

Pacific Journal of Mathematics

FIBER INTEGRATION IN SMOOTH BUNDLES

JAN WILLIAM AUER

FIBER INTEGRATION IN SMOOTH BUNDLES

J. W. AUER

The purpose of this paper is to comment on the operations of fiber integration, or integration over the fiber, which arise in the study of the cohomology of bundles.

Let $\xi = (E, \Pi_E, B, F, G)$ be a smooth (C^∞) bundle, where as usual E is the total space, B the base, F the (connected) fiber, G the group, and $\Pi_E: E \rightarrow B$ the projection. Assume $H^k(F)$ to be finite dimensional for all k . A form ω on the total space is said to have fiber-compact support if, and only if, for all $x \in B$, there is a neighbourhood U_x of x , a trivialization $\phi: U_x \times F \cong \pi_E^{-1}(U_x)$, and a compact set $K \subset F$ such that $(\text{support } \phi^*\omega) \cap (U_x \times F) \subset U_x \times K$. Denote these k -forms by $A_F^k(E)$, and their de Rham cohomology by $H_F^k(E)$. When F is compact $A_F^k(E) = A^k(E)$, the algebra of all k -forms on E ; if B is compact, $A_F(E)$ is the algebra $A_c(E)$ of forms on E with compact support. Now integration over fiber has been defined by various authors as a linear map

$$\Psi: H^k(E) \longrightarrow H^{k-m}(B; H^m(F)), \quad k \geq m,$$

where m is the dimension of F . These definitions are essentially algebraic in nature; for example Ψ has been defined by a spectral sequence. Using this idea when ξ is orientable, a linear map

$$\Psi_1: H_F^k(E) \longrightarrow H^{k-m}(B)$$

is defined, and called algebraic fiber integration on account of the origin of the definition.

On the other hand, when ξ is orientable, there is the geometrically defined linear map $\int_F: A_F^k(E) \rightarrow A^{k-m}(B)$ given roughly speaking by

$$\left(\int_F \omega \right) (x) = \int_{F_x} \omega / F_x,$$

where $\omega \in A_F^k(E)$, $x \in B$, and F_x denotes the fiber of ξ over x . The induced map Ψ_2 of cohomology is called geometric fiber integration:

$\Psi_2: H_F^k(E) \rightarrow H^{k-m}(B)$. The main purpose of the paper is to show

THEOREM. $\Psi_1 = \Psi_2$.

1. The spectral sequence in $A_F(E)$.

(a) In this section we obtain results analogous to a theorem of Borel [6], which we use to obtain an expression for \mathcal{P}_1 .

The action of G on F induces an action on $H^k(F)$ and on $H_c^k(F)$, $k = 0, 1, 2, \dots, m$, where $H_c^k(F)$ denotes the de Rham cohomology of forms on F with compact support. Denote by h^k the total space of the bundle over B with fiber $H_c^k(F)$; because this bundle has a discrete group, the exterior derivative $\delta_B^{p,q}$ in $A_B^p(h^q)$ (the p -forms on B with coefficients in h^q) is a differential operator. Denote $\text{Ker } \delta_B^{p,q} / \text{Im } \delta_B^{p-1,q}$ by $H^p(B; h^q)$.

(b) We filter $A_F^k(E)$ following Hattori [5] and Borel [6]: when $\{x_i\}$, $\{y_j\}$ are coordinates on neighbourhoods in B and F respectively, $i = 1, 2, \dots, n - m$, $j = 1, 2, \dots, m$, where $n = \dim E$, we use the same symbols to denote coordinates induced on sufficiently small open sets in E ; then according to Hattori and Borel, I^p consists of those forms which involve at least p base differentials dx_i .

It will be convenient to define this filtration by a bigradation of $A_F(E)$. For this purpose, it is necessary to assume that a fixed connection ([7], p. 63) has been prescribed in ξ . Of course, by the remarks above, the filtration will be independent of the connection. If X is a C^∞ vector field on E , then by definition its horizontal and vertical parts, HX and VX respectively, induced by the connection, are again C^∞ vector fields. Whenever $\{x_i\}$, $\{y_j\}$ are coordinates on an open set $W \subset E$, as described above, we denote by dy_j^v , $j = 1, 2, \dots, m$, the 1-forms on W defined by $dy_j^v(x) = dy_j(VX)$, where X is a vector field on W . Then $\{dx_i\} \cup \{dy_j^v\}$ generate $A^1(W)$ over $C^\infty(W)$.

DEFINITION 1. $C^{p,q} \equiv C^{p,q}(E) = \{\omega \in A_F^{p+q}(E) \mid i(X_1) \cdots i(X_{p+1})\omega = i(Y_1) \cdots i(Y_{q+1})\omega = 0 \text{ for all horizontal vector fields } X_i \text{ and vertical vector fields } Y_j \text{ on } E\}$.

