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**TWO PRIMARY FACTOR INEQUALITIES**

JOHN H. E. COHN

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In the theory of integral functions, the expressions

$$(1) \quad E(z, p) = (1 - z) \exp \left\{ \sum_1^p \frac{z^r}{r} \right\}, \quad p = 1, 2, \dots$$

called *primary factors*, are of some importance, and it is of interest to find upper bounds for  $|E(z, p)|$ . Clearly  $E(z, p) = 0$  only for  $z = 1$ , and so for other values, define  $f(z, p) = \log |E(z, p)|$ . It is known that for suitable constants  $a_p, b_p$  the inequalities

$$(2) \quad f(z, p) \leq a_p |z|^p, \quad |z| \geq 1, \quad z \neq 1$$

$$(3) \quad f(z, p) \leq b_p |z|^{p+1}, \quad |z| \leq 1, \quad z \neq 1$$

are satisfied; for instance Hille has shown that one may take  $a_p = 1 + \sum_1^p 1/r \leq 2 + \log p$  and  $b_p = 1$ .

In this paper, the smallest values of both  $a_p$  and  $b_p$  are determined, the latter in closed form.

Throughout, we shall write  $z = \rho e^{i\theta}$ , where without loss of generality  $\rho \geq 0, 0 \leq \theta \leq \pi$ . Then

$$(4) \quad f(z, p) = \frac{1}{2} \log (1 - 2\rho \cos \theta + \rho^2) + \sum_1^p \frac{\rho^r}{r} \cos r\theta.$$

Also, using the Taylor series for  $\log(1 - z)$  gives from (1)

$$(5) \quad f(z, p) = -\rho^{p+1} \sum_0^{\infty} \frac{\rho^r}{p+r+1} \cos(p+r+1)\theta,$$

provided  $\rho < 1$ . A further expression is obtained by writing  $\log E(z, p)$  as an integral of its derivative and taking real parts, to give

$$f(z, p) = \int_0^\rho \frac{t \cos p\theta - \cos(p+1)\theta}{1 - 2t \cos \theta + t^2} t^p dt,$$

provided  $\theta \neq 0$  or  $\rho < 1$ .

The problem considered in this paper is the determination of the maxima of the functions

$$(6) \quad g(z, p) = \rho^{-p} f(z, p) \quad \text{for } \rho \geq 1$$

and

$$h(z, p) = \rho^{-p-1} f(z, p) \quad \text{for } \rho \leq 1,$$

and to show where these occur.

1. **Summary of results.** Henceforth we use  $a_p$  and  $b_p$  to denote the smallest constants for which (2) and (3) hold. We shall show that both  $a_p$  and  $b_p$  are monotone decreasing functions of  $p$ . The value of  $a_1$  is given by  $a_1 = \log(\rho - 1)$  where  $\rho$  is the solution of the transcendental equation  $(\rho - 1) \log(\rho - 1) = \rho$ ,  $\rho > 1$  and the maximum occurs at  $z = \rho$ . Also  $a_2 = 1$ , the maximum occurring at  $z = 2$ , and  $a_\infty$  is given by the common value of  $x^{-1}$  and

$$e^{-x} \left( \gamma + \int_0^1 \frac{e^s - 1}{s} ds + \int_1^x \frac{e^s}{s} ds \right),$$

for the unique value of  $x$  which makes these expressions equal,  $\gamma$  denoting Euler's constant. For each  $p \geq 2$ , the maximum occurs at a point  $z$  on the real axis which satisfies  $1 < z \leq 2$ .

The  $z$  maximizing  $b_p$  occur on  $|z| = 1$ , with  $\theta = \pi/(2p + 1)$ ,  $p = 1, 2, 3, \dots$ . For  $p > 1$  the maximum is unique, but for  $p = 1$  it is attained at every point of the arc  $|z - 1| = 1$ ,  $|z| \leq 1$ . We derive the explicit bounds

$$\frac{1}{2} \geq b_p > \log \pi/2 + \gamma - \int_0^{\pi/2} \frac{1 - \cos x}{x} dx,$$

and both bounds are sharp. We also have an explicit formula

$$b_p = \log \left( 2 \sin \frac{1}{2} \theta \right) + \sum_1^p \frac{1}{r} \cos r\theta,$$

where  $\theta = \pi/(2p + 1)$ . In particular these results give

$$1.2785 \geq a_p > 0.7423, \quad \frac{1}{2} \geq b_p > 0.4719.$$

Since  $a_2 = 1$  we have therefore

$$\log |E(z, p)| \leq \min(|z|^p, |z|^{p+1}), \quad p = 2, 3, \dots,$$

and this is sharp.

