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FUNCTION ALGEBRAS OVER VALUED FIELDS

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In this paper we consider primarily algebras $F(T)$ of continuous funtions taking a topological space T into a complete nonarchimedean nontrivially valued field F. Some general properties of function algebras and topological algebras over valued fields are developed in §§1 and 2. Some principal results (Theorems 6 and 7) are analogs of theorems of Nachbin and Shirota, and Warner: Essentially that $F(T)$ with compact-open topology is F-barreled iff unbounded functions exist on closed noncompact subsets of T; and that full Fréchet algebras are realizable as function algebras $F(\mathscr{M})$ where $\mathscr{M}$ denotes the space of nontrivial continuous homomorphisms of the algebra.

Nachbin and Shirota's well-known result provides a necessary and sufficient condition for an algebra of realvalued continuous functions on a topological space to be barreled when it carries the compact-open topology. To develop an analog of Nachbin's theorem for F-valued functions, it is necessary to bypass the heavily real-number-oriented machinery on which his proof depends. We accomplish this in part by developing an ordering of the elements of a discretely valued field (Sec. 3, Def. 2) which serves to take the place of the usual ordering of the reals. We also consider a notion of "support" of a continuous F-valued linear functional on $F(T)$ (Sec. 3, Def. 3). The support notion is developed without measure theory or representation theorems for continuous linear functionals.

The results of the paper depend heavily on theorems proved by Ellis ([3]), Kaplansky ([7], [8]), and van Tiel ([14]), as well as the proofs of the major theorems as originally presented by Nachbin ([10]) and Warner ([15]) which provided the ideas for this line of approach.

Throughout the paper "algebra" (denoted by X or Y) includes the presence of an identity and commutativity. The underlying field F is assumed to be a complete nonarchimedean rank one nontrivially valued field. Unless otherwise stated, T denotes a 0-dimensional (a base for the topology consisting of closed and open sets exists) Hausdorff topological space and $F(T)$ the algebra of continuous functions from T into F with pointwise operations. The terms Banach space or Banach algebra are used throughout in the sense of [12].

1. Topological algebras over valued fields.

In this section we discuss some basic properties of topological algebras over fields with valuation. We assume throughout that the underlying field F is a complete nonarchimedean rank one nontrivially valued field.

DEFINITION 1. A topological algebra X over F is *nonarchimedean locally multiplicatively F-convex* (NLMC) if there exists a base $\mathscr{B}$ of neighborhoods U of 0 in X such that for each $U \in \mathscr{B}$, (1) U is F-convex (i.e. if λ and μ are scalars such that $|\lambda|, |\mu| \leqq 1$, then $\lambda U + \mu U \subset U$), and (2) $UU \subset U$.

DEFINITION 2. A seminorm p on X is *nonarchimedean* and *multiplicative* respectively if for all $x, y \in X$ (1) $p(x + y) \leqq \max [p(x), p(y)]$ and (2) $p(xy) \leqq p(x)p(y)$.

PROPOSITION 1. *A topological algebra X is an NLMC algebra iff the topology on X is generated by a family P of nonarchimedean multiplicative seminorms.*

Proof. Given such a family P generating the topology on X, the sets $\{x \mid p_i(x) \leqq \varepsilon, p_1, \cdots, p_n \in P, 0 < \varepsilon \leqq 1\}$ form a base at 0 satisfying the condition of Definition 1.

Conversely, if $\mathscr{B}$ is a base at 0 satisfying the conditions of Definition 1, then, letting $p_U(x) = \inf \{|\mu| \mid x \in \mu U, \mu \in F\}$ the seminorms $(p_U)_{U \in \mathscr{B}}$ constitute the desired family P.

PROPOSITION 2. *If the valuation on F is discrete and X is an NLMC algebra, then there exists a family P' or nonarchimedean multiplicative seminorms generating the topology on X such that $p'(X) \subset |F|$ for each $p' \in P'$.*

Proof. Let P be a family of nonarchimedean multiplicative seminorms generating X's topology. For each $p \in P$ let $p'(x) = \inf \{|\mu| \mid |\mu| \geqq p(x)\}$. Each such p' is clearly nonarchimedean and multiplicative. Moreover since $p(x) \leqq p'(x) \leqq |\mu^{-1}| p(x)$ for any nonzero $\mu \in F$ such that $|\mu| < 1$ and $|\mu|$ generates the value group of F, P' will also generate the topology on X.

DEFINITION 3. An NLMC algebra X is *discrete* if there exists a family P of nontrivial nonarchimedean multiplicative seminorms generating the topology on X such that each p in P is discrete [the only limit point of $p(X)$ is 0].

PROPOSITION 3. *A Hausdorff NLMC algebra X is discrete iff F is discretely valued.*

Proof. Use Prop. 2.