Here, as usual, $i(Z)$ is the substitution operator with respect to the vector field Z on E :

$$i(Z)\omega(Z_1, \dots, Z_{p+q-1}) = \omega(Z, Z_1, \dots, Z_{p+q-1})$$

when Z_1, \dots, Z_{p+q-1} are vector fields on E , $\omega \in A^{p+q}(E)$. Thus $i(Z): A_F^{p+q}(E) \rightarrow A_F^{p+q-1}(E)$, because $A_F(E)$ is clearly stable under $i(Z)$.

Now we let Greek letters α, β , etc., represent sequences of positive integers of the form (i_1, \dots, i_p) , for some positive integer p , with $i_1 < i_2 < \dots < i_p$. Then dx_α will denote $dx_{i_1} \wedge \dots \wedge dx_{i_p}$, and so on. We put $|\alpha| = p$ when $\alpha = (i_1, \dots, i_p)$. Then if W is a coordinate neighbourhood in E , $\omega \in A^k(E)$, we may write ω/W as

$$(1) \quad \omega/W = \sum_{p+q=k} \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta^v$$

where $\omega^{\alpha\beta} \in C^\infty(W)$, the C^∞ -functions on W . Consequently any $\omega \in C^{p,q}(W)$ may be written

$$(2) \quad \omega = \sum_{\substack{|\alpha|+|\beta|=p \\ |\beta|=q}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta$$

with $\omega^{\alpha\beta} \in C^\infty(W)$.

DEFINITION 2.

$$I^{p,q} = \bigoplus_{r \geq p} C^{r,p+q-r}, \quad I^p = \bigoplus_{q \geq 0} I^{p,q}.$$

It is easily seen that

PROPOSITION 1. $\{I^p\}_{p=0}^\infty$ is a decreasing filtration of $A_F(E)$. It is the filtration associated with the gradation $\{C^p\}_{p=0}^\infty$ of $A_F(E)$, where $C^p = \bigoplus_{q \geq 0} C^{p,q}$, because $I^p = \bigoplus_{r \geq p} C^r$.

Note in particular that $I^0 = A_F(E)$, and $I^p = \{0\}$ for $p > \dim B = n - m$.

REMARK. If the above definitions are carried out for $A(E)$, one obtains the filtrations of *Hattori* [5] and *Borel* [6]. In particular, the filtration is independent of the choice of connection in ξ .

(c) Denote by $\{E_r^{p,q}\}$ the spectral sequence defined by the filtration $\{I^{p,q}\}$ of $A_F(E)$ defined above (see § 2). Then we will next show

PROPOSITION 2.

$$E_2^{p,q} \cong H^p(B; h^q), \quad p, q \geq 0.$$

Let \mathcal{A}_B^p be the sheaf of germs of p -forms on B , and \mathcal{C}_B^∞ the sheaf of germs of C^∞ -functions on B .

DEFINITION 3. The sheaf \mathcal{F}^q of fiber-compact q -forms along the fiber of ξ .

We first define the presheaf $\underline{\mathcal{F}}^q$ on B by $\underline{\mathcal{F}}^q(U) = C^{0,q}(\Pi_E^{-1}(U))$, when U is open in B . Then \mathcal{F}^q is the sheaf induced by $\underline{\mathcal{F}}^q$.

LEMMA 1. $E_0^{p,q} \cong \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^q)$ ($\Gamma(\mathcal{S})$ denotes the module of sections of any sheaf \mathcal{S} , and all tensor products are over the sheaf \mathcal{C}_B^∞).

Proof. From the definitions we have that

$$(3) \quad E_0^{p,q} = C^{p,q}$$

DEFINITION 4.

$$\Omega: \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^q) \longrightarrow C^{p,q}.$$

Let $s \in \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^q)$; then if $U \subset B$ is sufficiently small, we may write s locally as a sum of terms of the form ω , where

$$\omega = \left(\sum_{|\alpha|=p} \omega_1^\alpha dx_\alpha \right) \otimes \left(\sum_{|\beta|=q} \omega_2^\beta dy_\beta^v \right)$$

with $\omega_1^\alpha \in C^\infty(U)$, $\omega_2^\beta \in C^\infty(\Pi_E^{-1}(U))$. Put

$$\Omega(\omega) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} (\omega_1^\alpha \Pi_E) \omega_2^\beta dx_\alpha \wedge dy_\beta^v,$$

and extend linearly to define $\Omega(s)$.

LEMMA 2. Ω is well-defined, independent of the choice of coordinates.

