Pacific Journal of Mathematics

FUNCTION ALGEBRAS OVER VALUED FIELDS

George Bachman, Edward Beckenstein and Lawrence Narici

Vol. 44, No. 1 May 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 1, 1973

FUNCTION ALGEBRAS OVER VALUED FIELDS

G. BACHMAN, E. BECKENSTEIN AND L. NARICI

In this paper we consider primarily algebras $F(T)$ of continuous funtions taking a topological space T into a complete nonarchimedean nontrivially valued field F. Some general properties of function algebras and topological algebras over valued fields are developed in §§1 and 2. Some principal results (Theorems 6 and 7) are analogs of theorems of Nachbin and Shirota, and Warner: Essentially that $F(T)$ with compact-open topology is F-barreled iff unbounded functions exist on closed noncompact subsets of T; and that full Fréchet algebras are realizable as function algebras $F(\mathscr{M})$ where $\mathscr{M}$ denotes the space of nontrivial continuous homomorphisms of the algebra.

Nachbin and Shirota's well-known result provides a necessary and sufficient condition for an algebra of realvalued continuous functions on a topological space to be barreled when it carries the compact-open topology. To develop an analog of Nachbin's theorem for F-valued functions, it is necessary to bypass the heavily real-number-oriented machinery on which his proof depends. We accomplish this in part by developing an ordering of the elements of a discretely valued field (Sec. 3, Def. 2) which serves to take the place of the usual ordering of the reals. We also consider a notion of "support" of a continuous F-valued linear functional on $F(T)$ (Sec. 3, Def. 3). The support notion is developed without measure theory or representation theorems for continuous linear functionals.

The results of the paper depend heavily on theorems proved by Ellis ([3]), Kaplansky ([7], [8]), and van Tiel ([14]), as well as the proofs of the major theorems as originally presented by Nachbin ([10]) and Warner ([15]) which provided the ideas for this line of approach.

Throughout the paper "algebra" (denoted by X or Y) includes the presence of an identity and commutativity. The underlying field F is assumed to be a complete nonarchimedean rank one nontrivially valued field. Unless otherwise stated, T denotes a 0-dimensional (a base for the topology consisting of closed and open sets exists) Hausdorff topological space and $F(T)$ the algebra of continuous functions from T into F with pointwise operations. The terms Banach space or Banach algebra are used throughout in the sense of [12].

1. Topological algebras over valued fields. In this section we discuss some basic properties of topological algebras over fields with valuation. We assume throughout that the underlying field F is a complete nonarchimedean rank one nontrivially valued field.

DEFINITION 1. A topological algebra X over F is *nonarchimedean locally multiplicatively F-convex* (NLMC) if there exists a base $\mathscr{B}$ of neighborhoods U of 0 in X such that for each $U \in \mathscr{B}$, (1) U is F-convex (i.e. if λ and μ are scalars such that $|\lambda|, |\mu| \leqq 1$, then $\lambda U + \mu U \subset U$), and (2) $UU \subset U$.

DEFINITION 2. A seminorm p on X is *nonarchimedean* and *multiplicative* respectively if for all $x, y \in X$ (1) $p(x + y) \leqq \max [p(x), p(y)]$ and (2) $p(xy) \leqq p(x)p(y)$.

PROPOSITION 1. *A topological algebra X is an NLMC algebra iff the topology on X is generated by a family P of nonarchimedean multiplicative seminorms.*

Proof. Given such a family P generating the topology on X, the sets $\{x \mid p_i(x) \leqq \varepsilon, p_1, \cdots, p_n \in P, 0 < \varepsilon \leqq 1\}$ form a base at 0 satisfying the condition of Definition 1.

Conversely, if $\mathscr{B}$ is a base at 0 satisfying the conditions of Definition 1, then, letting $p_U(x) = \inf \{|\mu| \mid x \in \mu U, \mu \in F\}$ the seminorms $(p_U)_{U \in \mathscr{B}}$ constitute the desired family P.

PROPOSITION 2. *If the valuation on F is discrete and X is an NLMC algebra, then there exists a family P' or nonarchimedean multiplicative seminorms generating the topology on X such that $p'(X) \subset |F|$ for each $p' \in P'$.*

Proof. Let P be a family of nonarchimedean multiplicative seminorms generating X's topology. For each $p \in P$ let $p'(x) = \inf \{|\mu| \mid |\mu| \geqq p(x)\}$. Each such p' is clearly nonarchimedean and multiplicative. Moreover since $p(x) \leqq p'(x) \leqq |\mu^{-1}| p(x)$ for any nonzero $\mu \in F$ such that $|\mu| < 1$ and $|\mu|$ generates the value group of F, P' will also generate the topology on X.

DEFINITION 3. An NLMC algebra X is *discrete* if there exists a family P of nontrivial nonarchimedean multiplicative seminorms generating the topology on X such that each p in P is discrete [the only limit point of $p(X)$ is 0].

PROPOSITION 3. *A Hausdorff NLMC algebra X is discrete iff F is discretely valued.*

Proof. Use Prop. 2.