If X is a topological algebra over C, the complex numbers, then we can identify the nontrivial continuous homomorphisms of X into C with the closed maximal ideals in X ([9, p. 13]). This is no longer true for noncomplex algebras, and we single out those algebras in which the $1-1$ correspondence still obtains for special attention.

DEFINITION 4. A commutative Hausdorff NLMC algebra X with identity e is a *Gelfand algebra* if for every closed maximal ideal $M \subset X$ the factor algebra X/M (with quotient topology) is topologically isomorphic to F.

Associated with the nontrivial nonarchimedean multiplicative seminorms p generating the topology on an NLMC algebra X, are nonarchimedean normed algebras X/N_p where N_p is the ideal $p^{-1}(0)$ where X/N_p is normed by taking $\|x + N_p\| = p(x)$. The completions X_p of these normed algebras are referred to as *factor algebras*.

PROPOSITION 4. *If X is a Gelfand algebra and X/N_p is complete, then X/N_p is a Gelfand algebra.*

Proof. Let π_p denote the continuous homomorphism $x \rightarrow x + N_p$ from X onto X/N_p. We observe that if M is a maximal ideal in the Banach algebra X/N_p, then M is closed; thus $\pi_p^{-1}(M)$ is a closed maximal ideal in X containing N_p. For any $x \in X$ there exists $\mu \in F$ such that $x\text{-}\mu e \in \pi_p^{-1}(M)$ (X is a Gelfand algebra), so that $\pi_p(x) - \mu\pi_p(e) \in M$ where e is the identity of X. Thus $(X/N_p)/M$ is algebraically isomorphic to F. Since M is closed, the factor structure is a one-dimensional Hausdorff topological vector space and is therefore topologically isomorphic to F.

PROPOSITION 5. *Let P be a saturated family of seminorms generating the topology on the NLMC algebra X and let $(X_p)_{p \in P}$ denote the associated factor algebras. If each X_p is a Gelfand algebra, then X is a Gelfand algebra.*

Proof. Let M be a closed maximal ideal in X. By [1, p. 466] there exists $p \in P$ such that $M \supset N_p$ and $\inf\{p(e - x) \mid x \in M\} > 0$. Consequently $\pi_p(M)$ is a proper ideal in X/N_p and $\pi_p(e)$ is not an adherence point of $\pi_p(M)$. Thus $\overline{\pi_p(M)}$ is a proper ideal in X_p and is therefore contained in a closed maximal ideal $N \subset X_p$. Since X_p is a Gelfand algebra, N is the kernel of a continuous nontrivial homomorphism f_p taking X_p into F. Hence $f = f_p\pi_p$ is a continuous nontrivial homomorphism taking X into F. It follows from elementary considerations that the kernel of f is equal to M. Consequently X/M is seen to be algebraically—hence topologically—isomorphic to F.

A result similar in spirit to this can be found in [2, p. 175]. We turn next to some examples.

EXAMPLE 1. Let F be a local field, let T be a 0-dimensional Hausdorff space and let $F(T)$ carry the topology of uniform convergence on compact sets. The topology on $F(T)$ is generated by the nonarchimedean multiplicative seminorms p_K where K is a compact subset of T and for any $x \in F(T)$, $p_K(x) = \sup_{t \in K} |x(t)|$. We may identify $F(T)/p_K^{-1}(0)$ with a subalgebra of $F(K)$. Moreover we may construct a'Stone-Cech' compactification $\beta_F T$ of T as is done in [3, p. 243] utilizing the compact valuation ring V of F in place of the compact interval [0, 1]. Since V is Hausdorff and 0-dimensional, $\beta_F T$ will be compact, Hausdorff and 0-dimensional. Thus the Ellis-Tietze extension theorem ([4]) applies and any function continuous on K may be extended to a function continuous on $\beta_F T$. It follows that $F(T)/p_K^{-1}(0) = F(K)$.

The continuous nontrivial homomorphisms of $F(K)$ into F are in $1-1$ correspondence with the points t of K ([11]) and using this result it can be shown [9, p. 31] that the points of T generate the continuous nontrivial homomorphisms of $F(T)$ into F.*

Topological algebras X for which all homomorphisms of X into F are continuous are called *functionally continuous* [9, p. 51]). What follows is an example of such an algebra.

EXAMPLE 2. Let F be any complete nonarchimedean nontrivially valued field and T a 0-dimensional Hausdorff space. $F(T)$ carries the compact-open topology. A subalgebra X of $F(T)$ is "closed under inverses" if when $x \in X$ and $x^{-1} \in F(T)$, $x^{-1} \in X$. We apply Michael's proof [9, p. 54] and observe that if Conditions 1 and 2 below are satisfied, then the homomorphisms of X are generated by the points of T and therefore X is functionally continuous.