Proof. This follows from the fact that if $\{\bar{x}_i\}$, $\{\bar{y}_j\}$ are coordinates defined on an open set in E overlapping the domain of definition of the coordinates $\{x_i\}$, $\{y_j\}$, then

$$d\bar{x}_\alpha = \sum_{|\mu|=p} \frac{\partial \bar{x}_\alpha}{\partial x_\mu} dx_\mu$$

and

$$(4) \quad d\bar{y}_\beta^v = \sum_{|\alpha|=q} \frac{\partial \bar{y}_\beta^v}{\partial y_\alpha} dy_\alpha^v$$

where $\partial \bar{x}_\alpha / \partial x_\mu$ represents the $p \times p$ sub-matrix with rows α , columns μ of the $(n-m) \times (n-m)$ matrix with entries $\partial \bar{x}_i / \partial x_j$, and analogously for $\partial \bar{y}_\beta^v / \partial y_\alpha$. Note that equation (4) is *not* the equation of transformation for the dy_β^v 's; the latter equation also involves linear combinations of the dx_α 's. Since Ω is easily seen to be an isomorphism, this completes the proof of Lemma 1.

DEFINITION 5. The homomorphism

$$\delta_F^{p,q}: \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^q) \longrightarrow \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^{q+1}).$$

We first define the presheaf homomorphism $\delta_F^q: \mathcal{F}^q \rightarrow \mathcal{F}^{q+1}$. Let $U \subset B$ be open, $\phi \in \mathcal{F}^q(U) = C^{0,q}(\Pi_E^{-1}(U))$, and Y_1, \dots, Y_{q+1} be vertical vector fields on $\Pi_E^{-1}(U)$. Then

$$\begin{aligned}
 & \delta_F^q(U)(\phi)(Y_1, \dots, Y_{q+1}) \\
 (5) \quad &= \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i-1} Y_i(\phi(Y_1, \dots, \hat{Y}_i, \dots, Y_{q+1})) \\
 &+ \frac{1}{q+1} \sum_{i < j} (-1)^{i+j} \phi([Y_i, Y_j], Y_1, \dots, \hat{Y}_i, \dots, \hat{Y}_j, \dots, Y_{q+1})
 \end{aligned}$$

where the symbol \hat{Y}_i indicates that Y_i is to be omitted. In terms of coordinates, if $\phi = \sum_{|\beta|=q} \omega^\beta dy_\beta^v$, with $\omega^\beta \in C^\infty(\Pi_E^{-1}(U))$, then

$$(6) \quad \underline{\delta}_F^q(U)(\phi) = \sum_{|\beta|=q} \sum_k \frac{\partial \omega^\beta}{\partial y_k} dy_k^v \wedge dy_\beta^v$$

Now δ_F^q is the sheaf homomorphism induced by $\underline{\delta}_F^q$, and $\delta_F^{p,q}$ is the homomorphism of modules of sections induced by $(-1)^p 1_{\mathcal{A}_B^p} \otimes \delta_F^q$.

Let $\Delta_0^{p,q}: E_0^{p,q} \rightarrow E_0^{p,q+1}$ be the map induced by the exterior derivative on E ; that is, $\Delta_0^{p,q}$ is defined by the diagram below, where $\rho_0^{p,q}$ is the canonical projection:

$$\begin{array}{ccc}
 I^{p,q} & \xrightarrow{\delta} & I^{p,q+1} \\
 \rho_0^{p,q} \downarrow & & \downarrow \rho_0^{p,q+1} \\
 E_0^{p,q} & \xrightarrow{\Delta_0^{p,q}} & E_0^{p,q+1}
 \end{array}$$

Now in view of $E_0^{p,q} = C^{p,q}$, $\Delta_0^{p,q}$ is exactly the differential operator induced in C^p by the exterior derivative on E :

$$\rho_0^{p,q+1} \delta = \Delta_0^{p,q}: C^{p,q} \longrightarrow C^{p,q+1},$$

where $\rho_0^{p,q+1}: A_F^{p+q+1}(E) \rightarrow C^{p,q+1}$ is the canonical projection induced by $A_F(E) = \bigoplus_{p,q} C^{p,q}$.

