PROXIMITY APPROACH TO SEMI-METRIC AND DEVELOPABLE SPACES

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In this paper we study semi-metric and developable spaces via generalized proximities and uniformities. We find sufficient conditions for a compatible semi-metric d on a space X to induce a Lodato proximity and also study the effects on a space X when d satisfies various weaker forms of continuity. We present two new characterizations of developable spaces, one of which reads: A T_1-space is developable if and only if it has a compatible upper semi-continuous semi-metric.

We give improved versions of two known metrization theorems. Finally, we generalize the concepts: T_1-map, uniform map, completely uniform map, pseudo-open map, etc., to apply to proximity spaces and improve some of the known results; for example, an open uniform image of a developable space is developable.

Our motivation for this study was a result mentioned by Arkhangel'skii [4]: A semimetric space (X, d) is metrizable if the induced nearness relation $\delta(d)$ is an EF-proximity (where $A\, \delta(d)\, B$ iff $d(A, B) = \text{Inf} \{d(a, b): a \in A, b \in B\} = 0$). One naturally wonders as to what properties are satisfied by X if $\delta(d)$ satisfies weaker proximity axioms such as those of an S-, a LO- or an R-proximity. We soon found that $\delta(d)$ is always an S-proximity and that, if d is the natural semi-metric associated with a developable space, then $\delta(d)$ is a LO-proximity. Further impetus was provided by a result due to Pareek [15]: A T_1-space is semi-metrizable iff there exists a countable family $\{W_n: n \in N\}$ of symmetric subsets of $X \times X$ satisfying (a) $\bigcap_{n=1}^{\infty} W_n = \Delta$ and (b) for each $x \in X$, $\{W_n[x]: n \in N\}$ forms a neighbourhood base at x. This result is analogous to the well-known Alexandroff-Urysohn theorem: A T_1-space is metrizable iff it has a compatible uniformity with a countable base. This provided us with a motivation to find one of our new characterizations of developable spaces.

The basic definitions and results are given in this section. In the next section, we study semi-metrizable spaces having compatible semi-metric which satisfy various weaker forms of continuity, e.g., those which are lower semi-continuous or are separately continuous. We also find sufficient conditions for $\delta(d)$ to be a LO-proximity.

In the third section, we obtain two new characterizations of developable spaces and also find a sufficient condition for a space to

be a totally bounded developable space. In §4, we prove two new
metrization theorems. Finally, in the fifth section, we study the
images of some specific proximity spaces under some special mappings.

We now give some basic definitions and results which are needed
in this paper. Let X be a nonempty set and let δ be a binary relation
on the power set of X. Consider the following axioms:

1.1. (i) $A\delta B$ implies $B\delta A$,

(ii) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$,

(iii) $A\delta B$ implies $A \neq \phi$, $B \neq \phi$,

(iv) $A \cap B \neq \phi$ implies $A\delta B$,

(v) $a\delta B$ and $b\delta C$ for every $b\varepsilon B$ implies $a\delta C$,

(vi) $A\delta B$ and $b\delta C$ for every $b\varepsilon B$ implies $A\delta C$,

(vii) $a\hat{\delta}B$ implies the existence of an $E \subset X$ such that $a\hat{\delta}E$
and $(X - E)\hat{\delta}B$,

(viii) $A\hat{\delta}B$ implies the existence of an $E \subset X$ such that $A\hat{\delta}E$
an $(X - E)\hat{\delta}B$.

DEFINITION 1.2. The relation δ is called

(a) an *S-proximity* if it satisfies 1.1 (i)-(iv) and (v) (see [13]),

(b) a *LO-proximity* if it satisfies 1.1 (i)-(iv) and (vi) (see [13]),

(c) an *R-proximity* if it satisfies 1.1 (i)-(iv) and (vii) (see [11]),

(d) an *EF-proximity* if it satisfies 1.1 (i)-(iv) and (viii) (see [13]).

By a "proximity δ", we mean any one of the four proximities
defined above and in this case we call the pair (X, δ) a *proximity space*.
A binary relation δ that satisfies 1.1 (i)—(iv) induces an operator
$A \to \bar{A} = \{x \in X : x\delta A\}$ on the power set of X, and this will be a
Kuratowski closure operator if and only if δ is an S-proximity. Since
(viii) implies (vii), (viii) implies (vi), and (vii) or (vi) implies (v), then
every proximity is an S-proximity. The topology induced by the
closure operator of a proximity δ will be written $\tau(\delta)$.