1. For any $x_1, \cdots, x_n \in X$ such that $\bigcap_{i=1}^n x_i^{-1}(0) = \varnothing$, there exists $y_1, \cdots, y_n \in X$ such that $\Sigma x_i y_i = e$ where e is the constant function $e(t) = 1$ for all $t \in T$.

2. For some positive integer m there exists $x_1, \cdots, x_m \in X$ such that for all $\mu_1, \cdots, \mu_m \in F$, $\cap (x_i - \mu_i e)^{-1}(0)$ is compact.

We note that if $X = F(T)$, then by the results of a sequel to this paper [16], it follows that X satisfies statement 1. If, in addition, there exists a bijection $x \in X$, then X satisfies 2. Hence if we take $T = F$ and let T carry any 0-dimensional Hausdorff topology finer than

* The result of Example 1 actually obtains if F is any complete nonarchimedean nontrivially valued field as it can be shown in this case that a bounded continuous function defined on a compact subset K of T mapping into F can be extended to a bounded continuous function mapping T into F. The same comment applies to Example 1, parts (c), (d), and (e) of Sec. 2.

the valuation topology on F, the nontrivial homomorphisms of the algebra $F(T)$ taking values in F are generated by the points of T.

2. Function algebras over valued fields. In this section we discuss function algebras over valued fields. First we prove a version of a theorem of Kaplansky ([7, p. 173]) which is relevant to the material to follow; we include this proof because there seems to be an inconsistency in the use of "totally disconnected" in [7].

LEMMA 1. (Kaplansky) *Let T be a topological space and let $F(T)$ be endowed with the topology of uniform convergence on compact sets. I is a closed ideal in $F(T)$, iff there is some closed subset H of T such that $I = \{f \in F(T) \mid f(H) = \{0\}\}$. I is a closed maximal ideal in $F(T)$ iff there is some $t \in T$ such that $I = \{f \mid f(t) = 0\}$.*

Proof. Suppose I is closed in $F(T)$ and let $H = \bigcap_{g \in I} g^{-1}(0)$. Letting $J(H) = \{f \mid f(H) = \{0\}\}$, we see that $I \subset J(H)$, and that $J(H)$ is a closed ideal. We show that if $f \in J(H)$, then $f \in I$.

Let K be any compact subset of T. If $y \in K$, then as I is an ideal, there exists $g_y \in I$ such that $g_y(y) = f(y)$. Since the clopen sets $\{U_y \mid y \in K\}$ where $U_y = \{x \in T \mid |f(x) - g_y(x)| < \varepsilon\}$ cover K for any fixed $\varepsilon > 0$, there exist $y_1, \cdots, y_n$ such that $K \subset \bigcup_{i=1}^n U_{y_i}$. Since the sets U_{y_i} are clopen, we see that there exist pairwise disjoint clopen sets W_i such that $K \subset \bigcup_{i=1}^n W_i$ where $W_i \subset U_{y_i}$ for each i. Letting k_A denote the characteristic function of the set A, we see that if $h = \sum_{i=1}^n g_{y_i} k_{W_i}$, then $h \in I$ and $\sup_{t \in K} |h(t) - f(t)| < \varepsilon$. As $\varepsilon > 0$ can be made arbitrarily small, it follows that $f \in \bar{I} = I$..

In the proof to follow, "totally disconnected" is used as in [13, p. 380]: distinct points may be separated by clopen sets.

THEOREM 1. (Kaplansky) *Let S and T be 0-dimensional Hausdorff spaces. Let $F(S)$ and $F(T)$ carry their compact-open topologies and suppose that $F(T)$ is topologically isomorphic to $F(S)$. Then S and T are homeomorphic.*

Proof. Let A be a topological isomorphism from $F(S)$ onto $F(T)$. If K is a closed subset of S and $J(K)$ denotes the ideal of functions that vanish on K, note that a mapping A' is defined by $A(J(\{s\})) = J(\{t\}) = J(\{A'(s)\})$ for some $t \in T$; i.e. $A' \colon S \to T$ is such that $A'(s) = t$, and is well-defined as T is totally disconnected. Since A is injective and S is totally disconnected, then A' is injective as well. For any $t \in T$, $J(\{t\}) = A(M)$ where M is a closed maximal ideal in $F(S)$.

Since $M = J(\{s\})$ for some $s \in S$, A' is seen to be surjective.

Clearly $(A')^{-1} = (A^{-1})'$ so to show that A' is a homeomorphism, it suffices to show that A' is a closed map. To this end, since S is 0-dimensional, $K = \bigcap_{g \in J(K)} g^{-1}(0)$; since $J(K) = \bigcap_{s \in K} J(\{s\})$, it follows that $A(J(K)) = J(A'(K)) = \bigcap_{s \in K} J(\{A'(s)\})$. If $t \notin A'(K)$, then $t = A'(s)$ where $s \notin K$. Thus $J(K) \not\subset J(\{s\})$ and $J(A'(K)) \not\subset J(\{t\})$. As $J(A'(K)) = J(\overline{A'(K)}) \not\subset J(\{t\})$, we see that $t \notin \overline{A'(K)}$ and therefore $A'(K) = \overline{A'(K)}$.