LEMMA 3. *The diagram below is commutative:*

$$\begin{array}{ccc}
 \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^q) & \xrightarrow{\Omega} & C^{p,q} = E_0^{p,q} \\
 \delta_F^{p,q} \downarrow & & \downarrow \Delta_0^{p,q} \\
 \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^{q+1}) & \xrightarrow{\Omega} & C^{p,q+1} = E_0^{p,q+1}
 \end{array}$$

Proof. Any $s \in \Gamma(\mathcal{A}_B^p \otimes \mathcal{F}^q)$ is locally expressible on $W \subset E$ as a sum of terms of the form

$$\omega = \left(\sum_{|\alpha|=p} \omega_i^\alpha dx_\alpha \right) \otimes \left(\sum_{|\beta|=q} \omega_i^\beta dy_\beta^v \right)$$

so that

$$\Omega(\omega) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} (\omega_1^\alpha \Pi_E) \omega_2^\beta dx_\alpha \wedge dy_\beta^v.$$

Now we may write on W

$$(7) \quad dy_j^v = dy_j + \sum_i \theta_j^i dx_i$$

$$(8) \quad dy_k = dy_k^v + \sum_j \zeta_k^j dx_j$$

where $\theta_j^i, \zeta_k^j \in C^\infty(W)$.

Replacing dy_β^v using equation (7) one obtains

$$\Omega(\omega) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} (\omega_1^\alpha \Pi_E) \omega_2^\beta dx_\alpha \wedge dy^\beta + \sum_{\substack{|\alpha|>p \\ \mu}} (\dots)^{\alpha\mu} dx_\alpha \wedge dy_\mu$$

(where $(\dots)^{\alpha\mu}$ indicate coefficients of terms with $|\alpha| > p$) so that using the definition of $\rho^{p,q+1}$, one obtains on account of $\partial/\partial y_k(\omega_1^\alpha \Pi_E) = 0$,

$$(9) \quad \Delta_0^{p,q} \Omega(\omega) = \rho^{p,q+1} \delta \Omega(\omega) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_k \frac{\partial \omega_2^\beta}{\partial y_k} (\omega_1^\alpha \Pi_E) dy_k^v \wedge dx_\alpha \wedge dy_\beta^v.$$

One sees immediately that this is the same as the expression for $\Omega \delta_F^{p,q}(\omega)$, proving Lemma 3.

As a consequence we have

$$E_1^{p,q} = \text{Ker } \Delta_0^{p,q} / \text{Im } \Delta_0^{p,q-1} = \text{Ker } \delta_F^{p,q} / \text{Im } \delta_F^{p,q-1}.$$

Let \mathcal{L}^q be the sheaf of germs of smooth sections of the bundle h^q . Now one can show by an argument identical to the one employed by Borel ([6], pp. 206, 207) that there is an isomorphism

$$(10) \quad \mathcal{L}^q = \text{Ker } \delta_F^q / \text{Im } \delta_F^{q-1},$$

and hence also $\text{Ker } \delta_F^{p,q} / \text{Im } \delta_F^{p,q-1} \cong A_B^p(h^q)$. Thus we have

LEMMA 4. $E_1^{p,q} \cong A_B^p(h^q)$.

Proof of Proposition 2. Because $H^p(B; h^q) = \text{Ker } \delta_B^{p,q} / \text{Im } \delta_B^{p-1,q}$ (see § 1 (a) for the definition of $\delta_B^{p,q}$) and $E_2^{p,q} \cong \text{Ker } \Delta_1^{p,q} / \text{Im } \Delta_1^{p-1,q}$ it suffices to show that the diagram (11) below is commutative:

$$(11) \quad \begin{array}{ccc} A_B^p(h^q) & \xrightarrow{\Omega^1} & E_1^{p,q} \\ \downarrow \Delta_B^{p,q} & & \downarrow \Delta_1^{p,q} \\ A_B^{p+1}(h^q) & \xrightarrow{\Omega^1} & E_1^{p+1,q}. \end{array}$$

Here Ω^1 is the isomorphism induced by virtue of Lemma 4.
 Consider the exact sequence of cochain complexes

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & C^{p+1, q-1} & \xrightarrow{i} & C^{p, q} \oplus C^{p+1, q-1} & \xrightarrow{j} & C^{p, q} \longrightarrow 0 \\
 & & \downarrow \Delta_0^{p+1, q-1} & & \downarrow \partial^{p, q} & & \downarrow \Delta_0^{p, q} \\
 0 & \longrightarrow & C^{p+1, q} & \xrightarrow{i} & C^{p, q+1} \oplus C^{p+1, q} & \xrightarrow{j} & C^{p, q+1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

where $\partial^{p, q}$ is induced by the exterior derivative $\delta: A(E) \rightarrow A(E)$ and $i(d) = (0, d)$, $d \in C^{p+1, q-1}$, $j(c, d) = c$, $c \in C^{p, q}$.