If X is a topological algebra over C, the complex numbers, then we can identify the nontrivial continuous homomorphisms of X into C with the closed maximal ideals in X ([9, p. 13]). This is no longer true for noncomplex algebras, and we single out those algebras in which the $1-1$ correspondence still obtains for special attention.

DEFINITION 4. A commutative Hausdorff NLMC algebra X with identity e is a *Gelfand algebra* if for every closed maximal ideal $M \subset X$ the factor algebra X/M (with quotient topology) is topologically isomorphic to F.

Associated with the nontrivial nonarchimedean multiplicative seminorms p generating the topology on an NLMC algebra X, are nonarchimedean normed algebras X/N_p where N_p is the ideal $p^{-1}(0)$ where X/N_p is normed by taking $\|x + N_p\| = p(x)$. The completions X_p of these normed algebras are referred to as *factor algebras.*

PROPOSITION 4. *If X is a Gelfand algebra and X/N_p is complete, then X/N_p is a Gelfand algebra.*

Proof. Let π_p denote the continuous homomorphism $x \rightarrow x + N_p$ from X onto X/N_p. We observe that if M is a maximal ideal in the Banach algebra X/N_p, then M is closed; thus $\pi_p^{-1}(M)$ is a closed maximal ideal in X containing N_p. For any $x \in X$ there exists $\mu \in F$ such that $x\text{-}\mu e \in \pi_p^{-1}(M)$ (X is a Gelfand algebra), so that $\pi_p(x) - \mu\pi_p(e) \in M$ where e is the identity of X. Thus $(X/N_p)/M$ is algebraically isomorphic to F. Since M is closed, the factor structure is a one-dimensional Hausdorff topological vector space and is therefore topologically isomorphic to F.

PROPOSITION 5. *Let P be a saturated family of seminorms generating the topology on the NLMC algebra X and let $(X_p)_{p \in P}$ denote the associated factor algebras. If each X_p is a Gelfand algebra, then X is a Gelfand algebra.*

Proof. Let M be a closed maximal ideal in X. By [1, p. 466] there exists $p \in P$ such that $M \supset N_p$ and $\inf\{p(e-x) \mid x \in M\} > 0$. Consequently $\pi_p(M)$ is a proper ideal in X/N_p and $\pi_p(e)$ is not an adherence point of $\pi_p(M)$. Thus $\overline{\pi_p(M)}$ is a proper ideal in X_p and is therefore contained in a closed maximal ideal $N \subset X_p$. Since X_p is a Gelfand algebra, N is the kernel of a continuous nontrivial homomorphism f_p taking X_p into F. Hence $f = f_p\pi_p$ is a continuous nontrivial homomorphism taking X into F. It follows from elementary considerations that the kernel of f is equal to M. Consequently X/M is seen to be algebraically—hence topologically—isomorphic to F.

A result similar in spirit to this can be found in [2, p. 175]. We turn next to some examples.

EXAMPLE 1. Let F be a local field, let T be a 0-dimensional Hausdorff space and let $F(T)$ carry the topology of uniform convergence on compact sets. The topology on $F(T)$ is generated by the nonarchimedean multiplicative seminorms p_K where K is a compact subset of T and for any $x \in F(T)$, $p_K(x) = \sup_{t \in K} |x(t)|$. We may identify $F(T)/p_K^{-1}(0)$ with a subalgebra of $F(K)$. Moreover we may construct a 'Stone-Cech' compactification $\beta_F T$ of T as is done in [3, p. 243] utilizing the compact valuation ring V of F in place of the compact interval [0, 1]. Since V is Hausdorff and 0-dimensional, $\beta_F T$ will be compact, Hausdorff and 0-dimensional. Thus the Ellis-Tietze extension theorem ([4]) applies and any function continuous on K may be extended to a function continuous on $\beta_F T$. It follows that $F(T)/p_K^{-1}(0) = F(K)$.

The continuous nontrivial homomorphisms of $F(K)$ into F are in $1-1$ correspondence with the points t of K ([11]) and using this result it can be shown [9, p. 31] that the points of T generate the continuous nontrivial homomorphisms of $F(T)$ into F.*

Topological algebras X for which all homomorphisms of X into F are continuous are called *functionally continuous* [9, p. 51]). What follows is an example of such an algebra.

EXAMPLE 2. Let F be any complete nonarchimedean nontrivially valued field and T a 0-dimensional Hausdorff space. $F(T)$ carries the compact-open topology. A subalgebra X of $F(T)$ is "closed under inverses" if when $x \in X$ and $x^{-1} \in F(T), x^{-1} \in X$. We apply Michael's proof [9, p. 54] and observe that if Conditions 1 and 2 below are satisfied, then the homomorphisms of X are generated by the points of T and therefore X is functionally continuous.

1. For any $x_1, \cdots, x_n \in X$ such that $\bigcap_{i=1}^n x_i^{-1}(0) = \varnothing$, there exists $y_1, \cdots, y_n \in X$ such that $\Sigma x_i y_i = e$ where e is the constant function $e(t) = 1$ for all $t \in T$.