EXAMPLE 1. Let T be a totally disconnected Hausdorff space and let $F(T)$ carry the compact-open topology. We note immediately that the set of evaluation maps constitutes a set of distinct continuous homomorphisms of $F(T)$ into F. Moreover properties (a)—(e) also hold.

(a) If K is a compact subset of T, p_K is as in Ex. 1 of Sec. 1, and $N_K = p_K^{-1}(0)$, then the completion of the normed algebra $F(T)/N_K$ is $F(K)$.

Proof. Since T is totally disconnected, the characteristic functions in $F(T)$ separate the points of T. Thus the functions $f|_K$ as f runs through $F(T)$ separate points in K. The desired result now follows from an application of Kaplansky's Stone-Weierstrass theorem ([8] or [12] p. 161).

(b) With "V^*-algebra" as in [12, p. 148], if T is locally compact, then $F(T)$ is the projective limit of V^*-algebras as in [9, p. 17].

Proof. The complete NLMC algebra $F(T)$ is the projective limit of the factor algebras $F(K)$ as K runs through the compact subsets of T and each $F(K)$ is a V^*-algebra.

(c) If T is ultranormal and F is a local field, then $F(T)/N_K = F(K)$.

Proof. Use the Ellis-Tietze extension theorem of [4].

(d) If T is 0-dimensional and F is a discretely valued field, then $F(T)/N_K = F(K)$ for any compact subset K of T.

Proof. Apply a modification of the Ellis-Tietze extension theorem to functions $f \in F(K)$ and thereby extend f continuously to a 'Stone-Cech' compactification $\beta_H T$ where H is any local field. Where Ellis used local compactness of the field F, we use discreteness of the valuation on F, and compactness of $\beta_H T$.

(e) The points of T constitute all continuous homomorphisms of $F(T)$ into F when F is discretely valued.

Proof. See Ex. 1 of Sec. 1 and use (d).

3. Main results. Let X be a NLMC algebra over a discretely valued F. Then, as in the classical case ([9, p. 33]), if X is the projective (dense inverse) limit of a family $(F(K_n))$ of Gelfand V^*-algebras by mappings $\pi_{mn}: F(K_n) \to F(K_n)$, $m > n$, where (K_n) is a family of compact 0-dimensional Hausdorff spaces (it following that K_n is homeomorphically embedded in K_m), then X is topologically isomorphic to $F(\cup K_n)$ where* $F(\cup K_n)$ carries the compact-open topology. Moreover in this case $\cup K_n$ can and will be identified with the set of all nontrivial continuous homomorphisms of X into F and carries the weak topology generated by (K_n).

DEFINITION 1. Let $\mathscr{M}$ denote the nontrivial continuous homomorphisms of an MLHC algebra X over F into F, and let $\mathscr{M}$ carry the weak-* topology. Let $F(\mathscr{M})$ denote the algebra of continuous functions mapping $\mathscr{M}$ into F with compact open topology and consider the map $\psi: X \to F(\mathscr{M})$ where, for any $x \in X$, $\psi(x)(h) = h(x)$ for each $h \in \mathscr{M}$. X is called a *full* algebra if the homomorphism ψ is an isomorphism of X onto $F(\mathscr{M})$.

In [9] E. A. Michael stated that he did not know whether or not ψ was a topological isomorphism in the case where X is a Fréchet full algebra. S. Warner proved that this was true in the classical case ([15, p. 269]). In this section we show that ψ is a topological isomorphism if F is a local field (Theorem 7). It then follows according to some results of van Tiel [14] that X is the projective limit of a sequence $(F(K_n))$ of Gelfand V^*-algebras where $K_n = V_n^\circ \cap \mathscr{M}$ (V_n° is the polar of a neighborhood V_n of 0 in X coming from a base of F-convex closed neighborhoods of 0). Thus we will have a partial converse of the result which was described in the opening paragraphs of this section. We also note that by Prop. 5 of Sec. 1, X is a Gelfand algebra under the hypothesis just mentioned.

In what follows F is assumed to be discretely valued. In some cases it will also be assumed that F is a local field so that certain standard results from the duality theory of topological vector spaces ([14]) may be used.