*We assume that all proximities δ considered in this paper are
separated, i.e., $x\delta y$ implies $x = y$. Obviously in this case $\tau(\delta)$ is T_1.*

If (X, τ) is a topological space and δ is a proximity on X such
that $\tau = \tau(\delta)$, then we say that τ and δ are compatible. It is known
that: (i) every T_1-space has a compatible LO- (and ipso facto S-)
proximity [13]; (ii) A topological space is T_3 iff it has a compatible
R-proximity [11]; (iii) A topological space is Tychonoff iff it has a
compatible EF-proximity.

DEFINITION 1.3. A *semi-metric space* (X, d) is a T_1-space X to-
gether with a real-valued function d on $X \times X$ such that

(a) $d(x, y) = d(y, x) \geq 0$,

(b) $d(x, y) = 0$ iff $x = y$,

(c) $x \in M^-$ iff $d(x, M) = 0$, where $d(A, B)$
$$= \text{Inf} \{d(a, b): a \in A, b \in B\}, \text{ if } A \neq \phi \neq B$$
$$= \infty \text{ if } A = \phi \text{ or } B = \phi.$$

If (X, d) is a semi-metric space, we define $\delta = \delta(d)$ by

$$A\delta B \quad \text{iff} \quad d(A, B) = 0 .$$

It is easily verified that $\delta(d)$ is an S-proximity on X.   For $\varepsilon > 0$, we set

$$V_\varepsilon = \{(x, y) \in X \times X: d(x, y) < \varepsilon\} .$$

Clearly $V_\varepsilon[x] = S(x, \varepsilon)$, the sphere with centre x and radius ε.   We set $\mathscr{U}_d = \{U = U^{-1} \subset X \times X: V_{1/n} \subset U \text{ for some } n \in N\}$.   Also for $\mathscr{U} \subset \mathscr{P}(X \times X)$ (the power set of $X \times X$), we define $\delta = \delta(\mathscr{U})$ by

$$A\delta B \quad \text{iff} \quad (A \times B) \cap U \neq \phi \quad \text{for each} \quad U \in \mathscr{U} .$$

Clearly, if d is a semi-metric on X, then $\delta(d) = \delta(\mathscr{U}_d)$.

DEFINITION 1.4.   A *refining family* Σ on a toplogical space (X, τ) is a family $\{\alpha_i: i \in I\}$ of open covers of X such that for each $x \in U \in \tau$, there exists an $i \in I$ such that $\text{St}(x, \alpha_i) \subset U$.   In the special case when $I = N$, Σ is called a *development* on X and the pair (X, Σ) is called a *developable space*.   In this case, it is well known that Σ may be replaced by another development $\Sigma' = \{\beta_j: j \in N\}$ such that if $i < j$ then $\beta_j \subset \beta_i$; we will assume that Σ already satisfies this condition. We will also assume that the developable spaces are T_1.

LEMMA 1.5.   *Every refining family* Σ *on a* T_1-*space* (X, τ) *induces a compatible LO-proximity* $\delta = \delta(\Sigma)$ *on* X, *where* δ *is defined by*

$$A\delta B \quad iff \quad \text{St}(A, \alpha_i) \cap B \neq \phi \quad \text{for each } i \in I .$$

Proof.   That δ satisfies 1.1 (i)—(iv), is separated, and is compatible with τ is obvious.   We now show that δ satisfies 1.1 (vi).   Suppose $A\delta B$ and $b\delta C$ for each $b \in B$.   Then $A\delta B$ implies that for each $i \in I$, there is a $b \in B$ such that $b \in \text{St}(A, \alpha_i)$.   Since $\text{St}(A, \alpha_i)$ is open, there is a $j \in I$ such that $b \in \text{St}(b, \alpha_j) \subset \text{St}(A, \alpha_i)$.   Since $b\delta C$, $C \cap \text{St}(b, \alpha_j) \neq \phi$ and this, in turn, implies $C \cap \text{St}(A, \alpha_i) \neq \phi$, i.e., $A\delta C$.