2. For some positive integer m there exists $x_1, \cdots, x_m \in X$ such that for all $\mu_1, \cdots, \mu_m \in F$, $\cap (x_i - \mu_i e)^{-1}(0)$ is compact.

We note that if $X = F(T)$, then by the results of a sequel to this paper [16], it follows that X satisfies statement 1. If, in addition, there exists a bijection $x \in X$, then X satisfies 2. Hence if we take $T = F$ and let T carry any 0-dimensional Hausdorff topology finer than

* The result of Example 1 actually obtains if F is any complete nonarchimedean nontrivially valued field as it can be shown in this case that a bounded continuous function defined on a compact subset K of T mapping into F can be extended to a bounded continuous function mapping T into F. The same comment applies to Example 1, parts (c), (d), and (e) of Sec. 2.

the valuation topology on F, the nontrivial homomorphisms of the algebra $F(T)$ taking values in F are generated by the points of T.

2. Function algebras over valued fields. In this section we discuss function algebras over valued fields. First we prove a version of a theorem of Kaplansky ([7, p. 173]) which is relevant to the material to follow; we include this proof because there seems to be an inconsistency in the use of "totally disconnected" in [7].

LEMMA 1. (Kaplansky) *Let T be a topological space and let $F(T)$ be endowed with the topology of uniform convergence on compact sets. I is a closed ideal in $F(T)$, iff there is some closed subset H of T such that $I = \{f \in F(T) \mid f(H) = \{0\}\}$. I is a closed maximal ideal in $F(T)$ iff there is some $t \in T$ such that $I = \{f \mid f(t) = 0\}$.*

Proof. Suppose I is closed in $F(T)$ and let $H = \bigcap_{g \in I} g^{-1}(0)$. Letting $J(H) = \{f \mid f(H) = \{0\}\}$, we see that $I \subset J(H)$, and that $J(H)$ is a closed ideal. We show that if $f \in J(H)$, then $f \in I$.

Let K be any compact subset of T. If $y \in K$, then as I is an ideal, there exists $g_y \in I$ such that $g_y(y) = f(y)$. Since the clopen sets $\{U_y \mid y \in K\}$ where $U_y = \{x \in T \mid |f(x) - g_y(x)| < \varepsilon\}$ cover K for any fixed $\varepsilon > 0$, there exist $y_1, \cdots, y_n$ such that $K \subset \bigcup_{i=1}^{n} U_{y_i}$. Since the sets U_{y_i} are clopen, we see that there exist pairwise disjoint clopen sets W_i such that $K \subset \bigcup_{i=1}^{n} W_i$ where $W_i \subset U_{y_i}$ for each i. Letting k_A denote the characteristic function of the set A, we see that if $h = \sum_{i=1}^{n} g_{y_i} k_{W_i}$, then $h \in I$ and $\sup_{t \in K} |h(t) - f(t)| < \varepsilon$. As $\varepsilon > 0$ can be made arbitrarily small, it follows that $f \in \bar{I} = I$..

In the proof to follow, "totally disconnected" is used as in [13, p. 380]: distinct points may be separated by clopen sets.

THEOREM 1. (Kaplansky) *Let S and T be 0-dimensional Hausdorff spaces. Let $F(S)$ and $F(T)$ carry their compact-open topologies and suppose that $F(T)$ is topologically isomorphic to $F(S)$. Then S and T are homeomorphic.*

Proof. Let A be a topological isomorphism from $F(S)$ onto $F(T)$. If K is a closed subset of S and $J(K)$ denotes the ideal of functions that vanish on K, note that a mapping A' is defined by $A(J(\{s\})) = J(\{t\}) = J(\{A'(s)\})$ for some $t \in T$; i.e. $A' \colon S \to T$ is such that $A'(s) = t$, and is well-defined as T is totally disconnected. Since A is injective and S is totally disconnected, then A' is injective as well. For any $t \in T$, $J(\{t\}) = A(M)$ where M is a closed maximal ideal in $F(S)$.

Since $M = J(\{s\})$ for some $s \in S$, A' is seen to be surjective.

Clearly $(A')^{-1} = (A^{-1})'$ so to show that A' is a homeomorphism, it suffices to show that A' is a closed map. To this end, since S is 0-dimensional, $K = \bigcap_{g \in J(K)} g^{-1}(0)$; since $J(K) = \bigcap_{s \in K} J(\{s\})$, it follows that $A(J(K)) = J(A'(K)) = \bigcap_{s \in K} J(\{A'(s)\})$. If $t \notin A'(K)$, then $t = A'(s)$ where $s \notin K$. Thus $J(K) \not\subset J(\{s\})$ and $J(A'(K)) \not\subset J(\{t\})$. As $J(A'(K)) = J(\overline{A'(K)}) \not\subset J(\{t\})$, we see that $t \notin \overline{A'(K)}$ and therefore $A'(K) = \overline{A'(K)}$.

EXAMPLE 1. Let T be a totally disconnected Hausdorff space and let $F(T)$ carry the compact-open topology. We note immediately that the set of evaluation maps constitutes a set of distinct continuous homomorphisms of $F(T)$ into F. Moreover properties (a)—(e) also hold.