The numerical values of  $a_p$  and  $b_p$  are as follows.

$p$	$a_p$	$b_p$
1	1.2785	0.5000
2	1.0000	0.4823
3	0.9123	0.4771
4	0.8691	0.4752
5	0.8435	0.4741
$\infty$	0.7423	0.4719

2. **Preliminaries.** It is clear that for (2) and (3) to hold, both  $a_p$  and  $b_p$  must be positive, since for example  $z = 2$  and  $z = \varepsilon \exp i\pi/(p+1)$  with  $\varepsilon$  sufficiently small give positive values of  $f(z, p)$ . We see therefore that not only the point  $z = 1$ , but also a neighbourhood of this point can be excluded from the discussion. We find by elementary means that

$$(8) \quad \frac{\partial g}{\partial \theta} = \frac{\rho\{\sin(p+1)\theta - \rho \sin p\theta\}}{1 - 2\rho \cos \theta + \rho^2}$$

$$(9) \quad \frac{\partial^2 g}{\partial \theta^2} = \frac{\partial\{(p+1)\cos(p+1)\theta - p\rho \cos p\theta\}}{1 - 2\rho \cos \theta + \rho^2} - \frac{2\rho \sin \theta}{1 - 2\rho \cos \theta + \rho^2} \frac{\partial g}{\partial \theta}$$

$$(10) \quad \frac{\partial g}{\partial \rho} = -\frac{pg}{\rho} + \frac{\rho \cos p\theta - \cos(p+1)\theta}{1 - 2\rho \cos \theta + \rho^2}$$

$$(11) \quad \frac{\partial h}{\partial \theta} = \frac{\sin(p+1)\theta - \rho \sin p\theta}{1 - 2\rho \cos \theta + \rho^2}$$

$$(12) \quad \frac{\partial^2 h}{\partial \theta^2} = \frac{(p+1)\cos(p+1)\theta - p\rho \cos \theta}{1 - 2\rho \cos \theta + \rho^2} - \frac{2\rho \sin \theta}{1 - 2\rho \cos \theta + \rho^2} \frac{\partial h}{\partial \theta}$$

$$(13) \quad \frac{\partial h}{\partial \rho} = -\frac{(p+1)h}{\rho} + \frac{\rho \cos p\theta - \cos(p+1)\theta}{\rho(1 - 2\rho \cos \theta + \rho^2)}.$$

3. **The case  $\rho = 1$ .** We consider first the unit circle on which of course  $f, g$  and  $h$  coincide, with  $0 \leq \theta \leq \pi$ . Then by (8) we find that  $\partial f/\partial \theta = 1/2 \cos(p+(1/2))\theta \operatorname{cosec}(1/2)\theta$ , and so local maxima occur at  $\theta = \beta, 5\beta, 9\beta, \dots$  where  $\beta = \pi/(2p+1)$ . We shall show that  $f(\beta) > f(5\beta) > f(9\beta) > \dots$  and hence that  $f(\beta)$  is the largest value taken by  $f(z, p)$  on  $|z| = 1$ . For, let  $n \geq 0$  with  $(4n+5)\beta \leq \pi$ . Then

$$\begin{aligned} f((4n+5)\beta) - f((4n+1)\beta) &= \int_{(4n+1)\beta}^{(4n+5)\beta} f'(\theta) d\theta \\ &= \frac{1}{2} \int_{(4n+1)\beta}^{(4n+3)\beta} \cos \frac{\pi\theta}{2\beta} \operatorname{cosec} \frac{1}{2}\theta d\theta + \frac{1}{2} \int_{(4n+3)\beta}^{(4n+5)\beta} \cos \frac{\pi\theta}{2\beta} \operatorname{cosec} \frac{1}{2}\theta d\theta \\ &= \frac{-\beta}{2\pi} \int_0^{2\pi} \frac{\sin \frac{1}{2}\phi}{\sin\left(4n+3 - \frac{\phi}{\pi}\right)\frac{\beta}{2}} d\phi + \frac{\beta}{2\pi} \int_0^{2\pi} \frac{\sin \frac{1}{2}\phi}{\sin\left(4n+3 + \frac{\phi}{\pi}\right)\frac{\beta}{2}} d\phi \\ &< 0, \end{aligned}$$

where we have substituted  $\phi = (4n+3)\pi - \pi\theta/\beta$  in the first integral, and  $\phi = -(4n+3)\pi + \pi\theta/\beta$  in the second.