Now it is known ([3], p. 85) that the differential operator $\Delta_1^{p, q}$ is the same as the connecting homomorphism $d^{p, q}$ induced by this exact sequence:

$$d^{p, q}: \text{Ker } \Delta_0^{p, q} / \text{Im } \Delta_0^{p, q-1} \longrightarrow \text{Ker } \Delta_0^{p+1, q} / \text{Im } \Delta_0^{p+1, p-1} .$$

Consequently, if $\omega \in \text{Ker } \Delta_0^{p, q}$

$$\Delta_1^{p, q}([\omega]_{E_1^{p, q}}) = [i^{-1} \partial^{p, q} j^{-1} \omega]_{E_1^{p+1, q}} ,$$

where $[\omega]_{E_1^{p, q}}$ indicates the class of ω in $E_1^{p, q}$, and so on; that is, $\Delta_1^{p, q}([\omega]_{E_1^{p, q}})$ is $\partial^{p, q} \omega$ modulo $\text{Im } \Delta_0^{p+1, q-1}$.

Now suppose that $\omega \in A_B^p(h^q)$; then for a coordinate neighbourhood $U \subset B$, ω/U may be written as

$$\omega/U(x) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} [\omega^{\alpha\beta} dy_\beta^v]_{E_c^p(F_x)} dx_\alpha$$

where $x \in U \subset B$ and $\omega^{\alpha\beta} \in C^\infty(\Pi_E^{-1}(U))$.

A consideration of the isomorphism Ω^1 yields

$$\begin{aligned}
 \Omega^1(\omega/U) &= \sum_{\substack{|\alpha|=p \\ |\beta|=q}} [\omega^{\alpha\beta} dx_\alpha \wedge dy_\beta^v]_{E_1^{p, q}} \\
 &= \left[\sum_{\substack{|\alpha|=p \\ |\beta|=q}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta + \sum_{\substack{|\alpha|>p \\ \mu}} (\dots)^{\alpha\mu} dx_\alpha \wedge dy_\mu \right]_{E_1^{p, q}} ,
 \end{aligned}$$

where we have again replaced dy_β^v by means of equation (7).

A short computation now shows, when all quotients have been taken, and using (8), that

$$(12) \quad \Delta_1^{p,q} \Omega^1(\omega/U) = \left[\sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_k \frac{\partial \omega^{\alpha\beta}}{\partial x_k} dx_k \wedge dx_\alpha \wedge dy_\beta^v \right]_{E_1^{p+1,q}} ;$$

on the other hand,

$$\begin{aligned} \Omega^1 \delta_B^{p,q}(\omega/U) &= \Omega^1 \delta_B^{p,q} \left(\sum_{\substack{|\alpha|=p \\ |\beta|=q}} [\omega^{\alpha\beta} dy_\beta^v]_{H_c^q(F)} dx_\alpha \right) \\ &= \Omega^1 \left(\sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_k \left[\frac{\partial \omega^{\alpha\beta}}{\partial x_k} dy_\beta^v \right]_{H_c^q(F)} dx_k \wedge dx_\alpha \right) \end{aligned}$$

where $[\omega^{\alpha\beta} dy_\beta^v]_{H_c^q(F)}$ indicates the section of h^q over U defined by

$$[\omega^{\alpha\beta} dy_\beta^v]_{H_c^q(F)}(x) = [\omega^{\alpha\beta} dy_\beta^v]_{H_c^q(F_x)}, \quad x \in U$$

(recall that $(h^q)_x = H_c^q(F_x)$).

The expression above clearly yields the right hand side of equation (12), as required.

2. Fiber integration, algebraic definition. As mentioned earlier, algebraic fiber integration is defined by using the definition of Borel and Hirzebruch [1] applied to the spectral sequence $\{E_r\}$ arising from the filtration $\{I^p\}$ of fiber-compact forms on the total space of ξ .

For convenience we recall some definitions from the theory of (decreasing) spectral sequences:

$$\begin{aligned} Z_s^{p,q} &= A_F^{p,q}(E) \cap \{a \in I^p \mid \delta a \in I^{p+s}\} \\ D_s^{p,q} &= A_F^{p+q}(E) \cap I^p \cap \delta I^{p-s} \\ E_s^{p,q} &= Z_s^{p,q} / (Z_{s-1}^{p+1,q-1} \oplus D_{s-1}^{p,q}) \end{aligned}$$

where $0 \leq p, q, s \leq \infty$.

Let $\Pi: Z_\infty^0 \rightarrow H_F(E)$ be the canonical projection; then

$$(13) \quad H^{p,q} = H_F^{p+q}(E) \cap \Pi(Z_\infty^p) \text{ filter } H_F(E), \text{ and } E_\infty^{p,q} = H_\infty^{p,q} / H_\infty^{p+1,q-1}.$$

Because $\dim F = m$ and $\dim B = n - m$, it follows that $E_r^{p,q} = 0$ for $q > m$, $p \geq 0$, $r \geq 0$ and that $I^p = 0$ for $p > n - m$.

