Pacific Journal of Mathematics

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Vol. 44, No. 1 May 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 1, 1973

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It is well known that a closed 3-manifold M contains a (piecewise linearly embedded) essential separating 2-sphere if and only if $\pi_1(M)$ is a nontrivial free product. In this paper necessary and sufficient conditions, in terms of $\pi_1(M)$, are given for the existence of a projective plane in M. If M is irreducible this condition is that $\pi_1(M)$ be an extension of Z or a nontrivial free product by Z_2. In particular this provides a criterion for deciding which irreducible closed 3-manifolds are not P^2-irreducible.

P^2-irreducible 3-manifolds have been studied in [2], [4]; if they are sufficiently large then their covering spaces are also P^2-irreducible. This property is not shared by irreducible but not P^2-irreducible manifolds; in [9] such manifolds are constructed having non prime covering spaces. This leads to the question as to which 3-manifolds are irreducible but not P^2-irreducible.

0. Notation and definitions.

We work in the piecewise linear category. A 3-manifold M is a compact, connected 3-manifold. *A surface F in M* is a compact 2-manifold embedded in M.

We denote by $U(X)$ a small regular neighborhood of X in M.

$F \subset \text{Int}(M)$ is *2-sided* in M if $U(F)$ is homeomorphic to $F \times I$. M is *irreducible* if every 2-sphere in M bounds a 3-cell in M. M is *P^2-irreducible* if M is irreducible and contains no 2-sided projective planes. M is *prime* if it is not the connected sum of two manifolds each different from the 3-sphere. (Here the connected sum $M_1 \# M_2$ is obtained by removing a 3-ball in the interior of M_1 and M_2 and identifying the boundary spheres under an orientation reversing homeomorphism.) F in M is *incompressible* if the following holds:

(a) if D is a disc in M such that $D \cap F = \partial D$, then ∂D bounds a disc in F, and

(b) if F is a 2-sphere, then S does not bound a 3-ball in M.

A *homotopy* N is a manifold that is homotopy equivalent to the manifold N.

Disjoint surfaces F and G in M are *pseudo parallel* if there exists an embedding of a homotopy $(F \times I)$ into M that has two boundary components, one of which is mapped onto F, the other one onto G. Finally, M is called π-trivial, if $\pi_1(M) = 1$.

REMARK. If the Poincaré conjecture is true, then pseudo parallel is the same as parallel.

1. Preliminaries. Let S^2, P^2 denote the 2-sphere and projective plane, resp.

LEMMA 1. *Let F be a closed surface, let M be an irreducible 3-manifold.*

(a) *If $F \neq S^2, P^2$ then M is a homotopy $(F \times I)$ if and only if M is homeomorphic to a line bundle over F.*

(b) *If M is nonorientable and $\pi_1(M) = Z_2$, then ∂M consists of two projective planes and M is a homotopy $(P^2 \times I)$.*

(c) If $\pi_1(M) = Z + Z_2$, *then $\partial M = \varnothing$ and M is a homotopy $(P^2 \times S^1)$.*

Proof. Part (a) follows from [5, Proposition 1]. Part (b) follows from [1, Theorem 5.1]. Part (c) follows from [11]: We map M onto a circle such that the inverse image of a point is a projective plane P^2 in M. Then, by (b), cl $(M - U(P^2))$ has as boundary two copies of P^2 and is a homotopy $(P^2 \times I)$.

LEMMA 2. *If M is irreducible and contains a 1-sided projective plane, then M is P^3 (the 3-dim. projective space).*

Proof. $U(P^2)$ is the twisted line bundle over P^2, with boundary a 2-sphere. Since this 2-sphere bounds a 3-cell in M, the result follows.

The next lemma is due to J. Tollefson [13, Lemma 1]:

LEMMA 3. *A non-irreducible closed 3-manifold M admitting a fixed point free involution T contains a 2-sphere S not bounding a 3-cell in M such that either $T(S) = S$ or $T(S) \cap S = \varnothing$.*

We will also need the following generalization of Tollefson's lemma.

LEMMA 4. *Let M be a 3-manifold (with or without boundary) admitting a fixed point free involution T. Suppose there exists a 2-sphere in M that does not separate M into two components one of which is π-trivial. Then there exists a 2-sphere S in M having the same property and such that either $T(S) \cap S = \varnothing$ or $T(S) = S$.*

Proof. Take a 2-sphere S in M with the following properties: S does not separate M into two components one of which is π-trivial, $T(S) \cap S$ is a system of disjoint simple closed curves at which the intersection is transversal, and the number $n(T(S) \cap S)$ of components $T(S) \cap S$ is minimal. We show that either $n = 0$ or there exists an S' with the desired properties such that $T(S') = S'$.