For a developable space (X, Σ), $\Sigma = \{\lambda_n: n \in N\}$, we define $d = d(\Sigma)$ by

$$d(x, y) = \text{Inf} \left\{ \frac{1}{n+1}: y \in \text{St}(x, \lambda_n) \right\} .$$

It is easily seen that d is a compatible semi-metric on X and that

$\delta(d) = \delta(\Sigma)$. For each $n \in N$, we set

$$B_n = \cup \{G \times G: G \in \lambda_n\} \, .$$

LEMMA 1.6. $B_n = V_{1/n}$.

Proof. $(x, y) \in V_{1/n}$ iff $d(x, y) \leqq 1/(n + 1) < 1/n$ iff $y \in \text{St}\,(x, \lambda_n)$ iff $(x, y) \in B_n$.

DEFINITION 1.7. An *M-uniformity base* $\mathscr{B}$ on X is a family of subsets of $X \times X$ such that,
   (i) $\cap \{U: U \in \mathscr{B}\} = \varDelta$,
   (ii) $U = U^{-1}$ for each $U \in \mathscr{B}$,
   (iii) for each $A \subset X$ and U, V in $\mathscr{B}$, there exists a $W \in \mathscr{B}$ such that $W[A] \subset U[A] \cap V[A]$,
   (iv) for each $U \in \mathscr{U}$, $U \subset V = V^{-1} \subset X \times X$ implies $V \in \mathscr{B}$,
   (v) for A, B in $P(X)$ and $U \in \mathscr{B}$, if $V[A] \cap B \neq \phi$ for each $V \in \mathscr{B}$; then there exists an $x \in B$ and a $W \in \mathscr{B}$ such that $W[x] \subset U[A]$. $\mathscr{U} \subset P(X \times X)$ is said to be an *M-uniformity* on X iff there exists a family $\mathscr{B} \subset \mathscr{U}$ satisfying conditions (i)—(v) above and for each $U \in \mathscr{U}$ there is a $B \in \mathscr{B}$ such that $B \subset U$.

REMARK 1.8. Mozzochi [12] has shown that an M-uniformity $\mathscr{U}$ need not have an open base and that if it does have one, then every convergent filter on X is $\mathscr{U}$-Cauchy.

DEFINITION 1.9. An *S-uniformity* $\mathscr{U}$ on X is a family of subsets of $X \times X$ which has a base $\mathscr{B}$ satisfying 1.7 (i)—(iv) and for each $p \in X$, $B \subset X$ and $U \in \mathscr{B}$, if $V[p] \cap B \neq \phi$ for each $V \in \mathscr{B}$, then there exists an $x \in B$ and a $W \in \mathscr{B}$ such that $W[x] \subset U[p]$.

If $\mathscr{U}$ is an M- or an S-uniformity, then $\tau(\mathscr{U})$ is defined as usual; viz. $G \in \tau(\mathscr{U})$ iff for each $x \in G$, there exists a $U \in \mathscr{U}$ such that $U[x] \subset G$. If $\tau = \tau(\mathscr{U})$, we say that τ and $\mathscr{U}$ are compatible.

THEOREM 1.10. (See [7], [12].) *If $\mathscr{U}$ consists of symmetric subsets of $X \times X$, then $\mathscr{U}$ is an M-uniformity base (resp. S-unformity base) if and only if $\delta(\mathscr{U})$ is a LO-proximity (resp. S-proximity).*

Let (X, d) be a semi-metric space and set $V_{1/n} = \{(x, y): d(x, y) < 1/n\}$. Then $\{V_{1/n}: n \in N\}$ is a countable base for a compatible S-uniformity.

2. **Semi-metric spaces.** In this section we suppose that (X, d) is a semi-metric space and consider the effects of various forms of continuity properties of d on the topology of X and on the proximity $\delta(d)$.