DEFINITION 2. Let F be discretely valued and let $(a_\mu)_{\mu \in H}$ be a system of distinct representatives of the cosets in the residue class field of F. We may assume that H is totally ordered where μ_0 corresponding to $a_{\mu_0} = 0$ is the first element. Let $\pi \in F$ be such that $|\pi| < 1$ and $|\pi|$ is a generator of the value group of F. If a and b are any two elements of F there exist (a_{μ_i}) and (a_{λ_i}) such that $a = \sum_{i=N}^{\infty} a_{\mu_i}\pi^i$ and $b = \sum_{i=N}^{\infty} a_{\lambda_i}\pi^i$. We now define the *supremum*, sup (a, b),

* We may assume $K_n \subset K_{n+1}$ as there exist sets K'_n such that $\mathscr{M} = \cup K'_n$ with $K'_n \subset K'_{n+1}$, and K'_n homeomorphic to K_n for all n.

of a and b as:

$$\sup(a, b) = \begin{cases} a \text{ if } |a| > |b| \\ b \text{ if } |b| > |a| \\ a \text{ if } a = b \\ a \text{ if } |a| = |b|, a_{\mu_i} = a_{\lambda_i} \text{ for } i = N, \cdots, j-1 \text{ and } \mu_j > \lambda_j \end{cases}$$

LEMMA 1. *Let T be a topological space and let f and g be continuous functions mapping T into F. Then the function defined at each $t \in T$ by* $\sup(f(t), g(t))$ *and denoted by* $\sup(f, g)$ *is continuous.*

Proof. Suppose (t_s) is a net in T converging to t. We show that $\sup(f, g)(t_s)$ converges to $\sup(f, g)(t)$. Letting $f(t) = a$ and $g(t) = b$, we need only consider the last possibility for $\sup(a, b)$, the first three being trivial. Choose $\varepsilon > 0$ such that $\varepsilon < |\pi|^j$. For r such that $|f(t_s) - f(t)| < \varepsilon$ and $|g(t_s) - g(t)| < \varepsilon$ for $s \geqq r$, it follows that

$$f(t_s) - f(t) = \sum_{i=M}^{\infty} a^s_{\mu_i}\pi^i \text{ and } g(t_s) - g(t) = \sum_{i=M}^{\infty} a^s_{\lambda_i}\pi^i$$

where $M > j$. We may also write

$$f(t_s) = \sum_{1=N}^{j} a_{\mu_i}\pi^i + \sum_{i=j+1}^{\infty} a^s_{\mu_i}\pi^i \text{ and } g(t_s) = \sum_{i=N}^{j} a_{\lambda_i}\pi^i + \sum_{i=j+1}^{\infty} a^s_{\lambda_i}\pi^i .$$

Thus, since $a_{\mu_i} = a_{\lambda_i}$ for $i = N, \cdots, j-1$, and $\mu_j > \lambda_j$, it follows that $\sup(f, g)(t_s) = f(t_s)$ for $s \geqq r$. Thus $\sup(f, g)(t_s) = f(t_s) \to f(t) = \sup(f, g)(t)$.

LEMMA 2. *Let $F(T)$ denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space T into the discretely valued F, with compact-open topology. If V is an F-barrel (closed absorbent F-convex set) in $F(T)$, then there is some $\delta > 0$ such that $\sup_{t \in T}|f(t)| \leqq \delta$ implies that $f \in V$.*

Proof. Let B be the sup-norm Banach space of all bounded functions from T into F. We note that $V \cap B$ is an F-barrel in B. Since B is F-barreled ([14, p. 268]) there is some $\delta > 0$ such that $\sup_{t \in T}|f(t)| \leqq \delta$ which implies that $f \in V \cap B$.

LEMMA 3. *Let V, F, T and $F(T)$ be as in Lemma 2, and suppose that for some compact subset K of T, $\{f \mid f(K) = \{0\}\} \subset V$. Then there is some $\mu > 0$ such that whenever $\sup_{t \in K}|f(t)| < \mu$, then $f \in V$. Thus V is a neighborhood of 0 in $F(T)$.*

Proof. Let $a \in F$ and denote the function sending each $t \in T$ into a by a. With δ as in Lemma 2, choose $a \in F$ such that $0 < |a| \leqq$

$\delta/2$. Choosing an integer n so that $\delta/n < |a|$, let $f \in F(T)$ be such that $\sup_{t \in K}|f(t)| \leqq \delta/n$. With $g = \sup(f, a) - a$, it follows that $g(t) = 0$ for each t in K. Thus $g \in V$. Since $|f(t) - g(t)| \leqq |a| \leqq \delta/2$ for all $t \in T$, it follows that $f - g \in V$. Since V is F-convex, $g + (f - g) = f \in V$, and the proof is complete.

We continue towards nonarchimedean analogs of theorems of Nachbin (Theorem 3) and Warner (Theorem 7). First we consider a notion of *support* of a linear functional which serves to replace the classical notion used by Nachbin.

