FUNCTION ALGEBRAS OVER VALUED FIELDS

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In this paper we consider primarily algebras $F(T)$ of continuous functions taking a topological space $T$ into a complete nonarchimedean nontrivially valued field $F$. Some general properties of function algebras and topological algebras over valued fields are developed in §§1 and 2. Some principal results (Theorems 6 and 7) are analogs of theorems of Nachbin and Shirota, and Warner: Essentially that $F(T)$ with compact-open topology is $F$-barreled iff unbounded functions exist on closed noncompact subsets of $T$; and that full Fréchet algebras are realizable as function algebras $F(\mathcal{M})$ where $\mathcal{M}$ denotes the space of nontrivial continuous homomorphisms of the algebra.

Nachbin and Shirota’s well-known result provides a necessary and sufficient condition for an algebra of real-valued continuous functions on a topological space to be barreled when it carries the compact-open topology. To develop an analog of Nachbin’s theorem for $F$-valued functions, it is necessary to bypass the heavily real-number-oriented machinery on which his proof depends. We accomplish this in part by developing an ordering of the elements of a discretely valued field (Sec. 3, Def. 2) which serves to take the place of the usual ordering of the reals. We also consider a notion of “support” of a continuous $F$-valued linear functional on $F(T)$ (Sec. 3, Def. 3). The support notion is developed without measure theory or representation theorems for continuous linear functionals.

The results of the paper depend heavily on theorems proved by Ellis ([3]), Kaplansky ([7],[8]), and van Tiel ([14]), as well as the proofs of the major theorems as originally presented by Nachbin ([10]) and Warner ([15]) which provided the ideas for this line of approach.

Throughout the paper “algebra” (denoted by $X$ or $Y$) includes the presence of an identity and commutativity. The underlying field $F$ is assumed to be a complete nonarchimedean rank one nontrivially valued field. Unless otherwise stated, $T$ denotes a 0-dimensional (a base for the topology consisting of closed and open sets exists) Hausdorff topological space and $F(T)$ the algebra of continuous functions from $T$ into $F$ with pointwise operations. The terms Banach space or Banach algebra are used throughout in the sense of [12].

1. Topological algebras over valued fields. In this section we discuss some basic properties of topological algebras over fields with valuation. We assume throughout that the underlying field $F$ is a
complete nonarchimedean rank one nontrivially valued field.

**Definition 1.** A topological algebra $X$ over $F$ is nonarchimedean locally multiplicatively $F$-convex (NLMC) if there exists a base $\mathcal{B}$ of neighborhoods $U$ of 0 in $X$ such that for each $U \in \mathcal{B}$, (1) $U$ is $F$-convex (i.e. if $\lambda$ and $\mu$ are scalars such that $|\lambda|, |\mu| \leq 1$, then $\lambda U + \mu U \subset U$), and (2) $UU \subset U$.

**Definition 2.** A seminorm $p$ on $X$ is nonarchimedean and multiplicative respectively if for all $x, y \in X$ (1) $p(x + y) \leq \max \{p(x), p(y)\}$ and (2) $p(xy) \leq p(x)p(y)$.

**Proposition 1.** A topological algebra $X$ is an NLMC algebra iff the topology on $X$ is generated by a family $P$ of nonarchimedean multiplicative seminorms.

**Proof.** Given such a family $P$ generating the topology on $X$, the sets $\{x| p_\epsilon(x) \leq \epsilon, p_\epsilon \in P, 0 < \epsilon \leq 1\}$ form a base at 0 satisfying the condition of Definition 1.

Conversely, if $\mathcal{B}$ is a base at 0 satisfying the conditions of Definition 1, then, letting $p_\mu(x) = \inf \{|\mu|\,|x \in \mu U, \mu \in F\}$ the seminorms $(p_\mu)_{\mu \in \mathcal{B}}$ constitute the desired family $P$.

**Proposition 2.** If the valuation on $F$ is discrete and $X$ is an NLMC algebra, then there exists a family $P'$ of nonarchimedean multiplicative seminorms generating the topology on $X$ such that $p'(X) \subset |F|$ for each $p' \in P'$.

**Proof.** Let $P$ be a family of nonarchimedean multiplicative seminorms generating $X$’s topology. For each $p \in P$ let $p'(x) = \inf \{|\mu|\,|x \in \mu U, \mu \in F\}$. Each such $p'$ is clearly nonarchimedean and multiplicative. Moreover since $p(x) \leq p'(x) \leq |\mu^{-1}|p(x)$ for any nonzero $\mu \in F$ such that $|\mu| < 1$ and $|\mu|$ generates the value group of $F$, $P'$ will also generate the topology on $X$.

**Definition 3.** An NLMC algebra $X$ is discrete if there exists a family $P$ of nontrivial nonarchimedean multiplicative seminorms generating the topology on $X$ such that each $p$ in $P$ is discrete [the only limit point of $p(X)$ is $0$].

**Proposition 3.** A Hausdorff NLMC algebra $X$ is discrete iff $F$ is discretely valued.

**Proof.** Use Prop. 2.

If $X$ is a topological algebra over $C$, the complex numbers, then we can identify the nontrivial continuous homomorphisms of $X$ into $C$ with the closed maximal ideals in $X$ ([9, p. 13]). This is no longer
true for noncomplex algebras, and we single out those algebras in which the $1 - 1$ correspondence still obtains for special attention.