LEMMA 5. (a) $E_r^{p,m} \subset E_{r-1}^{p,m}$, $r \geq 3$, $p \geq 0$.

(b) $E_r^{k-m,q} = E_\infty^{k-m,q}$, $r > \sup(n-k, k-m)$, $k \geq m$, $q \geq 0$.

(c) $E_\infty^{k-m,m} = H_F^k(E) / H_\infty^{k-m+1,m-1}$, $k \geq m$.

As a consequence of Lemma 5 we now have an injection

$$h_1: E_\infty^{k-m,m} = E_{r_0}^{k-m,m} \subset \dots \subset E_2^{k-m,m}$$

where $r_0 = \sup(n-k, k-m) + 1$, and a projection $h_2: H_F^k(E) \rightarrow E_\infty^{k-m,m}$. Let $\chi: E_2^{k-m,m} \cong H^{k-m}(B; h^m)$ be the isomorphism induced by Ω (Proposition 2). Then the definition of Borel, Hirzebruch yields $\chi h_1 h_2$:

$H_F^k(E) \rightarrow H^{k-m}(B; h^m)$. We now define $\sigma: H^{k-m}(B; h^m) \rightarrow H^{k-m}(B)$, under the assumption that ξ is *orientable*, in the following sense (see [4]):

DEFINITION 6. ξ is orientable if, and only if, there exists an m -form ψ on E such that for all $x \in B$, $i_x^* \psi$ is an orientation on F_x , the fiber over x ; $i_x: F_x \subset E$. If such a ψ has been chosen, ξ is called oriented.

Clearly F is orientable when ξ is.

Recall that $\mathcal{L}^m = \text{Ker } \delta_F^m / \text{Im } \delta_F^{m-1} = \mathcal{F}^m / \text{Im } \delta_F^{m-1}$.

We first define a map of presheaves, $\underline{\mathcal{S}}: \mathcal{F}^m \rightarrow \mathcal{E}_B^\infty$, where \mathcal{E}_B^∞ denotes the presheaf of germs of C^∞ -functions on B : Let $x \in U$, U open in B , and $\omega \in \mathcal{F}^m(U)$; then $(\underline{\mathcal{S}}(U)(\omega))(x) = \int_{F_x} \omega(x)$. To show this is well defined, let $\{U_j | j \in J\}$ be a covering of B by open sets such that E is trivial over each U_j , with $\phi_j: U_j \times F \cong \Pi_E^{-1}(U_j)$, $j \in J$. Define $\psi_{j,x}: F \rightarrow F_x$ by $\psi_{j,x}(f) = \phi_j(x, f)$, $x \in U_j$, $f \in F$. Then

$$(14) \quad (\underline{\mathcal{S}}(U_j)(\omega))(x) = \text{deg } \psi_{j,x} \int_F \psi_{j,x}^*(\omega(x));$$

this shows that $\underline{\mathcal{S}}(U)(\omega)$ is C^∞ in x , because, since ω has fiber-compact support, the forms $\psi_{j,x}^*(\omega(x))$ on F have supports contained in a common compact set for x in sufficiently small open sets in B .

Thus there is a sheaf homomorphism $\mathcal{S}: \mathcal{F}^m \rightarrow \mathcal{E}_B^\infty$; if $\omega \in \text{Im } \delta_F^{m-1}(U)$, then by Stokes' Theorem, $(\underline{\mathcal{S}}(U)(\omega))(x) = 0$, for all $x \in U$. Consequently, \mathcal{S} induces a sheaf homomorphism, also denoted by \mathcal{S} , $\mathcal{S}: \mathcal{F}^m / \text{Im } \delta_F^{m-1} = \mathcal{L}^m \rightarrow \mathcal{E}_B^\infty$. Lastly,

$$\sigma: H^{k-m}(B; h^m) \longrightarrow H^{k-m}(B)$$

is canonically induced by \mathcal{S} on account of the commutative diagram below:

$$\begin{array}{ccc} A_B^{k-m}(h^m) \cong \Gamma(\mathcal{A}^{k-m} \otimes \mathcal{L}^m) & \xrightarrow{\Gamma(1 \otimes \mathcal{S})} & \Gamma(\mathcal{A}_B^{k-m}) = A_B^{k-m} \\ \delta_B^{k-m,m} \downarrow & & \downarrow \delta_B^{k-m} \\ A_B^{k-m+1}(h^m) \cong \Gamma(\mathcal{A}_B^{k-m+1} \otimes \mathcal{L}^m) & \xrightarrow{\Gamma(1 \otimes \mathcal{S})} & \Gamma(\mathcal{A}_B^{k-m+1}) = A_B^{k-m+1}. \end{array}$$

Combining σ with the map $\chi h_1 h_2$ we obtain algebraic fiber integration $\Psi_1 = \sigma \chi h_1 h_2$; $\Psi_1: H_F^k(E) \rightarrow H^{k-m}(B)$.

