leq (3\mathcal{A})^{k+1}\|g\|^{(k, k)}. \end{aligned}$$

This concludes the proof in the case when  $g$  is of class  $BC^\infty$ .

Suppose now that  $g$  satisfies the hypotheses of Theorem 2.2. By Theorem 2.1 and what has been proved so far, we can find a sequence of open sets  $\{G_\nu\}$  and a constant  $C$  independent of  $\nu$  such that

- (i)  $\bar{G}_\nu \subset G_{\nu+1} \subset G$
- (ii)  $G = \bigcup G_\nu$
- (iii) there exists  $f_\nu \in BC^\infty(G_\nu)$

such that  $\bar{\partial}f_\nu = g$  on  $G_\nu$  and

$$\|f_\nu\|_{G_\nu} \leq C\|g\|_G^{(r-1)}.$$

For each  $\mu$  let  $S_\mu = \{f_\nu | G_\mu: \nu \geq \mu\}$ . Since  $f_\nu - f_\mu$  is holomorphic on  $G_\mu$  for each  $\nu$ , and  $\{f_\nu - f_\mu | G_\mu\}$  is uniformly bounded,  $S_\mu$  is relatively compact by Montel's theorem. Thus we may choose, for each  $\mu$ , a subsequence  $\{f_{\nu, \mu}\}$  of  $S_\mu$  which converges uniformly on  $G_\mu$ , such that  $\{f_{\nu, \mu+1}\}$  is a subsequence of  $\{f_{\nu, \mu}\}$ . Then the diagonal sequence  $\{f_{\mu, \mu}\}$  converges uniformly on each  $G_\mu$  to a continuous function  $f$  defined on all of  $G$ . But  $f$  is in fact in  $C^\infty(G)$  since, if  $\nu > \mu$ ,  $f_{\nu\nu} - f_{\mu\mu}$  is holomorphic on  $G_\mu$  and  $f_{\nu\nu} - f_{\mu\mu} \rightarrow f - f_{\mu\mu}$  on  $G_\mu$ . Thus  $f - f_{\mu\mu}$  is holomorphic, hence in  $C^\infty(G_\mu)$  so  $f \in C^\infty(G_\mu)$  for each  $\mu$ . This also shows that  $\bar{\partial}f = g$  on  $G$ . Finally, if  $z \in G$  then there exists  $\mu$  such that  $z \in G_\nu$  for  $\nu \geq \mu$ . This means that

$$|f_{\mu\mu}(z)| \leq C\|g\|_G^{(r-1)}$$

but  $f(z) = \lim f_{\mu}(z)$  so

$$\|f\|_G \leq C \|g\|_G^{(r-1)}.$$

### 3. Proofs of Theorems 1.1 and 1.2.

*Proof of Theorem 1.1.* We may suppose that  $f$  has compact support in  $U$ . Choose  $\phi \in C^\infty(C^n)$  such that  $\phi \geq 0$ ,  $\int \phi = 1$  and  $\phi = 0$  outside the closed unit ball. For each  $\delta > 0$  define  $f_\delta$  by

$$f_\delta(z) = \int f(z - \delta w) \phi(w) dw.$$

Then  $f_\delta \in C^\infty(C^n)$ ,  $f_\delta \rightarrow f$  uniformly as  $\delta \rightarrow 0$  and for each  $\alpha$ ,

$$(\partial^\alpha f_\delta / \partial \bar{z}^\alpha)(z) = \int (\partial^\alpha f_\delta / \partial \bar{z}^\alpha)(z - \delta w) \phi(w) dw$$

so if  $G$  is an open set in  $C^n$ ,

$$\|f_\delta\|_G^{(s)} \leq \|f\|_{G^\delta}^{(s)} \quad s = 1, 2, \dots$$

where  $G^\delta = \{z - \delta w: z \in G, |w| \leq 1\}$ .

For each  $i$ ,  $1 \leq i \leq r$  we can find a sequence  $\{G_i^\nu\}$  of admissible neighborhoods such that  $K_i = \bigcap G_i^\nu$  and such that  $\{\mathcal{A}(G_i^\nu)\}$  is a constant sequence. Let us denote the constant by  $\mathcal{A}_i$ . Choose  $\mathcal{A} \geq 1$  such that  $\mathcal{A} \geq \mathcal{A}_i$  for  $1 \leq i \leq r$ .