**Definition 4.** A commutative Hausdorff NLMC algebra $X$ with identity $e$ is a Gelfand algebra if for every closed maximal ideal $M \subset X$ the factor algebra $X/M$ (with quotient topology) is topologically isomorphic to $F$.

Associated with the nontrivial nonarchimedean multiplicative seminorms $p$ generating the topology on an NLMC algebra $X$, are nonarchimedean normed algebras $X/N_p$ where $N_p$ is the ideal $p^{-1}(0)$ where $X/N_p$ is normed by taking $\| x + N_p \| = p(x)$. The completions $X_p$ of these normed algebras are referred to as factor algebras.

**Proposition 4.** If $X$ is a Gelfand algebra and $X/N_p$ is complete, then $X/N_p$ is a Gelfand algebra.

**Proof.** Let $\pi_p$ denote the continuous homomorphism $x \rightarrow x + N_p$ from $X$ onto $X/N_p$. We observe that if $M$ is a maximal ideal in the Banach algebra $X/N_p$, then $M$ is closed; thus $\pi_p^{-1}(M)$ is a closed maximal ideal in $X$ containing $N_p$. For any $x \in X$ there exists $\mu \in F$ such that $x - \mu e \in \pi_p^{-1}(M)$ ($X$ is a Gelfand algebra), so that $\pi_p(x) - \mu \pi_p(e) \in M$ where $e$ is the identity of $X$. Thus $(X/N_p)/M$ is algebraically isomorphic to $F$. Since $M$ is closed, the factor structure is a one-dimensional Hausdorff topological vector space and is therefore topologically isomorphic to $F$.

**Proposition 5.** Let $P$ be a saturated family of seminorms generating the topology on the NLMC algebra $X$ and let $(X_p)_{p \in P}$ denote the associated factor algebras. If each $X_p$ is a Gelfand algebra, then $X$ is a Gelfand algebra.

**Proof.** Let $M$ be a closed maximal ideal in $X$. By [1, p. 466] there exists $p \in P$ such that $M \supset N_p$ and $\inf \{ p(e - x) | x \in M \} > 0$. Consequently $\pi_p(M)$ is a proper ideal in $X/N_p$ and $\pi_p(e)$ is not an adherence point of $\pi_p(M)$. Thus $\pi_p(M)$ is a proper ideal in $X_p$ and is therefore contained in a closed maximal ideal $N \subset X_p$. Since $X_p$ is a Gelfand algebra, $N$ is the kernel of a continuous nontrivial homomorphism $f_p$ taking $X_p$ into $F$. Hence $f = f_p \pi_p$ is a continuous nontrivial homomorphism taking $X$ into $F$. It follows from elementary considerations that the kernel of $f$ is equal to $M$. Consequently $X/M$ is seen to be algebraically—hence topologically—isomorphic to $F$.

A result similar in spirit to this can be found in [2, p. 175]. We turn next to some examples.
**Example 1.** Let $F$ be a local field, let $T$ be a 0-dimensional Hausdorff space and let $F(T)$ carry the topology of uniform convergence on compact sets. The topology on $F(T)$ is generated by the nonarchimedean multiplicative seminorms $p_K$ where $K$ is a compact subset of $T$ and for any $x \in F(T)$, $p_K(x) = \sup_{t \in K} |x(t)|$. We may identify $F(T)/p_K(0)$ with a subalgebra of $F(K)$. Moreover we may construct a Stone-Cech compactification $\beta_T$ of $T$ as is done in [3, p. 243] utilizing the compact valuation ring $V$ of $F$ in place of the compact interval $[0,1]$. Since $V$ is Hausdorff and 0-dimensional, $\beta_T$ will be compact, Hausdorff and 0-dimensional. Thus the Ellis-Tietze extension theorem ([4]) applies and any function continuous on $K$ may be extended to a function continuous on $\beta_T$. It follows that $F(T)/p_K(0) = F(K)$.

The continuous nontrivial homomorphisms of $F(K)$ into $F$ are in 1–1 correspondence with the points $t$ of $K$ ([11]) and using this result it can be shown [9, p. 31] that the points of $T$ generate the continuous nontrivial homomorphisms of $F(T)$ into $F$.*

Topological algebras $X$ for which all homomorphisms of $X$ into $F$ are continuous are called functionally continuous [9, p. 51].

What follows is an example of such an algebra.

**Example 2.** Let $F$ be any complete nonarchimedean nontrivially valued field and $T$ a 0-dimensional Hausdorff space. $F(T)$ carries the compact-open topology. A subalgebra $X$ of $F(T)$ is “closed under inverses” if when $x \in X$ and $x^{-1} \in F(T)$, $x^{-1} \in X$. We apply Michael’s proof [9, p. 54] and observe that if Conditions 1 and 2 below are satisfied, then the homomorphisms of $X$ are generated by the points of $T$ and therefore $X$ is functionally continuous.

1. For any $x_1, \ldots, x_n \in X$ such that $\bigcap_{t=1}^n x_t^{-1}(0) = \emptyset$, there exists $y_1, \ldots, y_n \in X$ such that $\sum x_i y_i = e$ where $e$ is the constant function $e(t) = 1$ for all $t \in T$.