3. Fiber integration, geometric definition ([4], chapter 7). For arbitrary manifolds B , F , and $x \in B$, $y \in F$, define

$$i_x: F \longrightarrow B \times F \text{ by } i_x(y) = (x, y), \quad y \in F,$$

$$i_y: B \longrightarrow B \times F \text{ by } i_y(x) = (x, y), \quad x \in B.$$

When $\xi \in T_x(B)$, put $\hat{\xi} = (i_y)_* \xi \in T_{(x,y)}(B \times F)$ and when $\zeta \in T_y(F)$, let $\hat{\zeta} = (i_x)_* \zeta \in T_{(x,y)}(B \times F)$.

Define $\lambda_x = C^{p,q}(B \times F) \rightarrow A_c^q(F; \wedge^p T_x^*(B))$ (with the trivial connection in the product bundle $B \times F$) by

$$(\lambda_x \omega)(y; \zeta_1, \dots, \zeta_q)(\hat{\xi}_1, \dots, \hat{\xi}_p) = \omega((x, y); \hat{\xi}_1, \dots, \hat{\xi}_p, \hat{\zeta}_1, \dots, \hat{\zeta}_q)$$

where $x \in B$, $\omega \in C^{p,q}(B \times F)$, $\zeta_i \in T_y(F)$ and $\xi_j \in T_x(B)$. For the product bundle $B \times F$ geometric fiber integration is the linear map

$$\int_F: C^{p,r}(B \times F) \longrightarrow A^p(B), \quad p \geq 0,$$

defined by $\left(\int_F \omega\right)(x) = \begin{cases} \int_F \lambda_x \omega, & x \in B, \quad r = m \\ 0, & r \neq m, \end{cases}$ where $\omega \in C^{p,r}(B \times F)$.

For an arbitrary oriented bundle ξ , let $\{U_j, \phi_j\}$ be a family of trivialisations as before with $\phi_j: U_j \times F \cong \Pi_E^{-1}(U_j)$. If $\omega \in A_F^k(E)$, $\phi_j^* \omega$ is a fiber-compact form on $U_j \times F$, so that ω_j defined by $\omega_j(x) = \deg \psi_{f_j, x} \left(\int_F \phi_j^* \omega \right)(x)$ is a $k-m$ form on U_j . Define $\int_F A_F^k(E) \rightarrow A^{k-m}(B)$ by $\left(\int_F \omega\right)(x) = \omega_j(x)$ when $x \in U_j$.

It is easily shown that this is independent of the choice of U_j , so that \int_F is well defined ([4]).

Furthermore, $\int_F \delta_E = \delta_B \int_E$, so that there is an induced map

$$\Psi_2: H_F^k(E) \longrightarrow H^{k-m}(B),$$

$k \geq m$, called geometric fiber integration.

4. Proof of Theorem. $\Psi_1 = \Psi_2$.

Let $[\omega] \in H_F^k(E)$ be represented by $\omega \in A_F^k(E)$.

If $W_j = \Pi_E^{-1}(U_j)$ is sufficiently small we may write

$$\omega/W_j = \sum_{|\alpha|+|\beta|=k} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta, \quad \omega^{\alpha\beta} \in C^\infty(W_j),$$

or, upon substitution of equation (8) for dy_β ,

$$(15) \quad \omega/W_j = \sum_{\substack{|\alpha|=k-m \\ |\beta|=m}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta^v + \sum_{\substack{|\alpha|>k-m \\ \beta}} (\dots)^{\alpha\beta} dx_\alpha \wedge dy_\beta^v.$$

Since $h_2: H_F^k(E) \rightarrow H_F^k(E)/H_\infty^{k-m+1, m-1} = E_\infty^{k-m, m}$ is merely the projection,

$$h_2([\omega]) = \left[\sum_{\substack{|\alpha|=k-m \\ |\beta|=m}} \omega^{\alpha\beta} dx_\alpha \wedge dy_\beta^v \right]_{E_\infty}^{k-m, m}.$$

Hence,

$$\chi h_1 h_2([\omega]) = \left[\sum_{\substack{|\alpha|=k-m \\ |\beta|=m}} [\omega^{\alpha\beta} dy_\beta^v]_{h^m} dx_\alpha \right]_{H^{k-m}(B; h^m)}$$

and

$$\Psi_1([\omega]) = \sigma \chi h_1 h_2([\omega]) = [\mu]_{H^{k-m}(B)},$$

where $\mu \in A^{k-m}(B)$ with

$$(16) \quad \mu(x) = \sum_{|\alpha|=k-m} (\deg \psi_{j,x}^*) \left(\sum_{|\beta|=m} \omega^{\alpha\beta} dy_\beta^v \right) dx^\alpha$$

when $x \in U_j$.