LEMMA 2.1.   *In the following* (i) *and* (ii) *are equivalent and each implies* (iii):

(i)   *d is separately upper semi-continuous.*

(ii)   *For each $\varepsilon > 0$ and $x \in X$, $S(x, \varepsilon)$ is open.*

(iii)   *$\delta(d)$ is a LO-proximity on X.*

Proof.   (i)$\Rightarrow$(ii).   Let $y \in S(x, \varepsilon)$, i.e., $d(x, y) < \varepsilon$. Since d is upper semi-continuous in y, for every $\eta > 0$ there is a nbhd. N_y of y such that $d(x, z) < d(x, y) + \eta$ for each $z \in N_y$. Choose $\eta < \varepsilon - d(x, y)$. Then clearly $N_y \subset S(x, \varepsilon)$, showing thereby that $S(x, \varepsilon)$ is open.

(ii) $\Rightarrow$ (i).   Suppose $d(x, y) = r$ and $\varepsilon > 0$. Clearly $y \in S(x, r + \varepsilon)$, which is open, and hence there exists a nbhd. N_y of y such that $N_y \subset S(x, r + \varepsilon)$. But this means that for each $z \in N_{y'}, d(x, z) < d(x, y) + \varepsilon$, i.e., d is separately upper semi-continuous.

(i) $\Rightarrow$ (iii).   Let $\delta = \delta(d)$ and suppose $A \delta B$ and $b \delta C$ for each $b \in B$. Then for each $\varepsilon > 0$, there exists an $a \in A$ and a $b \in B$ such that $d(a, b) < \varepsilon$. Since d is upper semi-continuous at b, there exists a nbhd. N_b of b such that $d(a, x) < \varepsilon$ for each $x \in N_b$. Also $b \delta C$ implies the existence of a point $c \in C \cap N_b$ and hence, $d(a, c) < \varepsilon$, i.e., $A \delta C$.

COROLLARY 2.2.   *If a semi-metric d is separately upper semi-continuous, then $\mathscr{U}_d$ is an M-uniformity (obviously with a countable base).*

LEMMA 2.3.   *If d is separately lower semi-continuous, then X is regular.*

Proof.   Let A be closed in X and $p \in X - A$. Then $d(p, A) = r > 0$ and hence for each $a \in A, d(p, a) \geqq r$. Since d is lower semi-continuous at a, there exists a nbhd. N_a of a such that for each $x \in N_a, d(p, x) > r/2$. Set $N_A = \cup\{N_a: a \in A\}$. Then N_A is a nbhd. of A and $N_A \cap S(p, r/2) = \phi$, thereby showing that X is regular.

The following result is an improvement of a similar one stated by Cook [6]; whereas Cook require the semi-metric d to be continuous we want d to be only *separately* continuous.

THEOREM[1] 2.4.   *If d is separately continuous, then X is Tychonoff.*

Proof.   Suppose A is a closed subset of X and $x \in X - A$. We may assume that $d(x, A) = 1$. As in the proof of Urysohn's lemma, we now show how to construct, for each positive rational $r \in [0, 1]$, an open set V_r such that $x \in V_r$ for each such r, $V_r^- \subset V_s$ whenever $r < s$ and each $V_r \subset X - A$. We set $V_r = \{y \in X: d(x, y) < r\}$ which is open

by 2.1 (ii). Also the separate lower semi-continuity of d is equivalent to the set $\{y \in X : d(x, y) \leqq r\}$, being closed for each r.

The following theorem is an analogue of the result: A T_1-space is uniformizable with a countable base iff it is metrizable (and hence has a metric d such that $\delta(d)$ is an EF-proximity.)

THEOREM 2.5. *A T_1-space is M-uniformizable with a countable base if and only if it has a compatible semi-metric d such that $\delta(d)$ is a LO-proximity.*

Proof. Sufficiency is evident from 1.10 and necessity follows from 1.10 and Theorem 4.1.7 of Pareek [15], making use of the remarks just preceding 1.4.

The followilng analogue of the above result is proved similarly.

THEOREM 2.6. *A T_1-space is S-uniformizable with a countable base if and only if it has a compatible semi-metric d (and obviously $\delta(d)$ is an S-proximity).*

3. **Developable spaces.** In this section we suppose that (X, Σ) is a developable space with $\Sigma = \{\lambda_n : n \in N\}$ where each λ_n is an open cover of X and $\lambda_{n+1} \subset \lambda_n$. Let $d = d(\Sigma)$ be the induced semi-metric on X. Then $\delta(d) = \delta(\Sigma)$ is a LO-proximity on X.