2. For some positive integer $m$ there exists $x_1, \ldots, x_m \in X$ such that for all $\mu_1, \ldots, \mu_m \in F$, $\cap (x_i - \mu_i e)^{-1}(0)$ is compact.

We note that if $X = F(T)$, then by the results of a sequel to this paper [16], it follows that $X$ satisfies statement 1. If, in addition, there exists a bijection $x \in X$, then $X$ satisfies 2. Hence if we take $T = F$ and let $T$ carry any 0-dimensional Hausdorff topology finer than

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* The result of Example 1 actually obtains if $F$ is any complete nonarchimedean nontrivially valued field as it can be shown in this case that a bounded continuous function defined on a compact subset $K$ of $T$ mapping into $F$ can be extended to a bounded continuous function mapping $T$ into $F$. The same comment applies to Example 1, parts (c), (d), and (e) of Sec. 2.
the valuation topology on $F$, the nontrivial homomorphisms of the algebra $F(T)$ taking values in $F$ are generated by the points of $T$.

2. Function algebras over valued fields. In this section we discuss function algebras over valued fields. First we prove a version of a theorem of Kaplansky ([7, p. 173]) which is relevant to the material to follow; we include this proof because there seems to be an inconsistency in the use of "totally disconnected" in [7].

Lemma 1. (Kaplansky) Let $T$ be a topological space and let $F(T)$ be endowed with the topology of uniform convergence on compact sets. $I$ is a closed ideal in $F(T)$, iff there is some closed subset $H$ of $T$ such that $I = \{f \in F(T) \mid f(H) = \{0\}\}$. $I$ is a closed maximal ideal in $F(T)$ iff there is some $t \in T$ such that $I = \{f \mid f(t) = 0\}$.

Proof. Suppose $I$ is closed in $F(T)$ and let $H = \bigcap_{g \in I} g^{-1}(0)$. Letting $J(H) = \{f \mid f(H) = \{0\}\}$, we see that $I \subseteq J(H)$, and that $J(H)$ is a closed ideal. We show that if $f \in J(H)$, then $f \in I$.

Let $K$ be any compact subset of $T$. If $y \in K$, then as $I$ is an ideal, there exists $g_y \in I$ such that $g_y(y) = f(y)$. Since the clopen sets $\{U_y \mid y \in K\}$ where $U_y = \{x \in T \mid |f(x) - g_y(x)| < \varepsilon\}$ cover $K$ for any fixed $\varepsilon > 0$, there exist $y_1, \ldots, y_n$ such that $K \subseteq \bigcup_{i=1}^n U_{y_i}$. Since the sets $U_{y_i}$ are clopen, we see that there exist pairwise disjoint clopen sets $W_i$ such that $K \subseteq \bigcup_{i=1}^n W_i$ where $W_i \subseteq U_{y_i}$ for each $i$. Letting $k_A$ denote the characteristic function of the set $A$, we see that if $h = \sum_{i=1}^n g_{y_i}k_{W_i}$, then $h \in I$ and $\sup_{t \in K} |h(t) - f(t)| < \varepsilon$. As $\varepsilon > 0$ can be made arbitrarily small, it follows that $f \in \overline{I} = I$.

In the proof to follow, "totally disconnected" is used as in [13, p. 380]: distinct points may be separated by clopen sets.

Theorem 1. (Kaplansky) Let $S$ and $T$ be 0-dimensional Hausdorff spaces. Let $F(S)$ and $F(T)$ carry their compact-open topologies and suppose that $F(T)$ is topologically isomorphic to $F(S)$. Then $S$ and $T$ are homeomorphic.

Proof. Let $A$ be a topological isomorphism from $F(S)$ onto $F(T)$. If $K$ is a closed subset of $S$ and $J(K)$ denotes the ideal of functions that vanish on $K$, note that a mapping $A'$ is defined by $A(J([s])) = J([t]) = J(A'(s))$ for some $t \in T$; i.e. $A': S \to T$ is such that $A'(s) = t$, and is well-defined as $T$ is totally disconnected. Since $A$ is injective and $S$ is totally disconnected, then $A'$ is injective as well. For any $t \in T$, $J([t]) = A(M)$ where $M$ is a closed maximal ideal in $F(S)$.
Since $M = J([s])$ for some $s \in S$, $A'$ is seen to be surjective.

Clearly $(A')^{-1} = (A^{-1})'$ so to show that $A'$ is a homeomorphism, it suffices to show that $A'$ is a closed map. To this end, since $S$ is 0-dimensional, $K = \bigcap_{s \in J(K)} g^{-1}(0)$; since $J(K) = \bigcap_{s \in S} J([s])$, it follows that $A(J(K)) = J(A'(K)) = \bigcap_{s \in S} J([A'(s)])$. If $t \in A'(K)$, then $t = A'(s)$ where $s \in K$. Thus $J(K) \notsubset J([s])$ and $J(A'(K)) \notsubset J([t])$. As $J(A'(K)) = J(A'(K)) \notsubset J([t])$, we see that $t \notin A'(K)$ and therefore $A'(K) = A'(K)$.

**Example 1.** Let $T$ be a totally disconnected Hausdorff space and let $F(T)$ carry the compact-open topology. We note immediately that the set of evaluation maps constitutes a set of distinct continuous homomorphisms of $F(T)$ into $F$. Moreover properties (a)—(e) also hold.