Thus we obtain in view of (4), that for  $|z| = 1$ ,

$$(14) \quad f(z, p) \leq \sigma_p = \log\left(2 \sin \frac{\pi}{4p+2}\right) + \sum_1^p \frac{1}{r} \cos \frac{r\pi}{2p+1}.$$

We now consider  $\sigma_p$ , and prove first that  $\sigma_p > \sigma_{p+1}$ . Define  $\delta$  by  $\pi = 2(2p+1)(2p+3)\delta$ . Then

$$\begin{aligned}\sigma_p - \sigma_{p+1} &= \log \frac{\sin(2p+3)\delta}{\sin(2p+1)\delta} + \sum_1^p \frac{1}{r} \{ \cos 2(2p+3)r\delta - \cos 2(2p+1)r\delta \} \\ &\quad - \frac{1}{p+1} \cos 2(2p+1)(p+1)\delta \\ &= \lambda(\delta), \text{ say.}\end{aligned}$$

Thus if  $\lambda(\phi)$  is defined for  $0 < \phi \leq \delta$  by the same formula with  $\delta$  replaced by  $\phi$ , we find that as  $\phi \rightarrow 0$ ,

$$\lambda(\phi) \rightarrow \log \frac{2p+3}{2p+1} - \frac{1}{p+1} > 0.$$

Also

$$\begin{aligned}\lambda'(\phi) &= \{(2p+3) \operatorname{cosec}(2p+3)\phi - (2p+1) \operatorname{cosec}(2p+1)\phi\} \cos \frac{\pi\phi}{2\delta} \\ &> 0,\end{aligned}$$

since  $x \operatorname{cosec} x$  is strictly increasing in  $(0, \pi/2)$ .

Thus  $\lambda(\delta) > 0$ , and so

$$(15) \quad \sigma_p > \sigma_{p+1}.$$

Also as  $p \rightarrow \infty$  we find that

$$\begin{aligned}\sigma_p &= \log \left( 2 \sin \frac{\pi}{4p+2} \right) + \sum_1^p \frac{1}{r} \cos \frac{r\pi}{2p+1} \\ &= \log \frac{\pi}{2p+1} + o(1) + \sum_1^p \frac{1}{r} + \sum_1^p \frac{1}{r} \left\{ \cos \frac{r\pi}{2p+1} - 1 \right\} \\ &= \log \pi/2 + \left\{ \sum_1^p \frac{1}{r} - \log p \right\} - \int_0^{\pi/2} \frac{1 - \cos x}{x} dx + o(1) \\ &\rightarrow \log \frac{1}{2} \pi + \gamma - \int_0^{\pi/2} \frac{1 - \cos x}{x} dx = 0.4719.\end{aligned}$$

Thus we find, since  $\sigma_1 = 1/2$ , that for all  $p$

$$(16) \quad \frac{1}{2} \geq \sigma_p > 0.4719.$$

4. The case  $\rho \leq 1$ . For  $\rho \leq 1$ , we consider first  $p = 1$ , where the situation is slightly different from the remaining values of  $p$ . Using (11) we see that if  $\rho \neq 1$ , then for fixed  $\rho$ ,  $h$  has turning values, regarded as a function of  $\theta$ , only for  $\theta = 0$ ,  $\theta = \pi$  and  $2 \cos \theta = \rho$ . Using (12) we find that both  $\theta = 0$  and  $\theta = \pi$  give minima, and so for each  $\rho \in (0, 1)$  we find that

$$h(z, 1) \leq \rho^{-2} \left( \frac{1}{2} \log 1 + \frac{1}{2} \rho^2 \right) = \frac{1}{2},$$

with equality if and only if  $2 \cos \theta = \rho$ . Thus we have  $b_1 = 1/2$  with equality attained at every point of the arc  $|z - 1| = 1, |z| \leq 1$ .

For  $p \geq 2$ , the situation is quite different. Clearly whatever  $b_p$  with turn out to be, there will be equality in (3) for  $z = 0$ . But for  $0 < \rho < 2/7$ , we find using (5) and (7) that

$$\begin{aligned} h(z, p) &= - \sum_0^{\infty} \frac{\rho^r}{p+r+1} \cos(p+r+1)\theta \\ &< \frac{1}{3} \sum_0^{\infty} \left(\frac{2}{7}\right)^r = \frac{7}{15} < \sigma_p, \text{ by (16)}. \end{aligned}$$

Thus the maximum of  $h(z, p)$  occurs in the closed annulus  $2/7 \leq \rho \leq 1$ .