Let  $\varepsilon > 0$  be given. Choose  $\delta_0$  such that  $\|f - f_\delta\|_K < \varepsilon/2$  if  $\delta < \delta_0$ . Choose  $\nu$  such that if  $G = G_1^\nu \times \dots \times G_r^\nu$ , then

$$\|f\|_G^{(r)} < (3\mathcal{A})^{-r} \cdot \frac{\varepsilon}{4}.$$

Then there exists  $\delta < \delta_0$  such that

$$\|f\|_{G^\delta}^{(r)} < (3\mathcal{A})^{-r} \frac{\varepsilon}{2}.$$

By Theorem 2.2 we can choose  $u \in C^\infty(G)$  such that  $\bar{\partial}u = \bar{\partial}f_\delta$  and

$$\|u\|_G \leq (3\mathcal{A})^r \|\bar{\partial}f_\delta\|_G^{(r-1)}.$$

Then  $h = f_\delta - u$  is holomorphic in a neighborhood of  $K$  and

$$\begin{aligned} \|f - h\|_K &\leq \|f - f_\delta\|_K + \|f_\delta - h\|_K \\ &< \varepsilon/2 + \|u\|_K \\ &< \varepsilon/2 + (3\mathcal{A})^r \|f_\delta\|_G^r \\ &< \varepsilon/2 + (3\mathcal{A})^r \|f\|_{G^\delta}^r \\ &< \varepsilon. \end{aligned}$$



*Proof of Theorem 1.2.* (Here we follow Lieb [8].) Since  $K$  is compact we can choose finitely many neighborhoods  $U_{x_1}, \dots, U_{x_m}$  which cover  $K$ . Let  $U = U_{x_1} \cup \dots \cup U_{x_m}$ . Choose sequences  $\{G_i^\nu\}$  of admissible neighborhoods of  $K_i$ ,  $1 \leq i \leq r$  as in the proof of Theorem 1.1 and let  $\mathcal{A}$  be as above.

Let  $\varepsilon > 0$  be given. Choose  $\nu$  such that  $G^\nu = G_1^\nu \times \dots \times G_r^\nu$  lies in  $U$ , and such that there exist holomorphic functions  $h_j$  on  $U_{x_j} \cap G^\nu$  with  $\|f - h_j\|_{U_{x_j} \cap K} < \varepsilon$ . Notice that  $|h_i - h_j| \leq |f - h_i| + |f - h_j| < 2\varepsilon$  on  $U_{x_i} \cap U_{x_j} \cap K$ . By choosing  $\nu$  larger if necessary we may suppose that  $|h_i - h_j| < 4\varepsilon$  on  $U_{x_i} \cap U_{x_j} \cap G^\nu$ .

Choose a  $C^\infty$  partition of unity  $\phi_1, \dots, \phi_m$  subordinate to the cover  $\{U_{x_i}\}$ . Let  $g_k = \sum_i \phi_i(h_k - h_i)$ . Notice that  $\|g_k\|_{U_{x_k} \cap K} < 2\varepsilon$ . Also  $g_j - g_k = h_j - h_k$  and  $\bar{\partial}g_j - \bar{\partial}g_k = 0$  on  $U_{x_j} \cap U_{x_k} \cap G^\nu$ . Thus  $\{\bar{\partial}g_j\}$  defines a  $(0, 1)$  form  $g$  on  $G^\nu$  which satisfies  $\bar{\partial}g = 0$  so by Theorem 2.2 we can find  $u \in C^\infty(G^\nu)$  that  $\bar{\partial}u = g$  on  $G^\nu$  and

$$\|u\|_{G^\nu} \leq (3\mathcal{A})^r \|g\|_{G^\nu}^{(r-1)}.$$

But

$$\|g\|_{G^\nu}^{(r-1)} \leq \max_{1 \leq k \leq n} \|g_k\|_{G^\nu \cap U_{x_k}}^{(r)}$$

and

$$\partial^\alpha g_k / \partial \bar{z}^\alpha = \sum_i (\partial^\alpha \phi_i / \partial \bar{z}^\alpha)(h_k - h_i).$$

Hence  $\|g\|_{G^\nu}^{(r-1)} \leq 4\varepsilon C$  where  $C$  is a constant depending only on the partition of unity  $\{\phi_i\}$ .