(a) If $K$ is a compact subset of $T$, $p_K$ is as in Ex. 1 of Sec. 1, and $N_K = p_K^{-1}(0)$, then the completion of the normed algebra $F(T)/N_K$ is $F(K)$.

*Proof.* Since $T$ is totally disconnected, the characteristic functions in $F(T)$ separate the points of $T$. Thus the functions $f |_K$ as $f$ runs through $F(T)$ separate points in $K$. The desired result now follows from an application of Kaplansky’s Stone-Weierstrass theorem ([8] or [12] p. 161).

(b) With “$V^*$-algebra” as in [12, p. 148], if $T$ is locally compact, then $F(T)$ is the projective limit of $V^*$-algebras as in [9, p. 17].

*Proof.* The complete NLMC algebra $F(T)$ is the projective limit of the factor algebras $F(K)$ as $K$ runs through the compact subsets of $T$ and each $F(K)$ is a $V^*$-algebra.

(c) If $T$ is ultranormal and $F$ is a local field, then $F(T)/N_K = F(K)$.

*Proof.* Use the Ellis-Tietze extension theorem of [4].

(d) If $T$ is 0-dimensional and $F$ is a discretely valued field, then $F(T)/N_K = F(K)$ for any compact subset $K$ of $T$.

*Proof.* Apply a modification of the Ellis-Tietze extension theorem to functions $f \in F(K)$ and thereby extend $f$ continuously to a ‘Stone-Cech’ compactification $\beta H T$ where $H$ is any local field. Where Ellis used local compactness of the field $F$, we use discreteness of the valuation on $F$, and compactness of $\beta H T$.

(e) The points of $T$ constitute all continuous homomorphisms of $F(T)$ into $F$ when $F$ is discretely valued.

*Proof.* See Ex. 1 of Sec. 1 and use (d).
3. Main results. Let \( X \) be a NLMC algebra over a discretely valued \( F \). Then, as in the classical case ([9, p. 33]), if \( X \) is the projective (dense inverse) limit of a family \( (F(K_n)) \) of Gelfand \( V^* \)-algebras by mappings \( \pi_{mn}: F(K_n) \rightarrow F(K_m), m > n \), where \( (K_n) \) is a family of compact 0-dimensional Hausdorff spaces (it following that \( K_n \) is homeomorphically embedded in \( K_m \)), then \( X \) is topologically isomorphic to \( F(\bigcup K_n) \) where* \( F(\bigcup K_n) \) carries the compact-open topology. Moreover in this case \( \bigcup K_n \) can and will be identified with the set of all nontrivial continuous homomorphisms of \( X \) into \( F \) and carries the weak topology generated by \( (K_n) \).

**Definition 1.** Let \( \mathcal{M} \) denote the nontrivial continuous homomorphisms of an MLHC algebra \( X \) over \( F \) into \( F \), and let \( \mathcal{M} \) carry the weak-* topology. Let \( F(\mathcal{M}) \) denote the algebra of continuous functions mapping \( \mathcal{M} \) into \( F \) with compact open topology and consider the map \( \psi: X \rightarrow F(\mathcal{M}) \) where, for any \( x \in X \), \( \psi(x)(h) = h(x) \) for each \( h \in \mathcal{M} \). \( X \) is called a full algebra if the homomorphism \( \psi \) is an isomorphism of \( X \) onto \( F(\mathcal{M}) \).

In [9] E. A. Michael stated that he did not know whether or not \( \psi \) was a topological isomorphism in the case where \( X \) is a Fréchet full algebra. S. Warner proved that this was true in the classical case ([15, p. 269]). In this section we show that \( \psi \) is a topological isomorphism if \( F \) is a local field (Theorem 7). It then follows according to some results of van Tiel [14] that \( X \) is the projective limit of a sequence \( (F(K_n)) \) of Gelfand \( V^* \)-algebras where \( K_n = V_n \cap \mathcal{M} (V_n^0 \) is the polar of a neighborhood \( V_n \) of 0 in \( X \) coming from a base of \( F \)-convex closed neighborhoods of 0). Thus we will have a partial converse of the result which was described in the opening paragraphs of this section. We also note that by Prop. 5 of Sec. 1, \( X \) is a Gelfand algebra under the hypothesis just mentioned.

In what follows \( F \) is assumed to be discretely valued. In some cases it will also be assumed that \( F \) is a local field so that certain standard results from the duality theory of topological vector spaces ([14]) may be used.