Again consider a fixed value of  $p < 1$ . By (11) the greatest value of  $h$ , regarded as a function of  $\theta$ , occurs at a solution of  $\sin(p+1)\theta = \rho \sin p\theta$ .  $\theta = 0$  is impossible since then  $\partial^2 h / \partial \theta^2 > 0$  by (12) and  $\theta = \pi$  can be neglected since then by (5) and (7) we get

$$h(-\rho, p) = \sum_0^{\infty} \frac{\rho^r}{p+r+1} (-1)^{p+r} \leq \frac{1}{p+1} \leq \frac{1}{3} < \sigma_p.$$

A glance at the sketch of  $y = \sin(p+1)x / \sin px$  for  $x \in (0, \pi)$ , shown in Figure 1, reveals that there are precisely  $p$  other values of  $\theta$  to consider, since it is readily shown that each branch of the curve is monotone strictly decreasing. Again we consider the sign of  $\partial^2 h / \partial \theta^2$ . Since  $\rho < 1$  we find that for given  $\rho$ , the intersection of  $y = \rho$  with the  $r$ th. branch of the curve satisfies

$$\frac{2r-1}{2p+1} \pi < \theta < \frac{r\pi}{p+1},$$

whence  $p\theta \in ((r-1)\pi, r\pi)$  and  $(p+1)\theta \in ((r-(1/2))\pi, r\pi)$ . Thus at such a point we find from (12) and substituting for  $\rho$ ,

$$\frac{\partial^2 h}{\partial \theta^2} = \frac{\sin p\theta}{\sin^2 \theta} \{ \sin p\theta \cos(p+1)\theta - p \sin \theta \}$$

and so the second factor is negative. Thus  $\partial^2 h / \partial \theta^2 < 0$  only if  $\sin p\theta > 0$ , i.e. if  $r$  is odd. Moreover at a local maximum we have using (13)

$$h = \frac{\sin p\theta}{(p+1) \sin \theta}.$$

Thus if  $\theta \leq \pi/2$  we we find that except on the first branch  $\theta > 2\pi/p$  and so

$$y = \frac{\sin(p+1)x}{\sin px}$$

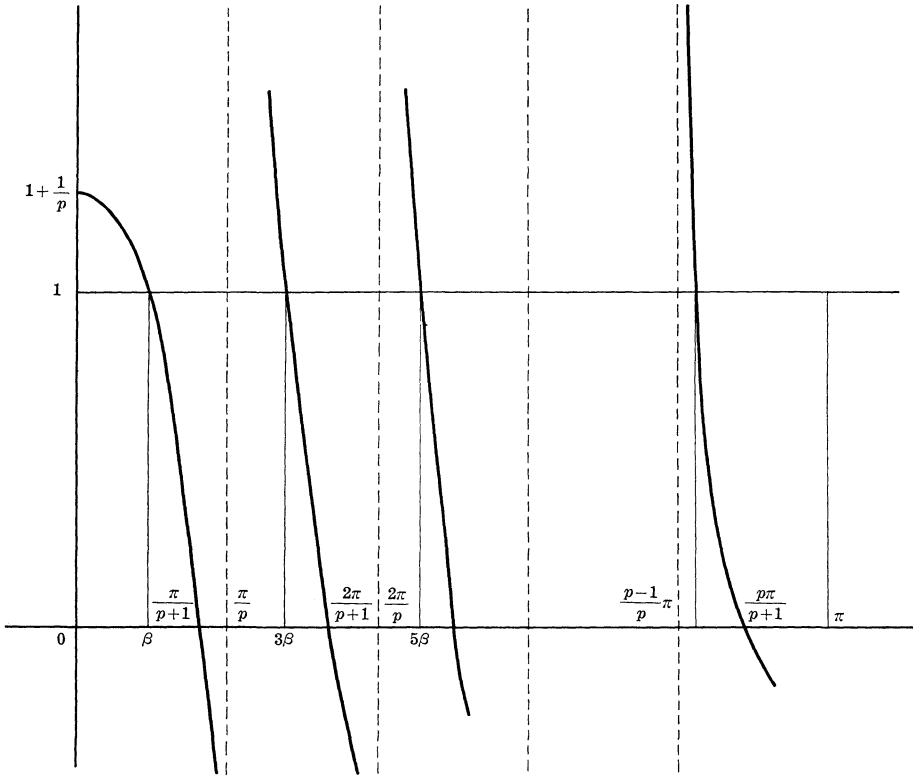


FIGURE 1

$$\begin{aligned} h &\leq \frac{1}{(p+1) \sin \theta} \\ &\leq \frac{\pi}{2(p+1)\theta} \\ &\leq \frac{p}{4(p+1)} < \sigma_p. \end{aligned}$$