Let  $b_j = g_j - u$ . Then  $\bar{\partial}b_j = 0$  on  $U_{x_j} \cap G^\nu$  and  $b_j - b_k = g_j - g_k = h_j - h_k$ . Thus  $\{h_j - b_j\}$  defines a holomorphic function  $h$  on  $G^\nu$  and

$$\begin{aligned} \|f - h\|_K &\leq \max_{1 \leq j \leq m} (\|f - h_j\|_K + \|g_j\|_K) + \|u\|_K \\ &\leq \varepsilon + 2\varepsilon + 4C(3\mathcal{A})^r \varepsilon, \end{aligned}$$

where  $C$  and  $\mathcal{A}$  are independent of  $\varepsilon$ . Since  $\varepsilon$  was arbitrary, this completes the proof.

**4. Annihilating Measures for  $H(K)$ .** If  $G$  is an open set in  $C^n$  we denote by  $A^{(0,1)}(G)$  the space of differential forms of type  $(0, 1)$  on  $G$  of class  $C^\infty$ . We topologize  $A^{(0,1)}(G)$  as the direct sum of  $n$  copies of the Frechet space  $C^\infty(G)$  with the topology of uniform convergence on compact subsets of  $G$  of derivatives of all orders. The dual space of  $C^\infty(G)$  is the space of distributions on  $C^n$  whose support is a compact subset of  $G$ . We will identify the dual space of  $A^{(0,1)}(G)$  with the space of distributions with compact support in  $G$ .

LEMMA 4.1. *Let  $G$  be a domain of holomorphy in  $C^n$ ,  $K$  a compact subset of  $G$ , and  $\mu$  an annihilating measure for  $H(K)$ . Then there exist distributions  $\lambda_1, \dots, \lambda_n$  with compact support in  $G$  such that  $\mu = -\sum \partial\lambda_j/\partial\bar{z}_j$ .*

*Proof.*  $C^\infty(G)$  and  $A^{(0,1)}(G)$  are Frechet spaces. Since  $G$  is a domain of holomorphy,  $\bar{\partial}$  maps  $C^\infty(G)$  onto the closed subspace of  $(0, 1)$  forms  $g$  satisfying  $\bar{\partial}g = 0$ . The measure  $\mu$ , considered as a continuous linear functional on  $C^\infty(G)$ , annihilates the kernel of  $\bar{\partial}$ , so by a theorem of Dieudonné and Schwartz [4],  $\mu$  is in the image of the adjoint,  $\bar{\partial}^*$ , of  $\bar{\partial}$ . But if  $\lambda = (\lambda_1, \dots, \lambda_n)$  is an  $n$ -tuple of distributions with compact support in  $G$  and  $f \in C^\infty(G)$ , then

$$\begin{aligned} (\bar{\partial}^*\lambda)(f) &= \lambda(\bar{\partial}f) = \sum \lambda_j(\partial f/\partial\bar{z}_j) \\ &= -\sum (\partial\lambda_j/\partial\bar{z}_j)(f). \end{aligned}$$

Thus for some  $\lambda_1, \dots, \lambda_n$  we have  $\mu = -\sum \partial\lambda_j/\partial\bar{z}_j$ .

We will also consider, for a bounded open set  $G$  in  $C^n$ , the Banach space  $C^r(\bar{G})$  of continuous functions on  $\bar{G}$  whose derivatives of order  $\leq r$  are bounded and continuous on  $G$ , with the norm

$$\|f\|_G^r = \max_{|\alpha| \leq r} \|D^\alpha f\|_G$$

where  $\alpha$  is a  $2n$ -tuple of non-negative integers and

$$D^\alpha = (\partial/\partial z_1)^{\alpha_1} \dots (\partial/\partial z_n)^{\alpha_n} (\partial/\partial \bar{z}_1)^{\alpha_{n+1}} \dots (\partial/\partial \bar{z}_n)^{\alpha_{2n}}.$$