**Definition 2.** Let \( F \) be discretely valued and let \( (a_{\mu})_{\mu \in H} \) be a system of distinct representatives of the cosets in the residue class field of \( F \). We may assume that \( H \) is totally ordered where \( \mu_0 \) corresponding to \( a_{\gamma_0} = 0 \) is the first element. Let \( \pi \in F \) be such that \( |\pi| < 1 \) and \( |\pi| \) is a generator of the value group of \( F \). If \( a \) and \( b \) are any two elements of \( F \) there exist \( (a_{\mu}) \) and \( (a_{\lambda}) \) such that \( a = \sum_{i=N}^0 a_{\mu_i} \pi^i \) and \( b = \sum_{i=N}^0 a_{\lambda_i} \pi^i \). We now define the supremum, \( \sup (a, b) \),

* We may assume \( K_n \subset K_{n+1} \) as there exist sets \( K_n' \) such that \( \mathcal{M} = \bigcup K_n' \) with \( K_n \subset K_{n+1} \), and \( K_n' \) homeomorphic to \( K_n \) for all \( n \).
of a and b as:

\[ \sup(a, b) = \begin{cases} 
  a & \text{if } |a| > |b| \\
  b & \text{if } |b| > |a| \\
  a & \text{if } a = b \\
  a & \text{if } |a| = |b|, a_{\mu_i} = a_{\lambda_i} \text{ for } i = N, \ldots, j - 1 \text{ and } \mu_j > \lambda_j
\end{cases} \]

**Lemma 1.** Let \( T \) be a topological space and let \( f \) and \( g \) be continuous functions mapping \( T \) into \( F \). Then the function defined at each \( t \in T \) by \( \sup(f(t), g(t)) \) and denoted by \( \sup(f, g) \) is continuous.

**Proof.** Suppose \( (t_s) \) is a net in \( T \) converging to \( t \). We show that \( \sup(f(t_s), g(t_s)) \) converges to \( \sup(f(t), g(t)) \). Letting \( f(t) = a \) and \( g(t) = b \), we need only consider the last possibility for \( \sup(a, b) \), the first three being trivial. Choose \( \varepsilon > 0 \) such that \( \varepsilon < |\pi|^j \). For \( r \) such that \( |f(t_s) - f(t)| < \varepsilon \) and \( |g(t_s) - g(t)| < \varepsilon \) for \( s \geq r \), it follows that

\[ f(t_s) - f(t) = \sum_{i=N}^{\infty} a_{\mu_i} \pi^i \quad \text{and} \quad g(t_s) - g(t) = \sum_{i=N}^{\infty} a_{\lambda_i} \pi^i \]

where \( M > j \). We may also write

\[ f(t_s) = \sum_{i=1}^{j} a_{\mu_i} \pi^i + \sum_{i=j+1}^{\infty} a_{\mu_i} \pi^i \quad \text{and} \quad g(t_s) = \sum_{i=1}^{j} a_{\lambda_i} \pi^i + \sum_{i=j+1}^{\infty} a_{\lambda_i} \pi^i. \]

Thus, since \( a_{\mu_i} = a_{\lambda_i} \) for \( i = N, \ldots, j - 1 \) and \( \mu_j > \lambda_j \), it follows that \( \sup(f(t_s), g(t_s)) = f(t_s) \) for \( s \geq r \). Thus \( \sup(f, g)(t_s) = f(t_s) \rightarrow f(t) = \sup(f, g)(t) \).

**Lemma 2.** Let \( F(T) \) denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space \( T \) into the discretely valued \( F \), with compact-open topology. If \( V \) is an \( F \)-barrel (closed absorbent \( F \)-convex set) in \( F(T) \), then there is some \( \delta > 0 \) such that \( \sup_{t \in T} |f(t)| \leq \delta \) implies that \( f \in V \).

**Proof.** Let \( B \) be the sup-norm Banach space of all bounded functions from \( T \) into \( F \). We note that \( V \cap B \) is an \( F \)-barrel in \( B \). Since \( B \) is \( F \)-barreled ([14, p. 268]) there is some \( \delta > 0 \) such that \( \sup_{t \in T} |f(t)| \leq \delta \) which implies that \( f \in V \cap B \).

**Lemma 3.** Let \( V, F, T \) and \( F(T) \) be as in Lemma 2, and suppose that for some compact subset \( K \) of \( T \), \( \{ f | f(K) = \{0\} \} \subset V \). Then there is some \( \mu > 0 \) such that whenever \( \sup_{t \in K} |f(t)| < \mu \), then \( f \in V \). Thus \( V \) is a neighborhood of 0 in \( F(T) \).

**Proof.** Let \( a \in F \) and denote the function sending each \( t \in T \) into \( a \) by \( a \). With \( \delta \) as in Lemma 2, choose \( a \in F \) such that \( 0 < |a| \leq \delta \).
δ/2. Choosing an integer \( n \) so that \( \delta/n < |a| \), let \( f \in F(T) \) be such that \( \sup_{t \in K} |f(t)| \leq \delta/n \). With \( g = \sup (f, a) - a \), it follows that \( g(t) = 0 \) for each \( t \) in \( K \). Thus \( g \in V \). Since \( |f(t) - g(t)| \leq |a| \leq \delta/2 \) for all \( t \in T \), it follows that \( f - g \in V \). Since \( V \) is \( F \)-convex, \( g + (f - g) = f \in V \), and the proof is complete.

We continue towards nonarchimedean analogs of theorems of Nachbin (Theorem 3) and Warner (Theorem 7). First we consider a notion of support of a linear functional which serves to replace the classical notion used by Nachbin.

In Lemmas 4 and 5 \( F(T) \) again denotes the algebra of continuous functions from the 0-dimensional Hausdorff space \( T \) into \( F \) with compact-open topology and \( \varphi \) denotes a member of the continuous dual \( F(T)' \) of \( F(T) \). For any subset \( S \) of \( T \), \( k_S \) denotes the characteristic function of \( S \) taking values in \( F \) and we note that \( k_S \in F(T) \) iff \( S \) is clopen. Let \( \mathcal{S} \) denote the family of subsets \( U \) of \( T \) such that \( U \) is clopen and \( \varphi(fk_U) = 0 \) for all \( f \in F(T) \).