(a) If K is a compact subset of T, p_K is as in Ex. 1 of Sec. 1, and $N_K = p_K^{-1}(0)$, then the completion of the normed algebra $F(T)/N_K$ is $F(K)$.

Proof. Since T is totally disconnected, the characteristic functions in $F(T)$ separate the points of T. Thus the functions $f|_K$ as f runs through $F(T)$ separate points in K. The desired result now follows from an application of Kaplansky's Stone-Weierstrass theorem ([8] or [12] p. 161).

(b) With "V^*-algebra" as in [12, p. 148], if T is locally compact, then $F(T)$ is the projective limit of V^*-algebras as in [9, p. 17].

Proof. The complete NLMC algebra $F(T)$ is the projective limit of the factor algebras $F(K)$ as K runs through the compact subsets of T and each $F(K)$ is a V^*-algebra.

(c) If T is ultranormal and F is a local field, then $F(T)/N_K = F(K)$.

Proof. Use the Ellis-Tietze extension theorem of [4].

(d) If T is 0-dimensional and F is a discretely valued field, then $F(T)/N_K = F(K)$ for any compact subset K of T.

Proof. Apply a modification of the Ellis-Tietze extension theorem to functions $f \in F(K)$ and thereby extend f continuously to a 'Stone-Cech' compactification $\beta_H T$ where H is any local field. Where Ellis used local compactness of the field F, we use discreteness of the valuation on F, and compactness of $\beta_H T$.

(e) The points of T constitute all continuous homomorphisms of $F(T)$ into F when F is discretely valued.

Proof. See Ex. 1 of Sec. 1 and use (d).

3. Main results. Let X be a NLMC algebra over a discretely valued F. Then, as in the classical case ([9, p. 33]), if X is the projective (dense inverse) limit of a family $(F(K_n))$ of Gelfand V^*-algebras by mappings $\pi_{mn}\colon F(K_m) \to F(K_n)$, $m > n$, where (K_n) is a family of compact 0-dimensional Hausdorff spaces (it following that K_n is homeomorphically embedded in K_m), then X is topologically isomorphic to $F(\cup K_n)$ where* $F(\cup K_n)$ carries the compact-open topology. Moreover in this case $\cup K_n$ can and will be identified with the set of all nontrivial continuous homomorphisms of X into F and carries the weak topology generated by (K_n).

DEFINITION 1. Let $\mathscr{M}$ denote the nontrivial continuous homomorphisms of an MLHC algebra X over F into F, and let $\mathscr{M}$ carry the weak-* topology. Let $F(\mathscr{M})$ denote the algebra of continuous functions mapping $\mathscr{M}$ into F with compact open topology and consider the map $\psi\colon X \to F(\mathscr{M})$ where, for any $x \in X$, $\psi(x)(h) = h(x)$ for each $h \in \mathscr{M}$. X is called a *full* algebra if the homomorphism ψ is an isomorphism of X onto $F(\mathscr{M})$.

In [9] E. A. Michael stated that he did not know whether or not ψ was a topological isomorphism in the case where X is a Fréchet full algebra. S. Warner proved that this was true in the classical case ([15, p. 269]). In this section we show that ψ is a topological isomorphism if F is a local field (Theorem 7). It then follows according to some results of van Tiel [14] that X is the projective limit of a sequence $(F(K_n))$ of Gelfand V^*-algebras where $K_n = V_n^\circ \cap \mathscr{M}$ (V_n° is the polar of a neighborhood V_n of 0 in X coming from a base of F-convex closed neighborhoods of 0). Thus we will have a partial converse of the result which was described in the opening paragraphs of this section. We also note that by Prop. 5 of Sec. 1, X is a Gelfand algebra under the hypothesis just mentioned.

In what follows F is assumed to be discretely valued. In some cases it will also be assumed that F is a local field so that certain standard results from the duality theory of topological vector spaces ([14]) may be used.

DEFINITION 2. Let F be discretely valued and let $(a_\mu)_{\mu \in H}$ be a system of distinct representatives of the cosets in the residue class field of F. We may assume that H is totally ordered where μ_0 corresponding to $a_{\mu_0} = 0$ is the first element. Let $\pi \in F$ be such that $|\pi| < 1$ and $|\pi|$ is a generator of the value group of F. If a and b are any two elements of F there exist (a_{μ_i}) and (a_{λ_i}) such that $a = \sum_{i=N}^{\infty} a_{\mu_i}\pi^i$ and $b = \sum_{i=N}^{\infty} a_{\lambda_i}\pi^i$. We now define the *supremum*, sup(a, b),

---

* We may assume $K_n \subset K_{n+1}$ as there exist sets K_n' such that $\mathscr{M} = \cup K_n'$ with $K_n' \subset K_{n+1}'$, and K_n' homeomorphic to K_n for all n.

of a and b as:

$$\sup(a, b) = \begin{cases} a \text{ if } |a| > |b| \\ b \text{ if } |b| > |a| \\ a \text{ if } a = b \\ a \text{ if } |a| = |b|, a_{\mu_i} = a_{\lambda_i} \text{ for } i = N, \cdots, j-1 \text{ and } \mu_j > \lambda_j \end{cases}$$