LEMMA 3.1. *d is upper semi-continuous.*

Proof. If $p, q \in X$, we have to consider two cases: (i) $d(p, q) = 0$ and (ii) $d(p, q) = 1/(m + 1)$ for some $m \in N$. Let $\varepsilon > 0$ be arbitrary. In case (i), $\delta(d)$ is a LO-proximity by 2.1 and $p = q$. Now choose $n \in N$ such that $1/n < \varepsilon$; then $p, q \in G$ for some $G \in \lambda_n$. For all $(x, y) \in G \times G, d(x, y) \leqq 1/(n + 1) < \varepsilon$. In case (ii), $p, q \in G$ for some $G \in \lambda_m$. Then for all $(x, y) \in G \times G, d(x, y) \leqq 1/(m + 1) < 1/(m + 1) + \varepsilon$. Thus the result is proved.

The above result (in conjunction with 2.1) provides an alternate proof of the fact that $\delta(\Sigma)$ is a compatible LO-proximity and also shows that $\mathscr{U}_d$ is a compatible M-uniformity with a countable *open* base $\{B_n : n \in N\}$. This provides a motivation for our next result.

THEOREM 3.2. *A T_1-space is developable if and only if it is M-uniformizable with a countable open base.*

Proof. Necessity follows from the remarks preceding 3.2 and sufficiency follows from Brown's result ([5], p. 65) that a space is developable iff it has a compatible semi-metric d for which every con-

vergent sequence is Cauchy, using 1.8.

In [6] Cook states that if a compatible semi-metric d on X is continuous, then X is developable. The following characterization of developable spaces is an improvement of this result.

THEOREM 3.3. *A T_1-space X is developable if and only if it has a compatible upper semi-continuous semi-metric.*

Proof. Since the necessity has been proved in (3.1), we need prove only sufficiency. Assuming d to be an upper semi-continuous semi-metric, we prove that $V_{1/n}$ is open for each $n \in N$ and then the result will follow from (3.2). If $(p, q) \in V_{1/n}$, then $d(p, q) < 1/n$. Choose $\varepsilon > 0$ such that $\varepsilon < 1/n - d(p, q)$. Then from the upper semi-continuity of d, there exist nbhds. N_p, N_q of p, q respectively, such that $d(x, y) < d(p, q) + \varepsilon < 1/n$ for each $(x, y) \in N_p \times N_q$. This shows that $(N_p \times N_q) \subset V_{1/n}$, showing thereby that $V_{1/n}$ is open.

DEFINITION 3.4. *A proximity base $\mathscr{B}$ for a proximity space (X, δ) is a family of subsets of X such that if $A\mathring{\delta}B$, then there exist $C, D \in \mathscr{B}$ such that $A \subset C$, $B \subset D$ and $C\mathring{\delta}D$.*

In a *LO*-space, we may assume that the members of $\mathscr{B}$ are closed. The following is an improvement of 2.23 of [12].

LEMMA 3.5. *If $\mathscr{B}$ is a proximity base for LO-space (X, δ), then $\{U_{A,B}: A, B \in \mathscr{B}$ and $A\mathring{\delta}B\}$, where $U_{A,B} = X \times X - [(A \times B) \cup (B \times A)]$, is a base for an M-uniformity $\mathscr{U}_1$ on X.*

A developable space (X, Σ) is said to be *totally bounded* iff for each $n \in N$, there exists a finite set $F \subset X$ such that $\mathrm{St}\,(F, \lambda_n) = X$. It is known that an *EF*-space (X, δ) has a countable base iff X has a compatible totally bounded metric (Theorem (8.19) of [13]). The following is a partial generalization.

THEOREM 3.6. *If a LO-space (X, δ) has a countable closed base $\mathscr{B}$, then X is a totally bounded developable space.*

Proof. $\{U_{A,B}: A, B \in \mathscr{B}, A\mathring{\delta}B\}$ is a countable open base for a compatible *M*-unifomity, which is also totally bounded. The result then follows from (3.2).

4. Metrizable spaces. Arkhangel'skii [4] proved that a semi-metric space (X, d) is metrizable if $\delta(d)$ is an *EF*-proximity. A glance at his proof shows that the following improved version is true.