**Lemma 4.** The family \( \mathcal{S} \) has the following properties: (1) If \( U \) is a clopen subset of \( G \in \mathcal{S} \), then \( U \in \mathcal{S} \); (2) \( \mathcal{S} \) is a ring of sets.

**Proof.** To prove (1) we observe that \( k_U = k_Gk_U \). (2) follows readily from (1).

**Definition 3.** The support of \( \varphi, F_\varphi \), is defined to be \( C(\cup \mathcal{S}) \).

We observe that since \( \varphi \) is continuous there is some compact set \( K \subset T \) and an integer \( N \) such that if \( f \in F(T) \), then \( |\varphi(f)| \leq N \sup_{t \in K} |f(t)| \). Thus, if \( f \) vanishes on \( K \), then \( \varphi(f) = 0 \).

**Theorem 1.** In the same notation as above (1) \( F_\varphi \subset K \) and therefore \( F_\varphi \) is compact, (2) if \( \varphi \) is nontrivial, then \( F_\varphi \) is not empty, and (3) if \( G \subset T \) is open and \( G \cap F_\varphi \) is not empty, then there exists \( f \in F(T) \) such that \( f(CG) = \{0\} \) and \( \varphi(f) = 1 \).

**Proof.** (1) If \( G \) is a clopen subset of \( CK \), then—since \( k_G \) vanishes on \( K \)—\( \varphi(fk_U) = 0 \) and \( G \in \mathcal{S} \).

(2) If \( F_\varphi \) is empty, \( T = \bigcup \mathcal{S} \), and it follows that for some \( U_i \in \mathcal{S} \), \( K \subset \bigcup_{i=1}^\infty U_i = G \). Since \( \mathcal{S} \) is a ring of sets, \( G \in \mathcal{S} \) and since \( CG \) is clopen and contained in \( CK \), \( \varphi(f) = \varphi(fk_U) = 0 \) for all \( f \in F(T) \). But then \( \varphi \) is trivial.

(3) If \( G \cap F_\varphi \neq \emptyset \), there is some \( t \in G \cap F_\varphi \). Since \( T \) is 0-dimensional, \( t \in U \subset G \) where \( U \) is clopen. Since \( U \cap F_\varphi \neq \emptyset \), then \( U \notin \mathcal{S} \) and there is some \( g \in F(T) \) such that \( \varphi(gk_U) \neq 0 \). We of course may assume that \( \varphi(gk_U) = 1 \). Letting \( gk_U = f \), (3) is seen to be proved.
In order to apply this notion of support to our version of Nachbin’s theorem (Theorem 3) we require that $F_\varphi$ have the property that if $f$ vanishes on $F_\varphi$, $\varphi(f) = 0$. We now develop a case where this is true and which makes the notion applicable to Theorem 3 as well as settling Michael’s question in this setting (Theorem 7).

**Lemma 5.** Suppose that $\varphi(g) = 0$ for any $g \in F(T)$ which vanishes on any clopen set $G$ containing $F_\varphi$. Then if $f$ vanishes on $F_\varphi$, $\varphi(f) = 0$.

**Proof.** Suppose that $f \in F(T)$ vanishes on $F_\varphi$, and let $A_n = \{ t \in T \mid |f(t)| < 1/n \}$ ($n = 1, 2, \ldots$). As $F_\varphi \subset A_n$ for any $n$ and $A_n$ is clopen $\varphi(f) = \varphi(fk_{A_n}) + \varphi(f(1 - k_{A_n}))$. By the hypothesis, since $f(1 - k_{A_n})$ vanishes on $A_n$, $\varphi(f) = \varphi(fk_{A_n})$. Let $K$ be a compact subset of $T$ such that $|\varphi(f)| \leq N \sup_{t \in K} |f(t)|$. Hence $|\varphi(f)| = |\varphi(fk_{A_n})| \leq N \sup_{t \in K} |f(t)|$, $|f(t)| < N/n$. Since this is true for every $n$, $\varphi(f) = 0$.

**Theorem 2.** Let $T$ be a Lindelöf space. Then if $f$ vanishes on $F_\varphi$, $\varphi(f) = 0$.

**Proof.** Let $G$ be a clopen subset containing $F_\varphi$. Since $CG$ is closed, $CG$ is Lindelöf. Since $CG \subset GF_\varphi = \bigcup \mathcal{S}$, there exist $U_i \in \mathcal{S}$ such that $CG \subset \bigcup_{i=1}^\infty U_i$. Since $\mathcal{S}$ is a ring, we may assume that the sets $U_i$ are pairwise disjoint. Since $CG \cap U_i = V_i$ is clopen and contained in $U_i$ then $V_i \in \mathcal{S}$. Thus $k_{CG} = \sum_{i=1}^\infty k_{V_i}$ in the topology of pointwise convergence on $F(T)$. We claim that the “pointwise convergence” of the preceding sentence may be replaced by “uniform convergence on compact sets.”