Similarly if  $\theta \geq \pi/2$  we find that

$$\pi - \theta \geq \pi - \frac{\pi p}{p+1} = \frac{\pi}{p+1}$$

and so

$$\begin{aligned} h &\leq \frac{1}{(p+1) \sin \theta} \\ &\leq \frac{1}{p+1} \operatorname{cosec} \frac{\pi}{p+1} \\ &\leq \frac{1}{3} \operatorname{cosec} \frac{1}{3}\pi < \sigma_p, \end{aligned}$$

since  $x \operatorname{cosec} x$  increases over  $(0, \pi/2)$ .

Thus we need only consider the first branch. Let  $k(\rho)$  be the value taken by  $h(z, p)$  with  $\rho \in (0, 1)$  and  $\theta$  defined by  $\rho \sin p\theta = \sin(p+1)\theta$ ,  $\theta \in (0, \pi/(p+1))$ . Then

$$(17) \quad \begin{aligned} \frac{dk}{d\rho} &= \frac{\partial h}{\partial \rho} + \frac{\partial h}{\partial \theta} \frac{d\rho}{d\theta} = \frac{\partial h}{\partial \rho} \\ &= -\frac{(p+1)k}{\rho} + \frac{1}{\rho} \frac{\sin p\theta}{\sin \theta}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{d\rho} \left\{ \rho \frac{dk}{d\rho} + (p+1)k \right\} &= \frac{d}{d\theta} \left( \frac{\sin p\theta}{\sin \theta} \right) / \frac{d}{d\theta} \left( \frac{\sin(p+1)\theta}{\sin p\theta} \right) \\ &> 0, \end{aligned}$$

since both  $(\sin p\theta)/(\sin \theta)$  and  $(\sin(p+1)\theta)/(\sin p\theta)$  decrease over  $(0, \pi/(p+1))$ . Therefore

$$\frac{d}{d\rho} \left\{ \rho^{p+2} \frac{dk}{d\rho} \right\} = \rho^{p+1} \frac{d}{d\rho} \left\{ \rho \frac{dk}{d\rho} + (p+1)k \right\} > 0,$$

and so  $\rho^{p+2} dk/d\rho$  increases. But as  $\rho \rightarrow 0$ ,  $\theta \rightarrow \pi/(p+1)$ , and so using (17) we see that  $\rho^{p+2} dk/d\rho \rightarrow 0$ . Thus for  $\rho > 0$ ,  $dk/d\rho > 0$ , whence  $k$  increases over  $(0, 1)$ . Thus for all such  $z$ ,  $h(z, p) \leq k(1) = \sigma_p$  with equality if and only if  $z = \exp i\pi/(2p+1)$ . This concludes the discussion of this case.

5. The case  $\rho \geq 1$ . We find that

$$\frac{\partial}{\partial \theta} \left\{ \frac{1}{2} \log(1 - 2\rho \cos \theta + \rho^2) + \rho \cos \theta \right\} = \frac{\rho^2 \sin \theta (2 \cos \theta - \rho)}{1 - 2\rho \cos \theta + \rho^2}$$

and so if  $\rho \geq 2$ ,  $1/2 \log(1 - 2\rho \cos \theta + \rho^2) + \rho \cos \theta \leq \log(\rho - 1) + \rho$ , whence for  $\rho \geq 2$ ,

$$\begin{aligned} g(z, p) &= \rho^{-p} \left\{ \frac{1}{2} \log(1 - 2\rho \cos \theta + \rho^2) + \sum_1^p \frac{\rho^r}{r} \cos r\theta \right\} \\ &\leq \rho^{-p} \left\{ \log(\rho - 1) + \sum_1^p \frac{\rho^r}{r} \right\} = g(\rho, p). \end{aligned}$$

Also  $g(\rho, 2)$  is decreasing for  $\rho > 2$ , for by (10) we find that

$$\frac{dg(\rho, 2)}{d\rho} = \frac{-2}{\rho^3} \log(\rho - 1) - \frac{\rho - 2}{\rho^2(\rho - 1)}.$$

But we now see from the definition of  $g(z, p)$  that

$$g(\rho, p+1) = \frac{1}{p+1} + \frac{1}{\rho} g(\rho, p)$$



and so by induction we see that  $g(\rho, p)$  decreases for  $\rho \geq 2$  for each  $p \geq 2$ . Thus

$$(18) \quad \text{for } \rho \geq 2, p \geq 2 \quad g(z, p) \leq g(2, p),$$

with equality only for  $z = 2$ .