THEOREM 4.1. *A T_1-space is uniformizable with a countable base if and only if it has a compatible semi-metric d such that $\delta(d)$ is an R-proximity if and only if it is metrizable.*

Nedev [14] proved that a semi-metric space (X, d) is metrizable if $d(A, x)$ is a continuous function of x for each $A \subset X$. The following is an improvement.

THEOREM 4.2. *A semi-metric space (X, d) is metrizable if for all closed subsets A of X, $d(A, x)$ is lower semi-continuous.*

Proof. Suppose A is closed, B is compact and $A \cap B = \phi$. Since the function $d(A, x)$ is lower semi-continuous and B is compact, it follows that $d(A, B) = d(A, x)$ for some $x \in B$. This implies that X is metrizable (see [4]).

We now give a table which shows the relationship of our results with the classical Alexandroff-Urysohn uniform metrization theorem. Suppose $\{W_n : n \in N\}$ is a countable family of symmetric subsets of $X \times X$ satisfying (a) $\bigcap_{n=1}^{\infty} W_n = \varDelta$ and (b) for each $x \in X$, $\{W_n[x] : n \in N\}$ forms a nbhd. base at x. Let d be the semi-metric by $\{W_n\}$, namely

$$d(x, y) = \text{Inf} \left\{ \frac{1}{n+1} : y \in W_n[x] \right\}.$$

(We assume without any loss of generality that $W_{n+1} \subset W_n$.) Finally set $\mathscr{U} = \{U \subset X \times X : W_n \subset U \text{ for some } n \in N\}$.

$\mathscr{U}$ is an S-uniformity	$\mathscr{U}$ is an M-uniformity	$\mathscr{U}$ is an M-uniformity. Each W_n is open	$\mathscr{U}$ is a uniformity
$\delta(d)$ is an S-proximity	$\delta(d)$ is a LO-proximity	d is USC $\delta(d)$ is a LO-proximity	$\delta(d)$ is an EF-proximity
X is semi-metrizable	X is semi-metrizable	X is developable	X is metrizable

5. Metric spaces, developable spaces, semi-metric spaces and mappings connected with them. In this section $f : X \to Y$ will denote a function from a proximity space (X, δ) onto a T_1-space Y. When X is developable, semi-metrizable or metrizable, δ will denote the corresponding naturally induced proximity relation as defined in §1. Several kinds of functions, which have been defined for the case in which

X is metrizable, can be redefined more generally when X is a proximity space; this is done by replacing the condition $d(A, B) > 0$ by $A \bar{\delta} B$. These mappings have been systematically discussed in the metric case by Arkhangel'skii [4]. Although we consider their generalizations, for the sake of simplicity we will keep the same terminology and attach δ-before each term[2].

Our strategy consists in defining a binary relation δ^1 on the power set of Y as follows.

$$(5.1) \qquad\qquad E \bar{\delta}^1 F \quad \text{iff} \quad f^{-1}(E) \bar{\delta} f^{-1}(F) .$$

It is easily verified that δ^1 satisfies 1.1 (i)—(iv) and so δ^1 is almost a quotient proximity. In order that δ^1 be a proximity, naturally we will have to put some additional conditions on f; also it is clear that if δ^1 is to satisfy stronger proximity conditions, so must δ.

Our first task is to find conditions on f which will make δ^1 an S-proximity compatible with the topology of Y.

DEFINITION 5.2. The function f is called δ-*pseudo-open* iff for each $y \in Y$ and $A \subset X$, if $f^{-1}(y) \bar{\delta} (X - A)$ then $y \in \text{Int} f(A)$.

It is easily verified that if f is open or closed, then it is also δ-pseudo-open.

LEMMA 5.3. *If δ is an S-proximity on X and f is δ-pseudo-open, then $y \bar{\delta}^1 E$ implies $y \notin E^-$.*

Proof. $y \bar{\delta}^1 E$ implies $f^{-1}(y) \bar{\delta} f^{-1}(E)$, and since f is δ-pseudo-open, it follows that $y \in \text{Inf} f[X - f^{-1}(E)]$. But $E \cap f[X - f^{-1}(E)] = \phi$ and hence $y \notin E^-$.