To prove this last statement, let $L$ be a compact subset of $T$ and consider $L \cap CG$. As $L \cap CG$ is compact and contained in $\bigcup_{i=1}^\infty V_i$, there is some integer $N_L$ such that $n \geq N_L$ implies that $L \cap CG$ is contained in $\bigcup_{i=1}^n V_i$. But $CG \subset \bigcup_{i=1}^n V_i$ so $L \cap CG = L \cap (\bigcup_{i=1}^n V_i)$. Thus for $n \geq N_L$, $CG$ and $\bigcup_{i=1}^n V_i$ have the same points in common with $L$, and $\sup_{t \in L} |(k_{CG} - \sum_{i=1}^n k_{V_i})(t)| = 0$ for $n \geq N_L$. Since $L$ was an arbitrary compact set, the series is seen to converge in the compact-open topology and $\varphi(fk_{CG}) = \sum_{i=1}^n \varphi(fk_{V_i}) = 0$.

We now present a version of a theorem of Nachbin ([10, p. 472])

**Theorem 3.** Let $F(T)$ denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space $T$ into the discretely valued field $F$, with compact-open topology. Suppose that for each $\varphi \in F(T)'$, $f$ vanishing on $F_\varphi$ implies $\varphi(f) = 0$. Then $F(T)$ is $F$-barreled iff for every $E \subset T$ which is closed and not compact there is some $f \in F(T)$...
which is unbounded on $E$.*

Proof. Suppose that the condition holds and let $V$ be an $F$-barrel in $F(T)$. To show that $V$ is a neighborhood of 0 in $F(T)$ we begin by letting $K = \bigcup_{\varphi \in \psi} F_{\varphi}$. If $K$ is not compact, let $f$ be unbounded on $K$ and consider the sets $A_n = \{ t \in T | |f(t)| > n \}$, $n = 1, 2, \ldots$. Each $A_n$ is clopen and $A_n \cap K \neq \emptyset$. Thus there is some $F_{\varphi_n} \subset K$ such that $A_n \cap F_{\varphi_n} \neq \emptyset$. By Theorem 1 (3) there exists $f_n \in F(T)$ such that $f_n$ vanishes outside of $A_n$ and $\varphi_n(f_n) = 1$. Since $\bigcap_{n=1}^{\infty} A_n = \emptyset$, the function $f = \sum_{n=1}^{\infty} a_n f_n$ is a continuous function for any choice of $a_n \in F$. As it is clear that $A_m \bigcap F_{\varphi_n} = \emptyset$ for all sufficiently large $m$, we may (by considering a subsequence) assume that $\varphi_n(f_m) = 0$ for all $m \geq n$. By a proper choice of $a_n$ we see that $|\varphi_n(f)| \to \infty$ and as $\varphi_n \in V^0$, $af$ cannot belong to $V^{\infty} = V$ no matter how small $|a|$ is. Thus we contradict the fact that $V$ is absorbent and $K$ must be compact. If $f$ vanishes on $K$, then $f$ vanishes on $F_{\varphi}$ for all $\varphi \in V^0$. Thus $f \in V^{\infty} = V$ so, by Lemma 3, $V$ is a neighborhood of 0.

To prove the converse, let $F(T)$ be $F$-barreled and $E$ be a closed noncompact subset of $T$. Let $V = \{ f | \sup_{t \in K} |f(t)| \leq \delta \}$, $\delta > 0$, and let $K$ be a compact subset of $T$. As $E \cap CK \neq \emptyset$, using $\beta_{11} T$ as in Sec. 2 Ex. 1 (d) we may assert the existence of a sequence $(f_n)$ of functions which vanishes on $K$ but $|f_n(t_N)| \geq N$ for any positive integer $N$ and some $t_N \in E$. Thus the set $\{ f | \sup_{t \in K} |f(t)| \leq \varepsilon \} \not\subset V$ for any $\varepsilon > 0$ and $V$ is not a neighborhood of 0. It follows that $V$ is not absorbing and there exists $f \in F(T)$ which is unbounded on $E$.

**Corollary.** Let $T$ be a 0-dimensional Hausdorff Lindelöf space and $F$ a discretely valued field. Then $F(T)$ is $F$-barreled.

**Proof.** We refer to Theorem 2 and the construction of the function in the proof of Theorem 6 for the proof of the corollary.

**Theorem 4.** Suppose the 0-dimensional Hausdorff space $T = \bigcup_{n=1}^{\infty} K_n$ where each $K_n$ is compact, $K_n \subset K_{n+1}$, and each compact subset of $T$ is contained in some $K_n$ (i.e. $T$ is hemicompact). Then denoting $T$ endowed with the weak topology ([3], p. 131) generated by the sets $(K_n)$ as $T_w$, $F(T)$ is dense in $F(T_w)$, each algebra carrying its compact-open topology.

**Proof.** Since the topology of $T_w$ is clearly stronger than that of $T$, $F(T) \subset F(T_w)$. We note that the topology of $T_w$ restricted to $K_n$ is

* In a sequel to this paper we show that Theorem 2 is true for any 0-dimensional Hausdorff space $T$ and any complete nonarchimedean nontrivially valued field $F$. Thus Theorem 3 is true for all spaces $T$. We also show that the result of Theorem 3 holds of $F$ is spherically complete ([16]).
equal to the topology \( K_n \) inherits from \( T \) and the compact subsets of \( T_w \) lie in the sets \( K_n \). Thus \( F(T) \) is a topological subspace of \( F(T_w) \).