Consider first the case  $p = 1$ . If  $\rho \leq 2$ , we find that for given  $\rho$ ,  $g$  is greatest when  $2 \cos \theta = \rho$ , or  $g(z, 1) \leq \rho/2 \leq 1$  for  $\rho \leq 2$ . For  $2 \leq \rho$ , we know that  $g(z, 1) \leq g(\rho, 1) = \rho^{-1} \log(\rho - 1) + 1$ , and it is easily seen that this expression has precisely one turning value, and that a maximum, which occurs where  $(\rho - 1) \log(\rho - 1) = \rho$ : This gives  $\rho = 4.5911$  and then  $g(\rho, 1) = 1.2785$ . Thus  $\alpha_1 = 1.2785$ .

Secondly, consider  $p = 2$ . For  $\rho \leq 2$  we have

$$\begin{aligned} \rho^2 g(z, 2) &= \frac{1}{2} \log(1 - 2\rho \cos \theta + \rho^2) + \rho \cos \theta + \frac{1}{2} \rho^2 \cos 2\theta \\ &\leq \frac{1}{2} \rho^2 + \frac{1}{2} \rho^2 \cos 2\theta, \text{ as before} \\ &\leq \rho^2, \end{aligned}$$

where equality occurs only if  $\rho = 2 \cos \theta$  and  $\cos 2\theta = 1$  are satisfied simultaneously; this does occur and at the single point  $z = 2$ . Thus  $\alpha_2 = 1$ .

Finally we consider  $p \geq 3$ , and then in view of (18) we need only consider the annulus  $1 \leq |z| \leq 2$ . At a local maximum, we obtain from (8),  $\rho \sin p\theta = \sin(p+1)\theta$ . In view of (9)  $\theta = 0$  arises only if  $\rho \geq 1 + p^{-1}$ , since otherwise  $\partial^2 g / \partial \theta^2$  is positive.  $\theta = \pi$  can be dismissed, since by (10) a local maximum at such a point would give  $g \leq p^{-1} \leq 1/3 < \sigma_p$ , by (16). Referring to the figure, we find therefore that we need to consider three cases

- (a)  $\theta = 0$  for  $\rho \geq 1 + p^{-1}$ ,
- (b)  $0 < \theta \leq \pi/(2p+1)$  for  $1 \leq \rho \leq 1 + p^{-1}$ ,
- (c) values of  $\theta$  between  $\pi/p$  and  $\pi - \pi/(p+1)$ .

As before the final case can be dismissed, since at such a local maximum we find from (10) that

$$\begin{aligned} g &= \frac{\sin(p+1)\theta}{p \sin \theta} \\ &\leq p^{-1} \operatorname{cosec} \frac{\pi}{p+1} \\ &= \frac{p+1}{\pi p} \frac{\pi}{p+1} \operatorname{cosec} \frac{\pi}{p+1} \\ &\leq \frac{p+1}{p} \frac{1}{4} \operatorname{cosec} \frac{1}{4} \pi \\ &\leq \frac{1}{3} \operatorname{cosec} \frac{1}{4} \pi = \frac{1}{3} 2^{1/2} < \sigma_p, \text{ in view of (16)}. \end{aligned}$$

Now in the second case, let  $m(\rho)$  be the value taken by  $g(z, p)$  when  $\rho \sin p\theta = \sin(p+1)\theta$  and  $0 < \theta \leq \pi/(2p+1)$ . Then using (8) and (10) we obtain similarly to (17),

$$(19) \quad \frac{dm}{d\rho} = \frac{-pm}{\rho} + \frac{\sin p\theta}{\sin \theta},$$

and so

$$\begin{aligned} \frac{d}{d\rho} \left\{ \rho^{p+1} \frac{dm}{d\rho} \right\} &= \rho^p \frac{d}{d\rho} \left\{ \rho \frac{dm}{d\rho} + pm \right\} \\ &= \rho^p \frac{d}{d\theta} \left\{ \frac{\sin(p+1)\theta}{\sin \theta} \right\} / \frac{d}{d\theta} \left\{ \frac{\sin(p+1)\theta}{\sin p\theta} \right\} \\ &> 0, \text{ as before.} \end{aligned}$$

Thus  $\rho^{p+1} dm/d\rho$  increases as  $\rho$  increases from 1 to  $1+p^{-1}$ . But using (19) we see that when  $\rho = 1$ ,

$$\begin{aligned} \frac{dm}{d\rho} &= -p\sigma_p + \sin \frac{p\pi}{2p+1} / \sin \frac{\pi}{2p+1} \\ &= -p\sigma_p + \frac{1}{2} \operatorname{cosec} \frac{\pi}{4p+2} \\ &> -p\sigma_p + (2p+1)/\pi \\ &> p(2/\pi - \sigma_p) > 0, \text{ in view of (16).} \end{aligned}$$

Thus  $m(\rho)$  is an increasing function of  $\rho$  as  $\rho$  increases from 1 to  $1+p^{-1}$ , and in particular  $g(1+p^{-1}, p) \geq g(z, p)$  for  $|z| \leq 1+p^{-1}$ . Thus we need only consider case (a).