COROLLARY 5.4. *If δ is an S-proximity and f is either open or closed, then $y \bar{\delta}^1 E$ implies $y \notin E^-$.*

DEFINITION 5.5. The function f is called δ-*uniform* iff for each $y \in Y$ and each nhbd. N_y of Y, $f^{-1}(y) \bar{\delta} (X - f^{-1}(N_y))$.

LEMMA 5.6. *If δ is an S-proximity, then f is δ-uniform if and only if $y \notin E^-$ implies $y \bar{\delta}^1 E$.*

Proof. If f is δ-uniform and $y \notin E^-$, then $f^{-1}(y) \bar{\delta} f^{-1}(E)$ and by 5.1, $y \bar{\delta}^1 E$. Conversely, if N_y is a nbhd. of y, then $y \notin (Y - N_y)^-$ implies $y \bar{\delta}^1 (Y - N_y)$ and this is equivalent to $f^{-1}(y) \bar{\delta} (X - f^{-1}(N_y))$. Thus f is δ-uniform.

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[2] This terminology was suggested by Dr. C. M. Pareek.

LEMMA 5.7. *If δ is an R-proximity and f is continuous and compact, then $y \notin E^-$ implies $y \bar{\delta}^1 E$.*

Proof. If $y \notin E^-$ then $f^{-1}(y) \cap f^{-1}(E^-) = \phi$. Since f is compact, $f^{-1}(y)$ is compact and since f is continuous, $f^{-1}(E^-)$ is closed. Finally, δ being an R-proximity on X, $f^{-1}(y) \bar{\delta} f^{-1}(E)$, which in turn implies $y \bar{\delta}^1 E$.

The following result is now obvious.

THEOREM 5.8. *The relation δ^1 is a compatible S-proximity on Y in the following cases:*
(a) *δ is an S-proximity, f is δ-pseudo-open and δ-uniform;*
(b) *δ is an R-proximity, f is δ-pseudo-open, continuous, and compact.*

Next we find out when δ^1 is a LO-proximity.

THEOREM 5.9. *If δ is a LO-proximity on X and f is δ-uniform and open, then δ^1 is a compatible LO-proximity on Y.*

Proof. By 5.8 (b), δ^1 is a compatible S-proximity and since f is δ-uniform, it follows that f is continuous. This together with the openness of f implies that for each $B \subset Y$, $f^{-1}(B^-) = f^{-1}(B)^-$. Now $A \delta^1 B$ iff $f^{-1}(A) \delta f^{-1}(B)$ iff $f^{-1}(A) \delta f^{-1}(B^-)$ iff $f^{-1}(A^-) \delta f^{-1}(B^-)$ iff $A^- \delta^1 B^-$, thereby showing that δ^1 is a LO-proximity.

To investigate as to when δ^1 is an R-proximity we introduce a stronger type of map.

DEFINITION 5.10. The function f is called δ-*completely uniform* iff for each nbhd. N_y of y in Y, there exists a nbhd. N_y' of y such that $f^{-1}(N_y') \bar{\delta}(X - f^{-1}(N_y))$.

Since the identity map on X is not δ-completely uniform unless δ is an R-proximity, it is clear that in order to have a meaningful discussion, we must have δ an R-proximity.

THEOREM 5.11. *If δ is an R-proximity on X and f is open and δ-uniform, then f is δ-completely uniform if and only if δ^1 is a compatible R-proximity.*

Proof. To prove the necessity, we first note that Theorem 5.9 shows that δ^1 is a compatible LO-proximity on Y. To see that δ^1 is an R-proximity, we note that $y \delta^1 E$ implies $y \in Y - E^-$ which is open. Since f is δ-completely uniform, there is a nbhd. N_y of y such that $f^{-1}(N_y) \bar{\delta} f^{-1}(E)$ i.e. $N_y \bar{\delta}^1 E$. Since trivially $y \bar{\delta}(Y - N_y)$, it follows that δ^1 is an R-proximity.

Conversely, suppose f is open δ-uniform and δ^1 is an R-proximity. If N_y is a nbhd. of $y \in Y$, $y \bar{\delta}(Y - N_y)$ and, since δ^1 is an R-proximity, there exist a set $A \subset Y$ such that $y \bar{\delta}^1(Y - A)$ and $A \bar{\delta}^1(Y - N_y)$, i.e., $y \in \text{Int}(A)$ and $f^{-1}(A) \bar{\delta} f^{-1}(Y - N_y) = X - f^{-1}(N_y)$. This proves that f is δ-completely uniform.