Using Sec. 2 Ex. 1 (d), \( F(T)/N_K = F(K) \) for any compact set \( K \subset T \) and it follows that \( F(T) \) is dense in \( F(T_w) \).

**Theorem 5.** Let everything be as in the preceding theorem. If \( F(T) \) is complete then \( T = T_w \) iff \( T_w \) is 0-dimensional.

**Proof.** If \( F(T) \) is complete, then \( F(T) = F(T_w) \). Since they are topologically isomorphic under the identity map by the proof of Theorem 4, if \( T_w \) is 0-dimensional, then \( T = T_w \) by Theorem 1 of Sec. 2. We may also observe that the functions of \( F(T) \) generate the topology of the space \( T \) while those of \( F(T_w) \) generate the topology of \( T_w \). Thus as \( F(T) = F(T_w) \), the topologies are equal.

**Theorem 6.** Let \( F(T) \) denote the algebra of continuous functions mapping the 0-dimensional Hausdorff space \( T \) into the local field \( F \) and suppose that \( F(T) \) is a complete locally \( F \)-convex metric space with topology \( T \). If the homomorphisms determined by the points of \( T \) are the \( T \)-continuous homomorphisms, then \( T \) is the compact-open topology.

**Proof.** Let the set of evaluation maps determined by \( T \) be denoted by \( T^* \) and let \( T^* \) carry the Gelfand topology (i.e. the weakest topology for \( T^* \) with respect to which the maps \( t \rightarrow x(t) \) of \( T^* \) into \( F \) are continuous for each \( x \in F(T) \)). Since \( T \) is 0-dimensional the Gelfand topology coincides with the original topology on \( T \), i.e. \( T \) and \( T^* \) are homeomorphic. Since \( (F(T), T) \) is \( F \)-barreled ([14, p. 268]), the polar of any compact subset of \( T^* \) is a neighborhood of 0 in \( F(T) \). Thus, identifying \( T \) and \( T^* \), \( T \) is seen to be stronger than the compact-open topology on \( F(T) \). If \( F(T) \) with compact-open topology could be shown to be \( F \)-barreled, the closed graph theorem could be applied to complete the proof. To show that \( F(T) \) is \( F \)-barreled, let \( E \) be a closed noncompact subset of \( T \). Since \( F(T) \) is a Frechet space, \( T^* \) is 0-dimensional and Lindelöf and therefore \( T \) is 0-dimensional and Lindelöf. Thus \( E \) is Lindelöf and there exists a denumerable clopen cover \( (U_n) \) from which no finite subcover can be extracted. We may assume the family \( (U_n) \) to be pairwise disjoint. Since \( CE \) is open in \( T \), \( CE = \cup V_\mu \) where each \( V_\mu \) is clopen so that \( T = (\bigcup_{n=1}^\infty U_n) \cup (\bigcup_{n=1}^\infty V_\mu) \) where the \( (V_\mu) \) may be assumed to be pairwise disjoint. Defining \( H_{2n} = V_{\mu_n}, H_{2n+1} = U_n \) and setting \( L_n = H_n - \bigcup_{i=1}^{n-1} H_i \) then \( T = \bigcup_{n=1}^\infty L_n \) where each \( L_n \) is clopen and \( (L_n) \) is pairwise disjoint. We note that \( E \) must intersect infinitely many \( L_n \)'s lest \( E \) turn out to be covered by finitely many of the \( U_i \). Now consider the function \( f: T \rightarrow F \)
defined by \( f(t) = \sum_{i=1}^{\infty} a^i k_{L_i}(t) \) where \( |a| > 1 \). We observe that \( f \) is unbounded on \( E \) and therefore \( F(T) \) with compact-open topology is \( F \)-barreled.*

We now prove a nonarchimedean version of a theorem of Warner ([15, p. 267]).

**Theorem 7.** Let the set of nontrivial continuous homomorphisms on the Frechet full algebra \( X \) be denoted by \( \mathcal{M} \). Let \( \mathcal{M} \) carry the weak-* (Gelfand) topology and \( F(\mathcal{M}) \) the compact-open topology. Then \( X \) is topologically isomorphic to \( F(\mathcal{M}) \).

**Proof.** Carrying the topology of \( X \) over to \( F(\mathcal{M}) \) via the isomorphism \( \psi \) (Def. 1 of Sec. 1) and noting that \( \mathcal{M} \) constitutes the set of nontrivial continuous homomorphisms of \( F(\mathcal{M}) \) into \( F \), we see by the previous theorem that the proof is done.

For complex algebras, Warner ([15]) has proved that the "\( \mathcal{M} \)" of Theorem 7 is a \( k \)-space (\( \mathcal{M} \) carries the weak topology generated by a sequence of compact sets). This question as well as an attempt to develop a substitute for concept of "\( Q \)-space" ([5, p. 271]) is investigated in subsequent papers ([16]).

**References**


*As shown here, the hypothesis of Theorem 6 implies \( T \) to be Lindelöf. \( T \) being Lindelöf however implies that all homomorphisms of \( F(T) \) into \( F \) are given by points of \( T \) ([16]).*

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