Let

$$\begin{aligned} \Delta_p &= f(1+p^{-1}, p) \\ (20) \quad &= -\log p + \sum_1^p \frac{1}{r} \left(1 + \frac{1}{p}\right)^r \\ &= -\log p + \sum_1^p \int_0^{1+p^{-1}} t^{r-1} dt \\ &= -\log p + \int_0^{1+p^{-1}} \frac{t^p - 1}{t-1} dt. \end{aligned}$$

Thus

$$\begin{aligned}
\Delta_{p+1} - \Delta_p &= -\log \frac{p+1}{p} + \int_0^{1+(p+1)^{-1}} \frac{t^{p+1} - t^p}{t-1} dt \\
&\quad - \int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{t^p - 1}{t-1} dt \\
&= -\log \frac{p+1}{p} + \frac{1}{p+1} \left\{ 1 + \frac{1}{p+1} \right\}^{p+1} + \log \frac{p+1}{p} \\
&\quad - \int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{t^p}{t-1} dt \\
&= \frac{1}{p+1} \left\{ 1 + \frac{1}{p+1} \right\}^{p+1} - I, \text{ say.}
\end{aligned}$$

To estimate  $I$  we observe that for  $t > 1 + (p+1)^{-1}$ ,

$$\frac{d}{dt} \left( \frac{t^{p+2}}{t-1} \right) = \frac{(p+1)t^{p+1}}{(t-1)^2} \left( t - 1 - \frac{1}{p+1} \right) > 0,$$

and so

$$\begin{aligned}
I &= \int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{t^{p+2}}{t-1} \frac{dt}{t^2} \\
&> (p+1) \left\{ 1 + \frac{1}{p+1} \right\}^{p+2} \int_{1+(p+1)^{-1}}^{1+p^{-1}} \frac{dt}{t^2} \\
&= (p+2) \left\{ 1 + \frac{1}{p+1} \right\}^{p+1} \left\{ \frac{p+1}{p+2} - \frac{p}{p+1} \right\} \\
&= \frac{1}{p+1} \left\{ 1 + \frac{1}{p+1} \right\}^{p+1},
\end{aligned}$$

and so

$$(21) \quad \Delta_{p+1} < \Delta_p.$$

From (10) we see that if  $\theta = 0$ ,  $\partial g / \partial \rho = -pg/\rho + (\rho-1)^{-1}$ , and so it is easily verified that  $\partial g / \partial \rho > 0$  at  $z = 1 + p^{-1}$ , and that  $\partial g / \partial \rho < 0$  at  $z = 2$ . Thus there exists at least one turning value of  $g$  on the real axis between these two points. At such a point  $g = \rho/p(\rho-1)$  and so

$$\begin{aligned}
\frac{\partial^2 g}{\partial \rho^2} &= -\frac{p}{\rho} \frac{\partial g}{\partial \rho} + \frac{pg}{\rho^2} - \frac{1}{(\rho-1)^2} \\
&= \frac{1}{\rho(\rho-1)} - \frac{1}{(\rho-1)^2} < 0,
\end{aligned}$$

and so there is exactly one such turning value, and that a maximum. Now let

$$(22) \quad \mu(x, p) = g(1 + xp^{-1}, p), \quad x \geq 1.$$

Then using (20) we find

$$\begin{aligned}
\mu(x, p) &= \left(1 + \frac{x}{p}\right)^{-p} \left\{ \log \frac{x}{p} + \sum_1^p \frac{1}{r} \left(1 + \frac{x}{p}\right)^r \right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p} \left\{ \Delta_p + \log x + \sum_1^p \frac{1}{r} \left\{ \left(1 + \frac{x}{p}\right)^r - \left(1 + \frac{1}{p}\right)^r \right\} \right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p} \left\{ \Delta_p + \log x + \sum_1^p \int_{1+p^{-1}}^{1+xp^{-1}} t^{r-1} dt \right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p} \left\{ \Delta_p + \log x + \int_{1+p^{-1}}^{1+xp^{-1}} \frac{t^p - 1}{t - 1} dt \right\} \\
&= \left(1 + \frac{x}{p}\right)^{-p} \left\{ \Delta_p + \int_1^x \left(1 + \frac{s}{p}\right)^p \frac{ds}{s} \right\} \\
&= A_p(x) + B_p(x), \text{ say.}
\end{aligned}$$