In Lemmas 4 and 5 $F(T)$ again denotes the algebra of continuous functions from the 0-dimensional Hausdorff space T into F with compact-open topology and φ denotes a member of the continuous dual $F(T)'$ of $F(T)$. For any subset S of T, k_S denotes the characteristic function of S taking values in F and we note that $k_S \in F(T)$ iff S is clopen. Let $\mathscr{S}$ denote the family of subsets U of T such that U is clopen and $\varphi(fk_U) = 0$ for all $f \in F(T)$.

LEMMA 4. *The family $\mathscr{S}$ has the following properties*: (1) *If U is a clopen subset of $G \in \mathscr{S}$, then $U \in \mathscr{S}$*; (2) *$\mathscr{S}$ is a ring of sets.*

Proof. To prove (1) we observe that $k_U = k_G k_U$. (2) follows readily from (1).

DEFINITION 3. The *support* of φ, F_φ, is defined to be $C(\cup \mathscr{S})$.

We observe that since φ is continuous there is some compact set $K \subset T$ and an integer N such that if $f \in F(T)$, then $|\varphi(f)| \leqq N \sup_{t \in K}|f(t)|$. Thus, if f vanishes on K, then $\varphi(f) = 0$.

THEOREM 1. *In the same notation as above* (1) *$F_\varphi \subset K$ and therefore F_φ is compact,* (2) *if φ is nontrivial, then F_φ is not empty, and* (3) *if $G \subset T$ is open and $G \cap F_\varphi$ is not empty, then there exists $f \in F(T)$ such that $f(CG) = \{0\}$ and $\varphi(f) = 1$.*

Proof. (1) If G is a clopen subset of CK, then—since k_G vanishes on K—$\varphi(fk_G) = 0$ and $G \in \mathscr{S}$.

(2) If F_φ is empty, $T = \cup \mathscr{S}$, and it follows that for some $U_i \in \mathscr{S}$, $K \subset \cup_{i=1}^n U_i = G$. Since $\mathscr{S}$ is a ring of sets, $G \in \mathscr{S}$ and since CG is clopen and contained in CK, $\varphi(f) = \varphi(fk_{CG}) = 0$ for all $f \in F(T)$. But then φ is trivial.

(3) If $G \cap F_\varphi \neq \emptyset$, there is some $t \in G \cap F_\varphi$. Since T is 0-dimensional, $t \in U \subset G$ where U is clopen. Since $U \cap F_\varphi \neq \emptyset$, then $U \notin \mathscr{S}$ and there is some $g \in F(T)$ such that $\varphi(gk_U) \neq 0$. We of course may assume that $\varphi(gk_U) = 1$. Letting $gk_U = f$, (3) is seen to be proved.

In order to apply this notion of support to our version of Nachbin's theorem (Theorem 3) we require that F_φ have the property that if f vanishes on F_φ, $\varphi(f) = 0$. We now develop a case where this is true and which makes the notion applicable to Theorem 3 as well as settling Michael's question in this setting (Theorem 7).

LEMMA 5. *Suppose that $\varphi(g) = 0$ for any $g \in F(T)$ which vanishes on any clopen set G containing F_φ. Then if f vanishes on F_φ, $\varphi(f) = 0$.*

Proof. Suppose that $f \in F(T)$ vanishes on F_φ, and let $A_n = \{t \in T \mid |f(t)| < 1/n\}$ $(n = 1, 2, \cdots)$. As $F_\varphi \subset A_n$ for any n and A_n is clopen $\varphi(f) = \varphi(fk_{A_n}) + \varphi(f(1 - k_{A_n}))$. By the hypothesis, since $f(1 - k_{A_n})$ vanishes on A_n, $\varphi(f) = \varphi(fk_{A_n})$. Let K be a compact subset of T such that $|\varphi(f)| \leqq N \sup_{t \in K} |f(t)|$. Hence $|\varphi(f)| = |\varphi(fk_{A_n})| \leqq N \sup_{t \in K} |fk_{A_n}(t)| < N/n$. Since this is true for every n, $\varphi(f) = 0$.

THEOREM 2. *Let T be a Lindelöf space. Then if f vanishes on F_φ, $\varphi(f) = 0$.*

Proof. Let G be a clopen subset containing F_φ. Since CG is closed, CG is Lindelöf. Since $CG \subset CF_\varphi = \cup \mathscr{S}$, there exist $U_i \in \mathscr{S}$ such that $CG \subset \bigcup_{i=1}^\infty U_i$. Since $\mathscr{S}$ is a ring, we may assume that the sets U_i are pairwise disjoint. Since $CG \cap U_i = V_i$ is clopen and contained in U_i then $V_i \in \mathscr{S}$. Thus $k_{CG} = \sum_{i=1}^\infty k_{V_i}$ in the topology of pointwise convergence on $F(T)$. We claim that the "pointwise convergence" of the preceding sentence may be replaced by "uniform convergence on compact sets."