LEMMA 1. *Let T be a topological space and let f and g be continuous functions mapping T into F. Then the function defined at each $t \in T$ by* $\sup(f(t), g(t))$ *and denoted by* $\sup(f, g)$ *is continuous.*

Proof. Suppose (t_s) is a net in T converging to t. We show that $\sup(f, g)(t_s)$ converges to $\sup(f, g)(t)$. Letting $f(t) = a$ and $g(t) = b$, we need only consider the last possibility for $\sup(a, b)$, the first three being trivial. Choose $\varepsilon > 0$ such that $\varepsilon < |\pi|^j$. For r such that $|f(t_s) - f(t)| < \varepsilon$ and $|g(t_s) - g(t)| < \varepsilon$ for $s \geqq r$, it follows that

$$f(t_s) - f(t) = \sum_{i=M}^{\infty} a^s_{\mu_i}\pi^i \text{ and } g(t_s) - g(t) = \sum_{i=M}^{\infty} a^s_{\lambda_i}\pi^i$$

where $M > j$. We may also write

$$f(t_s) = \sum_{1=N}^{j} a_{\mu_i}\pi^i + \sum_{i=j+1}^{\infty} a^s_{\mu_i}\pi^i \text{ and } g(t_s) = \sum_{i=N}^{j} a_{\lambda_i}\pi^i + \sum_{i=j+1}^{\infty} a^s_{\lambda_i}\pi^i .$$

Thus, since $a_{\mu_i} = a_{\lambda_i}$ for $i = N, \cdots, j-1$, and $\mu_j > \lambda_j$, it follows that $\sup(f, g)(t_s) = f(t_s)$ for $s \geqq r$. Thus $\sup(f, g)(t_s) = f(t_s) \rightarrow f(t) = \sup(f, g)(t)$.

LEMMA 2. *Let $F(T)$ denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space T into the discretely valued F, with compact-open topology. If V is an F-barrel (closed absorbent F-convex set) in $F(T)$, then there is some $\delta > 0$ such that $\sup_{t \in T}|f(t)| \leqq \delta$ implies that $f \in V$.*

Proof. Let B be the sup-norm Banach space of all bounded functions from T into F. We note that $V \cap B$ is an F-barrel in B. Since B is F-barreled ([14, p. 268]) there is some $\delta > 0$ such that $\sup_{t \in T}|f(t)| \leqq \delta$ which implies that $f \in V \cap B$.

LEMMA 3. *Let V, F, T and $F(T)$ be as in Lemma 2, and suppose that for some compact subset K of T, $\{f \mid f(K) = \{0\}\} \subset V$. Then there is some $\mu > 0$ such that whenever $\sup_{t \in K}|f(t)| < \mu$, then $f \in V$. Thus V is a neighborhood of 0 in $F(T)$.*

Proof. Let $a \in F$ and denote the function sending each $t \in T$ into a by a. With δ as in Lemma 2, choose $a \in F$ such that $0 < |a| \leqq$

$\delta/2$. Choosing an integer n so that $\delta/n < |a|$, let $f \in F(T)$ be such that $\sup_{t \in K} |f(t)| \leqq \delta/n$. With $g = \sup(f, a) - a$, it follows that $g(t) = 0$ for each t in K. Thus $g \in V$. Since $|f(t) - g(t)| \leqq |a| \leqq \delta/2$ for all $t \in T$, it follows that $f - g \in V$. Since V is F-convex, $g + (f - g) = f \in V$, and the proof is complete.

We continue towards nonarchimedean analogs of theorems of Nachbin (Theorem 3) and Warner (Theorem 7). First we consider a notion of *support* of a linear functional which serves to replace the classical notion used by Nachbin.

In Lemmas 4 and 5 $F(T)$ again denotes the algebra of continuous functions from the 0-dimensional Hausdorff space T into F with compact-open topology and φ denotes a member of the continuous dual $F(T)'$ of $F(T)$. For any subset S of T, k_S denotes the characteristic function of S taking values in F and we note that $k_S \in F(T)$ iff S is clopen. Let $\mathscr{S}$ denote the family of subsets U of T such that U is clopen and $\varphi(fk_U) = 0$ for all $f \in F(T)$.

LEMMA 4. *The family* $\mathscr{S}$ *has the following properties*: (1) *If* U *is a clopen subset of* $G \in \mathscr{S}$, *then* $U \in \mathscr{S}$; (2) $\mathscr{S}$ *is a ring of sets.*

Proof. To prove (1) we observe that $k_U = k_G k_U$. (2) follows readily from (1).

DEFINITION 3. The *support* of φ, F_φ, is defined to be $C(\cup \mathscr{S})$.

We observe that since φ is continuous there is some compact set $K \subset T$ and an integer N such that if $f \in F(T)$, then $|\varphi(f)| \leqq N \sup_{t \in K} |f(t)|$. Thus, if f vanishes on K, then $\varphi(f) = 0$.