Now,  $(1 + x/n)^n$  is an increasing sequence and by (21)  $\Delta_n$  is decreasing. Thus  $A_{p+1}(x) < A_p(x)$ . We shall show that  $B_{p+1}(x) \leq B_p(x)$  too. We find that for  $s < x$ ,

$$\begin{aligned}
\frac{\left(1 + \frac{s}{p}\right)^p}{\left(1 + \frac{x}{p}\right)^p} &\div \frac{\left(1 + \frac{s}{p+1}\right)^{p+1}}{\left(1 + \frac{x}{p+1}\right)^{p+1}} = \frac{(p+s)^p(p+1+x)^{p+1}}{(p+x)^p(p+1+s)^{p+1}} \\
&= \frac{p+1+x}{p+1+s} \left\{ 1 + \frac{x-s}{p^2 + p(x+s+1) + s+sx} \right\}^{-p} \\
&> \frac{p+1+x}{p+1+s} \left\{ 1 - \frac{p(x-s)}{p^2 + p(x+s+1) + s+sx} \right\} \\
&= \frac{p^2 + p(2s+1) + s+sx}{p^2 + p(2s+1) + s+s^2} > 1,
\end{aligned}$$

since  $(1 + \varepsilon)^{-p} > 1 - p\varepsilon$  for every positive  $\varepsilon$ . Thus

$$\left(1 + \frac{x}{p}\right)^{-p} \left(1 + \frac{s}{p}\right)^p > \left(1 + \frac{x}{p+1}\right)^{-p-1} \left(1 + \frac{s}{p+1}\right)^{p+1}$$

for  $1 \leq s < x$ , and so  $B_p(x) \geq B_{p+1}(x)$ .

We see therefore that

$$g\left(1 + \frac{x}{p+1}, p+1\right) < g\left(1 + \frac{x}{p}, p\right) \leq \alpha_p,$$

and so  $\alpha_{p+1} < \alpha_p$ .

Also, since  $\mu(x, p) > \mu(x, p+1)$ , we see that

$$\mu(x, p) > \mu(x) = \lim_{p \rightarrow \infty} \mu(x, p) = e^{-x} \left\{ \Delta + \int_1^x s^{-1} e^s ds \right\},$$

where  $\Delta = \lim_{p \rightarrow \infty} \Delta_p$ . Now from (20) we find that

$$\begin{aligned} \Delta_p &= -\log p + \sum_1^p \frac{1}{r} + \sum_1^p \frac{1}{r} \left\{ \left(1 + \frac{1}{p}\right)^r - 1 \right\} \\ &= -\log p + \sum_1^p \frac{1}{r} + \sum_1^p \int_0^1 \frac{1}{p} \left(1 + \frac{s}{p}\right)^{r-1} ds, \\ &= -\log p + \sum_1^p \frac{1}{r} + \int_0^1 \left\{ \left(1 + \frac{s}{p}\right)^p - 1 \right\} \frac{ds}{s}, \end{aligned}$$

and so

$$\Delta = \gamma + \int_0^1 (e^s - 1)s^{-1} ds = 1.895118.$$

Now

$$\begin{aligned} \mu'(x) &= -\mu(x) + x^{-1} \\ \mu''(x) &= -\mu'(x) - x^{-2}, \end{aligned}$$

and so  $\mu(x)$  has precisely one maximum, and at this point  $\mu(x) = x^{-1}$ , with

$$x^{-1}e^x = \Delta + \int_1^x s^{-1}e^s ds = \Delta + x^{-1}e^x - e + \int_1^x s^{-2}e^s ds,$$

or

$$\int_1^x s^{-2}e^s ds = e - \Delta = 0.823164,$$

whence  $x = 1.3472$  and so  $\mu_{\max} = 0.7423$ .

Thus we find that since  $\mu(x, p) > \mu(x)$ , we can always choose  $x$  such that  $\mu(x, p) > 0.7423$ , and so  $a_p > 0.7423$ . Thus as  $p$  increases from 2 to  $\infty$ ,  $a_p$  decreases from 1 to 0.7423.

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ROYAL HOLLOWAY COLLEGE, ENGLEFIELD GREEN, SURREY, ENGLAND

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