In view of Theorem 4.1, it is not necessary to investigate as to when δ^1 is an EF-proximity and we now turn our attention to the case when (X, d) is a semi-metric space. In this case we suppose that $\delta = \delta(d)$.

DEFINITION 5.12.   The function f is called a T_1-map iff for every pair of distinct points y_1, y_2 of Y, $f^{-1}(y_1) \bar{\delta} f^{-1}(y_2)$; equivalently, $d(f^{-1}(y_1), f^{-1}(y_2)) > 0$.

LEMMA 5.13.   *If (X, d) is a semi-metric space and f is a T_1-map, then*

$$\rho(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$$

is a semi-metric on Y (not necessarily compatible with the topology of Y).

COROLLARY 5.14.   *If either d is a metric and f is compact or if d is a semi-metric and f is δ-uniform, then ρ is a semi-metric on Y.*

LEMMA 5.15.   *Under the conditions of Lemma 5.13, for subsets A, B of Y,*

$$\rho(A, B) = d(f^{-1}(A), f^{-1}(B)) .$$

Proof. $\rho(A, B) = \text{Inf}\{d(f^{-1}(a), f^{-1}(b)): a \in A, b \in B\}$.   Now

$$d(f^{-1}(A), f^{-1}(B)) \leqq d(f^{-1}(a), f^{-1}(b))$$

for each $a \in A$, $b \in B$ and this implies that $(f^{-1}(A), f^{-1}(B)) \leqq \rho(A, B)$. Conversely, for every $\varepsilon > 0$, there exist $x \in f^{-1}(A)$, $y \in f^{-1}(B)$ such that $d(x, y) < d(f^{-1}(A), f^{-1}(B)) + \varepsilon$, and hence

$$\rho(A, B) \leqq d(x, y) < d(f^{-1}(A), f^{-1}(B)) + \varepsilon ,$$

i.e., $\rho(A, B) \leqq d(f^{-1}(A), f^{-1}(B))$, thus proving the result.

COROLLARY 5.16.   *Under the conditions of Lemma 5.15, $A \, \delta(\rho) B$ iff $f^{-1}(A) \delta f^{-1}(B)$.*

LEMMA 5.17.   *If d is (resp. separately) upper semi-continuous and*

f is open δ-uniform, then ρ is (resp. separately) upper semi-continuous.

Proof. Let $\rho(y_1, y_2) = r$ and let $\varepsilon > 0$. Then there are $x_i \in f^{-1}(y_i)$, $i = 1, 2$ such that $d(x_1, x_2) < r + \varepsilon/2$. Since d is upper semi-continuous, there exist nbhds. N_{x_i} of x_i, $i = 1, 2$ such that for all $z_i \in N_{x_i}$, $i = 1, 2$, $d(z_1, z_2) < r + \varepsilon$. But this shows that $\rho(p_1, p_2) < r + \varepsilon$ for $p_i \in f(N_{x_i})$, $i = 1, 2$ and the result follows since f is open. The case of separate upper semi-continuity is similarly handled.

It is well known that if X is metrizable and f is open δ-uniform, then Y is a developable space. We get an improvement and a more "symmetric" result from (3.3) and (5.17); viz.,

THEOREM 5.18. *If X is developable and f is open δ-uniform with respect to the proximity δ induced by the development on X, then Y is also developable.*

THEOREM 5.19. *If d is a metric on X and if f is continuous, open and compact, then Y is developable.*

Proof. By setting $\delta = \delta(d)$, $\delta^1 = \delta(\rho)$, the result follows from the continuity of d, (5.8), (5.17) and (3.3).

Finally, we conclude with a known result but which follows easily from our analysis.

THEOREM 5.20. *If d is a metric on X and if f is open and δ-completely uniform, then Y is metrizable.*

Proof. Set $\delta = \delta(d)$ and $\delta^1 = \delta(\rho)$. It follows from (5.11) that δ^1 is a compatible R-proximity on Y. Also from (5.14), ρ is a semi-metric on Y and the compatibility of δ^1 implies the compatibility of ρ. Hence by (4.1), Y is metrizable.

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