THEOREM 1. *In the same notation as above* (1) $F_\varphi \subset K$ *and therefore* F_φ *is compact,* (2) *if* φ *is nontrivial, then* F_φ *is not empty, and* (3) *if* $G \subset T$ *is open and* $G \cap F_\varphi$ *is not empty, then there exists* $f \in F(T)$ *such that* $f(CG) = \{0\}$ *and* $\varphi(f) = 1$.

Proof. (1) If G is a clopen subset of CK, then—since k_G vanishes on K—$\varphi(fk_G) = 0$ and $G \in \mathscr{S}$.

(2) If F_φ is empty, $T = \cup \mathscr{S}$, and it follows that for some $U_i \in \mathscr{S}$, $K \subset \cup_{i=1}^n U_i = G$. Since $\mathscr{S}$ is a ring of sets, $G \in \mathscr{S}$ and since CG is clopen and contained in CK, $\varphi(f) = \varphi(fk_{CG}) = 0$ for all $f \in F(T)$. But then φ is trivial.

(3) If $G \cap F_\varphi \neq \emptyset$, there is some $t \in G \cap F_\varphi$. Since T is 0-dimensional, $t \in U \subset G$ where U is clopen. Since $U \cap F_\varphi \neq \emptyset$, then $U \notin \mathscr{S}$ and there is some $g \in F(T)$ such that $\varphi(gk_U) \neq 0$. We of course may assume that $\varphi(gk_U) = 1$. Letting $gk_U = f$, (3) is seen to be proved.

In order to apply this notion of support to our version of Nachbin's theorem (Theorem 3) we require that F_φ have the property that if f vanishes on F_φ, $\varphi(f) = 0$. We now develop a case where this is true and which makes the notion applicable to Theorem 3 as well as settling Michael's question in this setting (Theorem 7).

LEMMA 5. *Suppose that $\varphi(g) = 0$ for any $g \in F(T)$ which vanishes on any clopen set G containing F_φ. Then if f vanishes on F_φ, $\varphi(f) = 0$.*

Proof. Suppose that $f \in F(T)$ vanishes on F_φ, and let $A_n = \{t \in T \mid |f(t)| < 1/n\}$ $(n = 1, 2, \cdots)$. As $F_\varphi \subset A_n$ for any n and A_n is clopen $\varphi(f) = \varphi(fk_{A_n}) + \varphi(f(1 - k_{A_n}))$. By the hypothesis, since $f(1 - k_{A_n})$ vanishes on A_n, $\varphi(f) = \varphi(fk_{A_n})$. Let K be a compact subset of T such that $|\varphi(f)| \leqq N \sup_{t \in K} |f(t)|$. Hence $|\varphi(f)| = |\varphi(fk_{A_n})| \leqq N \sup_{t \in K} |fk_{A_n}(t)| < N/n$. Since this is true for every n, $\varphi(f) = 0$.

THEOREM 2. *Let T be a Lindelöf space. Then if f vanishes on F_φ, $\varphi(f) = 0$.*

Proof. Let G be a clopen subset containing F_φ. Since CG is closed, CG is Lindelöf. Since $CG \subset CF_\varphi = \cup \mathscr{S}$, there exist $U_i \in \mathscr{S}$ such that $CG \subset \bigcup_{i=1}^\infty U_i$. Since $\mathscr{S}$ is a ring, we may assume that the sets U_i are pairwise disjoint. Since $CG \cap U_i = V_i$ is clopen and contained in U_i then $V_i \in \mathscr{S}$. Thus $k_{CG} = \sum_{i=1}^\infty k_{V_i}$ in the topology of pointwise convergence on $F(T)$. We claim that the "pointwise convergence" of the preceding sentence may be replaced by "uniform convergence on compact sets."

To prove this last statement, let L be a compact subset of T and consider $L \cap CG$. As $L \cap CG$ is compact and contained in $\bigcup_{i=1}^\infty V_i$ there is some integer N_L such that $n \geqq N_L$ implies that $L \cap CG$ is contained in $\bigcup_{i=1}^n V_i$. But $CG \subset \bigcup_{i=1}^n V_i$ so $L \cap CG = L \cap (\bigcup_{i=1}^n V_i)$. Thus for $n \geqq N_L$, CG and $\bigcup_{i=1}^n V_i$ have the same points in common with L, and $\sup_{t \in L} |(k_{CG} - \sum_{i=1}^n k_{V_i})(t)| = 0$ for $n \geqq N_L$. Since L was an arbitrary compact set, the series is seen to converge in the compact-open topology and $\varphi(fk_{CG}) = \sum_{i=1}^n \varphi(fk_{V_i}) = 0$.