On the other hand,

$$\Psi_2([\omega]) = \left[\int_F \omega \right]_{H^{k-m}(B)}$$

and

$$(17) \quad \left(\int_F \omega \right)(x) = \deg \psi_{j,x} \int_F \nu_x(\phi_j^* \omega) = \deg \psi_{j,x} \int_F \left(\sum_{\substack{|\alpha|=k-m \\ |\beta|=m}} \omega^{\alpha\beta} \psi_{j,x} dy_\beta \right) dx_\alpha$$

as a short computation shows.

A comparison of (16), (17) shows that $\Psi_1 = \Psi_2$ as required.

REFERENCES

1. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458-538.
2. S. S. Chern, *On the characteristic classes of complex sphere bundles and algebraic varieties*, Amer. J. Math., **75** (1953), 565-597.
3. R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann (1964).
4. W. H. Grueb, S. Halperin, and R. Vanstone, *Cohomology of Smooth Bundles I, II, III*, Academic Press (to appear).
5. A. Hattori, *A Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I (1960), 289-331.
6. F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer Verlag (1966) appendix II.
7. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, volume I, Interscience (1963).

Received September 30, 1971.

BROCK UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

C. R. HOBBY
University of Washington
Seattle, Washington 98105

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Jimmy T. Arnold, <i>Power series rings over Prüfer domains</i>	1
Maynard G. Arsove, <i>On the behavior of Pincherle basis functions</i>	13
Jan William Auer, <i>Fiber integration in smooth bundles</i>	33
George Bachman, Edward Beckenstein and Lawrence Narici, <i>Function algebras over valued fields</i>	45
Gerald A. Beer, <i>The index of convexity and the visibility function</i>	59
James Robert Boone, <i>A note on mesocompact and sequentially mesocompact spaces</i>	69
Selwyn Ross Caradus, <i>Semiclosed operators</i>	75
John H. E. Cohn, <i>Two primary factor inequalities</i>	81
Mani Gagrat and Somashekhar Amrith Naimpally, <i>Proximity approach to semi-metric and developable spaces</i>	93
John Grant, <i>Automorphisms definable by formulas</i>	107
Walter Kurt Hayman, <i>Differential inequalities and local valency</i>	117
Wolfgang H. Heil, <i>Testing 3-manifolds for projective planes</i>	139
Melvin Hochster and Louis Jackson Ratliff, Jr., <i>Five theorems on Macaulay rings</i>	147
Thomas Benton Hoover, <i>Operator algebras with reducing invariant subspaces</i>	173
James Edgar Keesling, <i>Topological groups whose underlying spaces are separable Fréchet manifolds</i>	181
Frank Leroy Knowles, <i>Idempotents in the boundary of a Lie group</i>	191
George Edward Lang, <i>The evaluation map and EHP sequences</i>	201
Everette Lee May, Jr., <i>Localizing the spectrum</i>	211
Frank Belsley Miles, <i>Existence of special K-sets in certain locally compact abelian groups</i>	219
Susan Montgomery, <i>A generalization of a theorem of Jacobson. II</i>	233
T. S. Motzkin and J. L. Walsh, <i>Equilibrium of inverse-distance forces in three-dimensions</i>	241
Arunava Mukherjea and Nicolas A. Tserpes, <i>Invariant measures and the converse of Haar's theorem on semitopological semigroups</i>	251
James Waring Noonan, <i>On close-to-convex functions of order β</i>	263
Donald Steven Passman, <i>The Jacobian of a growth transformation</i>	281
Dean Blackburn Priest, <i>A mean Stieltjes type integral</i>	291
Joe Bill Rhodes, <i>Decomposition of semilattices with applications to topological lattices</i>	299
Claus M. Ringel, <i>Socle conditions for $QF-1$ rings</i>	309
Richard Rochberg, <i>Linear maps of the disk algebra</i>	337
Roy W. Ryden, <i>Groups of arithmetic functions under Dirichlet convolution</i>	355
Michael J. Sharpe, <i>A class of operators on excessive functions</i>	361
Erling Stormer, <i>Automorphisms and equivalence in von Neumann algebras</i>	371
Philip C. Tonne, <i>Matrix representations for linear transformations on series analytic in the unit disc</i>	385