To prove this last statement, let L be a compact subset of T and consider $L \cap CG$. As $L \cap CG$ is compact and contained in $\bigcup_{i=1}^\infty V_i$ there is some integer N_L such that $n \geqq N_L$ implies that $L \cap CG$ is contained in $\bigcup_{i=1}^n V_i$. But $CG \subset \bigcup_{i=1}^n V_i$ so $L \cap CG = L \cap (\bigcup_{i=1}^n V_i)$. Thus for $n \geqq N_L$, CG and $\bigcup_{i=1}^n V_i$ have the same points in common with L, and $\sup_{t \in L} |(k_{CG} - \sum_{i=1}^n k_{V_i})(t)| = 0$ for $n \geqq N_L$. Since L was an arbitrary compact set, the series is seen to converge in the compact-open topology and $\varphi(fk_{CG}) = \sum_{i=1}^n \varphi(fk_{V_i}) = 0$.

We now present a version of a theorem of Nachbin ([10, p. 472])

THEOREM 3. *Let $F(T)$ denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space T into the discretely valued field F, with compact-open topology. Suppose that for each $\varphi \in F(T)'$, f vanishing on F_φ implies $\varphi(f) = 0$. Then $F(T)$ is F-barreled iff for every $E \subset T$ which is closed and not compact there is some $f \in F(T)$*

*which is unbounded on E.**

Proof. Suppose that the condition holds and let V be an F-barrel in $F(T)$. To show that V is a neighborhood of 0 in $F(T)$ we begin by letting $K = \overline{\bigcup_{\varphi \in V^0} F_\varphi}$. If K is not compact, let f be unbounded on K and consider the sets $A_n = \{t \in T \mid |f(t)| > n\}$, $n = 1, 2, \cdots$. Each A_n is clopen and $A_n \cap K \neq \emptyset$. Thus there is some $F_{\varphi_n} \subset K$ such that $A_n \cap F_{\varphi_n} \neq \emptyset$. By Theorem 1 (3) there exists $f_n \in F(T)$ such that f_n vanishes outside of A_n and $\varphi_n(f_n) = 1$. Since $\bigcap_{n=1}^\infty A_n = \emptyset$, the function $f = \sum_{n=1}^\infty a_n f_n$ is a continuous function for any choice of $a_n \in F$. As it is clear that $A_m \cap F_{\varphi_n} = \emptyset$ for all sufficiently large m, we may (by considering a subsequence) assume that $\varphi_n(f_m) = 0$ for all $m > n$. By a proper choice of a_n we see that $|\varphi_n(f)| \to \infty$ and as $\varphi_n \in V^0$, af cannot belong to $V^{00} = V$ no matter how small $|a|$ is. Thus we contradict the fact that V is absorbent and K must be compact. If f vanishes on K, then f vanishes on F_φ for all $\varphi \in V^0$. Thus $f \in V^{00} = V$ so, by Lemma 3, V is a neighborhood of 0.

To prove the converse, let $F(T)$ be F-barreled and E be a closed noncompact subset of T. Let $V = \{f \mid \sup_{t \in E} |f(t)| \leqq \delta\}$, $\delta > 0$, and let K be a compact subset of T. As $E \cap CK \neq \emptyset$, using $\beta_H T$ as in Sec. 2 Ex. 1 (d) we may assert the existence of a sequence (f_N) of functions which vanishes on K but $|f_N(t_N)| \geqq N$ for any positive integer N and some $t_N \in E$. Thus the set $\{f \mid \sup_{t \in K} |f(t)| \leqq \varepsilon\} \not\subset V$ for any $\varepsilon > 0$ and V is not a neighborhood of 0. It follows that V is not absorbing and there exists $f \in F(T)$ which is unbounded on E.

COROLLARY. *Let T be a 0-dimenional Hausdorff Lindelöf space and F a discretely valued field. Then $F(T)$ is F-barreled.*

Proof. We refer to Theorem 2 and the construction of the function in the proof of Theorem 6 for the proof of the corollary.

THEOREM 4. *Suppose the 0-dimensional Hausdorff space $T = \bigcup_{n=1}^\infty K_n$ where each K_n is compact, $K_n \subset K_{n+1}$, and each compact subset of T is contained in some K_n (i.e. T is hemicompact). Then denoting T endowed with the weak topology* ([3], p. 131) *generated by the sets (K_n) as T_w, $F(T)$ is dense in $F(T_w)$, each algebra carrying its compact-open topology.*

Proof. Since the topology of T_w is clearly stronger than that of T, $F(T) \subset F(T_w)$. We note that the topology of T_w restricted to K_n is

* In a sequel to this paper we show that Theorem 2 is true for any 0-dimensional Hausdorff space T and any complete nonarchimedean nontrivially valued field F. Thus Theorem 3 is true for all spaces T. We also show that the result of Theorem 3 holds of F is spherically complete ([**16**]).

equal to the topology K_n inherits from T and the compact subsets of T_w lie in the sets K_n. Thus $F(T)$ is a topological subspace of $F(T_w)$. Using Sec. 2 Ex. 1 (d), $F(T)/N_K = F(K)$ for any compact set $K \subset T$ and it follows that $F(T)$ is dense in $F(T_w)$.