We now present a version of a theorem of Nachbin ([10, p. 472])

THEOREM 3. *Let $F(T)$ denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space T into the discretely valued field F, with compact-open topology. Suppose that for each $\varphi \in F(T)'$, f vanishing on F_φ implies $\varphi(f) = 0$. Then $F(T)$ is F-barreled iff for every $E \subset T$ which is closed and not compact there is some $f \in F(T)$*

*which is unbounded on E.**

Proof. Suppose that the condition holds and let V be an F-barrel in $F(T)$. To show that V is a neighborhood of 0 in $F(T)$ we begin by letting $K = \overline{\bigcup_{\varphi \in V^0} F_\varphi}$. If K is not compact, let f be unbounded on K and consider the sets $A_n = \{t \in T \mid |f(t)| > n\}$, $n = 1, 2, \cdots$. Each A_n is clopen and $A_n \cap K \neq \emptyset$. Thus there is some $F_{\varphi_n} \subset K$ such that $A_n \cap F_{\varphi_n} \neq \emptyset$. By Theorem 1 (3) there exists $f_n \in F(T)$ such that f_n vanishes outside of A_n and $\varphi_n(f_n) = 1$. Since $\bigcap_{n=1}^{\infty} A_n = \emptyset$, the function $f = \sum_{n=1}^{\infty} a_n f_n$ is a continuous function for any choice of $a_n \in F$. As it is clear that $A_m \cap F_{\varphi_n} = \emptyset$ for all sufficiently large m, we may (by considering a subsequence) assume that $\varphi_n(f_m) = 0$ for all $m > n$. By a proper choice of a_n we see that $|\varphi_n(f)| \to \infty$ and as $\varphi_n \in V^0$, af cannot belong to $V^{00} = V$ no matter how small $|a|$ is. Thus we contradict the fact that V is absorbent and K must be compact. If f vanishes on K, then f vanishes on F_φ for all $\varphi \in V^0$. Thus $f \in V^{00} = V$ so, by Lemma 3, V is a neighborhood of 0.

To prove the converse, let $F(T)$ be F-barreled and E be a closed noncompact subset of T. Let $V = \{f \mid \sup_{t \in E} |f(t)| \leqq \delta\}$, $\delta > 0$, and let K be a compact subset of T. As $E \cap CK \neq \emptyset$, using $\beta_H T$ as in Sec. 2 Ex. 1 (d) we may assert the existence of a sequence (f_N) of functions which vanishes on K but $|f_N(t_N)| \geqq N$ for any positive integer N and some $t_N \in E$. Thus the set $\{f \mid \sup_{t \in K} |f(t)| \leqq \varepsilon\} \not\subset V$ for any $\varepsilon > 0$ and V is not a neighborhood of 0. It follows that V is not absorbing and there exists $f \in F(T)$ which is unbounded on E.

COROLLARY. *Let T be a 0-dimenional Hausdorff Lindelöf space and F a discretely valued field. Then $F(T)$ is F-barreled.*

Proof. We refer to Theorem 2 and the construction of the function in the proof of Theorem 6 for the proof of the corollary.

THEOREM 4. *Suppose the 0-dimensional Hausdorff space $T = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact, $K_n \subset K_{n+1}$, and each compact subset of T is contained in some K_n* (i.e. *T is hemicompact*). *Then denoting T endowed with the weak topology* ([3], p. 131) *generated by the sets* (K_n) *as T_w, $F(T)$ is dense in $F(T_w)$, each algebra carrying its compact-open topology.*

Proof. Since the topology of T_w is clearly stronger than that of T, $F(T) \subset F(T_w)$. We note that the topology of T_w restricted to K_n is

* In a sequel to this paper we show that Theorem 2 is true for any 0-dimensional Hausdorff space T and any complete nonarchimedean nontrivially valued field F. Thus Theorem 3 is true for all spaces T. We also show that the result of Theorem 3 holds of F is spherically complete ([**16**]).

equal to the topology K_n inherits from T and the compact subsets of T_w lie in the sets K_n. Thus $F(T)$ is a topological subspace of $F(T_w)$. Using Sec. 2 Ex. 1 (d), $F(T)/N_K = F(K)$ for any compact set $K \subset T$ and it follows that $F(T)$ is dense in $F(T_w)$.

THEOREM 5. *Let everything be as in the preceding theorem. If $F(T)$ is complete then $T = T_w$ iff T_w is 0-dimensional.*

Proof. If $F(T)$ is complete, then $F(T) = F(T_w)$. Since they are topologically isomorphic under the identity map by the proof of Theorem 4, if T_w is 0-dimensional, then $T = T_w$ by Theorem 1 of Sec. 2. We may also observe that the functions of $F(T)$ generate the topology of the space T while those of $F(T_W)$ generate the topology of T_W. Thus as $F(T) = F(T_W)$, the topologies are equal.