A continuous linear functional on  $C^r(\bar{G})$  is easily seen to define a distribution on  $C^n$  with support in  $\bar{G}$ . In addition, we will denote by  $B_r^{(0,1)}(\bar{G})$  the space of  $(0, 1)$  forms on  $G$  with coefficients in  $C^r(\bar{G})$ , topologized as the direct sum of  $n$  copies of  $C^r(\bar{G})$  with the norm

$$\|\sum g_j d\bar{z}_j\|_G^r = \max_{1 \leq j \leq n} \|g_j\|_G^r.$$

If  $G_1 \subset G_2$  are two bounded open sets in  $C^n$  let  $R: B_r^{(0,1)}(G_2) \rightarrow B_r^{(0,1)}(G_1)$  be the operator which restricts forms in  $G_2$  to  $G_1$ . Then  $R^*$  is a norm-decreasing embedding of the dual space of  $B_r^{(0,1)}(G_1)$  into  $B_r^{(0,1)}(G_2)^*$  since, if  $\lambda \in B_r^{(0,1)}(G_1)^*$ , and  $g = \sum g_j d\bar{z}_j$  is in  $B_r^{(0,1)}(G_2)$ ,

$$\begin{aligned} |(R^*\lambda)(g)| &= |\lambda(Rg)| \leq \|\lambda\| \|Rg\|_{G_1}^r \\ &\leq \|\lambda\| \|g\|_{G_2}^r, \end{aligned}$$

so  $\|R^*\lambda\| \leq \|\lambda\|$ .

With these preliminaries we can proceed to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Let  $\{G_\nu\}$  be a sequence of bounded open

neighborhoods of  $K$  such that

- (i)  $\bar{G}_\nu \subset G_{\nu-1}$
- (ii)  $K = \cap G_\nu$
- (iii) there exists a constant  $C$  independent of  $\nu$  such that if  $g \in A^{(0,1)}(G_\nu)$ ,  $\bar{\partial}g = 0$  in  $G_\nu$ , and  $\|g\|_{G_\nu^{(r-1)}} < \infty$ , then there exists  $f \in C^\infty(G_\nu)$  such that  $\bar{\partial}f = g$  and

$$\|f\|_{G_\nu} \leq C \|g\|_{G_\nu^{(r-1)}}.$$

If  $\mu$  is an annihilating measure for  $H(K)$  we can apply Lemma 4.1 to obtain, for each  $\nu$ , an  $n$ -tuple  $\lambda^\nu = (\lambda_1^\nu, \dots, \lambda_n^\nu)$  of distributions with compact support in  $G_\nu$  such that  $\mu = -\sum \partial\lambda_j^\nu/\partial\bar{z}_j$ . Let  $W_\nu$  be the subspace of  $C^{r-1}(\bar{G}_\nu)$  consisting of restrictions to  $G_\nu$  of  $C^\infty$  functions on  $C^n$ . If  $f \in W_\nu$  we can find  $h \in C^\infty(G_\nu)$  such that  $f - h$  is holomorphic on  $G_\nu$  and  $\|h\|_{G_\nu} \leq C \|\bar{\partial}f\|_{G_\nu^{(r-1)}}$  where  $C$  is the constant in (iii) above. Thus

$$\begin{aligned} |\lambda^\nu(\bar{\partial}f)| &= \left| \int f d\mu \right| = \left| \int h d\mu \right| \\ &\leq \|\mu\| \|h\|_K \leq C \|\mu\| \|\bar{\partial}f\|_{G_\nu^{(r-1)}} \end{aligned}$$

where  $\|\mu\|$  is the total variation of  $\mu$ . This means that  $\lambda^\nu$  defines a continuous linear functional on the subspace  $\bar{\partial}W_\nu$  of  $B_{r-1}^{(0,1)}(G_\nu)$  of norm  $\leq C\|\mu\|$ . By the Hahn-Banach theorem there is an  $n$ -tuple, which we will continue to denote by  $\lambda^\nu$ , of continuous linear functionals on  $C^{r-1}(\bar{G}_\nu)$  such that

- (a)  $\lambda^\nu(\bar{\partial}f) = \int f d\mu$  for all  $f \in C^\infty(C^n)$
- (b)  $\|\lambda^\nu\| \leq C\|\mu\|$ .