THEOREM 5. *Let everything be as in the preceding theorem. If $F(T)$ is complete then $T = T_w$ iff T_w is 0-dimensional.*

Proof. If $F(T)$ is complete, then $F(T) = F(T_w)$. Since they are topologically isomorphic under the identity map by the proof of Theorem 4, if T_w is 0-dimensional, then $T = T_w$ by Theorem 1 of Sec. 2. We may also observe that the functions of $F(T)$ generate the topology of the space T while those of $F(T_W)$ generate the topology of T_W. Thus as $F(T) = F(T_W)$, the topologies are equal.

THEOREM 6. *Let $F(T)$ denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space T into the local field F and suppose that $F(T)$ is a complete locally F-convex metric space with topology $\mathscr{T}$. If the homomorphisms determined by the points of T are the $\mathscr{T}$-continuous homomorphisms, then $\mathscr{T}$ is the compact-open topology.*

Proof. Let the set of evaluation maps determined by T be denoted by T^* and let T^* carry the Gelfand topology (i.e. the weakest topology for T^* with respect to which the maps $t \to x(t)$ of T^* into F are continuous for each $x \in F(T)$). Since T is 0-dimensional the Gelfand topology coincides with the original topology on T, i.e. T and T^* are homeomorphic. Since $(F(T), \mathscr{T})$ is F-barreled ([14, p. 268]), the polar of any compact subset of T^* is a neighborhood of 0 in $F(T)$. Thus, identifying T and T^*, $\mathscr{T}$ is seen to be stronger than the compact-open topology on $F(T)$. If $F(T)$ with compact-open topology could be shown to be F-barreled, the closed graph theorem could be applied to complete the proof. To show that $F(T)$ is F-barreled, let E be a closed noncompact subset of T. Since $F(T)$ is a Frechet space, T^* is 0-dimensional and Lindelöf and therefore T is 0-dimensional and Lindelöf. Thus E is Lindelöf and there exists a denumerable clopen cover (U_n) from which no finite subcover can be extracted. We may assume the family (U_n) to be pairwise disjoint. Since CE is open in T, $CE = \cup V_\mu$ where each V_μ is clopen so that $T = (\bigcup_{n=1}^\infty U_n) \cup (\bigcup_{n=1}^\infty V_{\mu_n})$ where the (V_{μ_n}) may be assumed to be pairwise disjoint. Defining $H_{2n} = V_{\mu_n}$, $H_{2n+1} = U_n$ and setting $L_m = H_m - \bigcup_{i=1}^{m-1} H_i$ then $T = \bigcup_{n=1}^\infty L_n$ where each L_n is clopen and (L_n) is pairwise disjoint. We note that E must intersect infinitely many L_n's lest E turn out to be covered by finitely many of the U_i. Now consider the function $f: T \to F$

defined by $f(t) = \sum_{i=1}^{\infty} a^n k_{L_i}(t)$ where $|a| > 1$. We observe that f is unbounded on E and therefore $F(T)$ with compact-open topology is F-barreled.*

We now prove a nonarchimedean version of a theorem of Warner ([15, p. 267]).

THEOREM 7. *Let the set of nontrivial continuous homomorphisms on the Frechet full algebra X be denoted by $\mathscr{M}$. Let $\mathscr{M}$ carry the weak-* (Gelfand) topology and $F(\mathscr{M})$ the compact-open topology. Then X is topologically isomorphic to $F(\mathscr{M})$.*

Proof. Carrying the topology of X over to $F(\mathscr{M})$ via the isomorphism ψ (Def. 1 of Sec. 1) and noting that $\mathscr{M}$ constitutes the set of nontrivial continuous homomorphisms of $F(\mathscr{M})$ into F, we see by the previous theorem that the proof is done.

For complex algebras, Warner ([15]) has proved that the "$\mathscr{M}$" of Theorem 7 is a k-space ($\mathscr{M}$ carries the weak topology generated by a sequence of compact sets). This question as well as an attempt to develop a substitute for concept of "Q-space" ([5, p. 271]) is investigated in subsequent papers ([16]).

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* As shown here, the hypothesis of Theorem 6 implies T to be Lindelöf. T being Lindelöf however implies that all homomorphisms of $F(T)$ into F are given by points of T ([**16**]).

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