THEOREM 6. *Let $F(T)$ denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space T into the local field F and suppose that $F(T)$ is a complete locally F-convex metric space with topology $\mathscr{T}$. If the homomorphisms determined by the points of T are the $\mathscr{T}$-continuous homomorphisms, then $\mathscr{T}$ is the compact-open topology.*

Proof. Let the set of evaluation maps determined by T be denoted by T^* and let T^* carry the Gelfand topology (i.e. the weakest topology for T^* with respect to which the maps $t \to x(t)$ of T^* into F are continuous for each $x \in F(T)$). Since T is 0-dimensional the Gelfand topology coincides with the original topology on T, i.e. T and T^* are homeomorphic. Since $(F(T), \mathscr{T})$ is F-barreled ([14, p. 268]), the polar of any compact subset of T^* is a neighborhood of 0 in $F(T)$. Thus, identifying T and T^*, $\mathscr{T}$ is seen to be stronger than the compact-open topology on $F(T)$. If $F(T)$ with compact-open topology could be shown to be F-barreled, the closed graph theorem could be applied to complete the proof. To show that $F(T)$ is F-barreled, let E be a closed noncompact subset of T. Since $F(T)$ is a Frechet space, T^* is 0-dimensional and Lindelöf and therefore T is 0-dimensional and Lindelöf. Thus E is Lindelöf and there exists a denumerable clopen cover (U_n) from which no finite subcover can be extracted. We may assume the family (U_n) to be pairwise disjoint. Since CE is open in T, $CE = \cup V_\mu$ where each V_μ is clopen so that $T = (\bigcup_{n=1}^{\infty} U_n) \cup (\bigcup_{n=1}^{\infty} V_{\mu_n})$ where the (V_{μ_n}) may be assumed to be pairwise disjoint. Defining $H_{2n} = V_{\mu_n}$, $H_{2n+1} = U_n$ and setting $L_m = H_m - \bigcup_{i=1}^{m-1} H_i$ then $T = \bigcup_{n=1}^{\infty} L_n$ where each L_n is clopen and (L_n) is pairwise disjoint. We note that E must intersect infinitely many L_n's lest E turn out to be covered by finitely many of the U_i. Now consider the function $f: T \to F$

defined by $f(t) = \sum_{i=1}^{\infty} a^n k_{L_i}(t)$ where $|a| > 1$. We observe that f is unbounded on E and therefore $F(T)$ with compact-open topology is F-barreled.*

We now prove a nonarchimedean version of a theorem of Warner ([15, p. 267]).

THEOREM 7. *Let the set of nontrivial continuous homomorphisms on the Frechet full algebra X be denoted by $\mathscr{M}$. Let $\mathscr{M}$ carry the weak-* (Gelfand) topology and $F(\mathscr{M})$ the compact-open topology. Then X is topologically isomorphic to $F(\mathscr{M})$.*

Proof. Carrying the topology of X over to $F(\mathscr{M})$ via the isomorphism ψ (Def. 1 of Sec. 1) and noting that $\mathscr{M}$ constitutes the set of nontrivial continuous homomorphisms of $F(\mathscr{M})$ into F, we see by the previous theorem that the proof is done.

For complex algebras, Warner ([15]) has proved that the "$\mathscr{M}$" of Theorem 7 is a k-space ($\mathscr{M}$ carries the weak topology generated by a sequence of compact sets). This question as well as an attempt to develop a substitute for concept of "Q-space" ([5, p. 271]) is investigated in subsequent papers ([16]).

REFERENCES

1. R. Arens, *A generalization of normed rings*, Pacific. J. Math., **2** (1952), 455-471.
2. ———, *Dense inverse limit rings*, Mich. J. Math., **5** (1958), 169-182.
3. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
4. R. L. Ellis, *A nonarchimedean analog of the Tietze-extension theorem*, Indag. Math., **29** (1967), 332-333.
5. L. Gillman and M. Jerison, *Rings of continuous functions*, van Nostrand, Princeton, N.J., 1960.
6. B. E. Johnson, *Continuity of homomorphisms of topological algebras*, Proc. Camb. Phil. Soc., (1964), 171-172.
7. I. Kaplansky, *Topological rings*, Amer. J. Math., **69** (1947), 153-183.
8. ———, *The Weierstrass theorem in fields with valuation*, Proc. Amer. Math. Soc., **1** (1950), 356-357.
9. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Memoirs Amer. Math. Soc., no. 11, 1952.
10. L. Nachbin, *Topological vector spaces of continuous functions*, Proc. Nat. Acad. Sci. U.S.A., **40** (1954), 471-474.
11. L. Narici, *On nonarchimedean Banach algebras*, Arch. Math. (Basel), **19** (1968), 428-435.
12. L. Narici, E. Beckenstein, and G. Bachman, *Functional analysis and valuation theory*, Marcel Dekker, New York, 1971.
13. M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., **41** (1937), 375-481.

* As shown here, the hypothesis of Theorem 6 implies T to be Lindelöf. T being Lindelöf however implies that all homomorphisms of $F(T)$ into F are given by points of T ([**16**]).

14. J. Van Tiel, *Espaces localement K-convexes*, Indag. Math., **27** (1965), 249-289.
15. S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J., **25** (1958), 265-282.
16. G. Bachman and E. Beckenstein, L. Narici, *Function Algebras over Valued Fields*, I-IV Atti Accad. Naz. dei Lincei, Rend. Cl. di fis. mat., e nat., (1971-1972).

Received June 7, 1971.

ST. JOHN'S UNIVERSITY
NOTRE DAME COLLEGE
STATEN ISLAND, NEW YORK
AND
POLYTECHNIC INSTITUTE OF BROOKLYN
AND
ST. JOHN'S UNIVERSITY
JAMAICA, NEW YORK 11439

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 44, No. 1 May, 1973