Now, by composing with the adjoint of the appropriate restriction operator we may consider each  $\lambda^\nu$  so obtained as a continuous linear functional on  $B_{r-1}^{(0,1)}(G_1)$ . Then the sequence  $\{\lambda^\nu\}$  constitutes a bounded sequence of elements in the dual space of a separable Banach space. Consequently, there is a subsequence  $\{\lambda^{\nu'}\}$  and an  $n$ -tuple  $\lambda = (\lambda_1, \dots, \lambda_n)$  of continuous linear functionals on  $C^{r-1}(\bar{G}_1)$  such that  $\lambda^{\nu'} \rightarrow \lambda$  in the weak star topology. Since each  $\lambda_j^{\nu'}$  is supported, as a distribution, on  $\bar{G}_\nu$ , and since  $K = \cap G_\nu$ , it follows that the support of each  $\lambda_j$ , as a distribution, lies in  $K$ . Moreover, if  $f$  is a  $C^\infty$  function on  $C^n$  then

$$\lambda(\bar{\partial}f) = \lim \lambda^{\nu'}(\bar{\partial}f) = \int f d\mu,$$

i.e.,  $\mu = -\sum \partial\lambda_j/\partial\bar{z}_j$ . Finally, it is clear that each  $\lambda_j$  is of order  $\leq r-1$ .

Conversely, suppose  $\mu$  is a measure on  $K$ , and  $\mu = \sum \partial\lambda_j/\partial\bar{z}_j$ , where  $\lambda_1, \dots, \lambda_n$  are distributions with compact support on  $K$ . If  $f$  is holo-

morphic in a neighborhood of  $K$ , then  $\partial f/\partial \bar{z}_j$ ,  $1 \leq j \leq n$ , are identically 0 in a neighborhood of  $K$  so

$$\begin{aligned} \int f d\mu &= \sum (\partial \lambda_j / (\partial \bar{z}_j))(f) \\ &= - \sum \lambda_j (\partial f / \partial \bar{z}_j) \\ &= 0 \end{aligned}$$

since the  $\lambda_j$  are supported on  $K$ .

*Proof of Theorem 1.4.* Let  $\{\phi_i\}$  be a  $C^\infty$  partition of unity subordinate to the open covering  $\{U_i\}$ , i.e., suppose  $\phi_i \in C^\infty(U_i)$ ,  $\phi_i$  has compact support,  $0 \leq \phi_i \leq 1$ , and  $\sum \phi_i = 1$  on a neighborhood of  $K$ . If  $\mu$  is an annihilating measure for  $H(K)$ , then by Theorem 1.3 there exist measures  $\lambda_1, \dots, \lambda_n$  on  $K$  such that

$$\begin{aligned} \mu &= - \sum \partial \lambda_j / \partial \bar{z}_j = - \sum_j \partial \left( \sum_i \phi_i \lambda_j \right) / \partial \bar{z}_j \\ &= - \sum_j \sum_i \phi_i (\partial \lambda_j / \partial \bar{z}_j) - \sum_j \sum_i (\partial \phi_i / \partial \bar{z}_j) \lambda_j \\ &= \sum_i \left\{ \phi_i \mu - \sum_j (\partial \phi_i / \partial \bar{z}_j) \lambda_j \right\} \\ &\equiv \sum_i \mu_i \end{aligned}$$

where each  $\mu_i$  is a measure compactly supported on  $U_i \cap K$ . If  $h$  is holomorphic on  $U_i \cap K$ , then

$$\partial(h\phi_i)/\partial \bar{z}_j = h(\partial \phi_i / \partial \bar{z}_j)$$

for  $1 \leq j \leq n$ . Thus

$$\begin{aligned} \int h d\mu_i &= \int h \phi_i d\mu - \sum_j \int h (\partial \phi_i / \partial \bar{z}_j) d\lambda_j \\ &= \int h \phi_i d\mu - \sum_j \int (\partial / \partial \bar{z}_j)(\phi_i h) d\lambda_j \\ &= 0. \end{aligned}$$

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