Suppose $n > 0$. Let D be an innermost disc on $T(S)$, with ∂D a component of $T(S) \cap S$, (that is, $\text{int}(D) \cap S = \emptyset$). D separates S into two discs D_1, D_2. Let $S_1 = D \cup D_1$, $S_2 = D \cup D_2$. It is easy to see that at least one of S_1 or S_2 does not separate M into two components one of which is π-trivial. Suppose S_1 has this property. If $T(S_1) = S_1$, we are done. If $T(S_1) \neq S_1$, then a component S' of $\partial U(S_1)$ (U is small wrt T) has the same property as S_1, but $n(T(S') \cap S') < n(T(S) \cap S)$ (since the component ∂D has vanished), a contradiction.

LEMMA 5. *If M is closed and $\pi_1(M) \approx \boldsymbol{Z}$, then M is a connected sum of a homotopy 3-sphere and a S^2-bundle over S^1.*

Proof. Write $M \approx M_1 \# M_2$, where M_1 is prime and $\pi_1(M_1) \approx \boldsymbol{Z}$, $\pi_1(M_2) = 1$ (see §5). An irreducible manifold with fundamental group $\boldsymbol{Z}$ is bounded (see e.g. [11]). Hence M_1 is not irreducible. Therefore M_1 is an S^2-bundle over S^1 (see §5).

2. The closed case.

THEOREM 1. *A closed irreducible 3-manifold M contains a 2-sided projective plane if and only if $\pi_1(M)$ is an extension of $\boldsymbol{Z}$ or a nontrivial free product by $\boldsymbol{Z}_2$.*

Proof. Suppose M contains a 2-sided P^2. Thus M is nonorientable and we let $p: M' \to M$ be the 2-fold orientable covering of M. Then $P^2 \subset M$ lifts to an essential 2-sphere $S^2 \subset M'$. If S^2 separates M' into M_1, M_2 then $\pi_1(M') \cong \pi_1(M_1) * \pi_1(M_2)$, a nontrivial free product. (Otherwise, if $\pi_1(M_1) = 1$, say, from $\partial M_1 = S^2$ it would follow that S^2 is contractible in M_1). If S^2 does not separate M', let k be a simple closed curve that intersects S^2 in exactly one point and let $U = U(S^2 \cup k)$. Then $\pi_1(M') = \boldsymbol{Z} * \pi_1(\text{cl}\,(M - U))$.

Conversely, assume $\pi_1(M)$ is an extension of $\boldsymbol{Z}$ or of a nontrivial free product G by $\boldsymbol{Z}_2$. Let $p: N \to M$ be the covering of M associated with $\boldsymbol{Z}$ or G, respectively, and let $T: N \to N$ be the covering transformation. By Lemma 5 and Kneser's conjecture [12] there exists an essential 2-sphere S^2 in N. Therefore, by Lemma 3 we can find a 2-sphere $S \subset N$ not bounding a 3-cell, such that either $T(S) \cap S = \emptyset$ or $T(S) = S$. The first case cannot occur, since M is irreducible. In the second case, $p(S)$ is a projective plane in M that is 2-sided, by Lemma 2.

3. The bounded case.

THEOREM 2. *Let M be an irreducible 3-manifold with (nonempty) incompressible boundary. M contains a 2-sided P^2 that is not pseudo parallel to a component of ∂M if and only if $\pi_1(M)$ is an extension of a nontrivial free product by $\boldsymbol{Z}_2$.*

Proof. Suppose M contains a 2-sided P^2 that is not pseudo parallel to a component of ∂M. Lift P^2 to S^2 in the 2-fold orientable cover M' of M, let $T: M' \to M'$ be the covering transformation. If S^2 separates M into M_1, M_2, we have that $T(M_1) = M_1$, $T(M_2) = M_2$, since P^2 is 2-sided in M. If $\pi_1(M_1) = 1$, say, then M_1 covers a submanifold M_{1*} having fundamental group $\boldsymbol{Z}_2$. By Lemma 1 (b), M_{1*} is a homotopy $(P^2 \times I)$, hence P^2 would be pseudo parallel to a component of ∂M, a contradiction. Therefore, in this case, $\pi_1(M') = \pi_1(M_1) * \pi_1(M_2)$, a nontrivial free product.

If S^2 does not separate M', then as in the proof of Theorem 1, $\pi_1(M') \cong \boldsymbol{Z} * \pi_1(\text{cl}\,(M' - U))$. If $\pi_1(\text{cl}\,(M' - U))$ would be trivial, then $\pi_1(M) = \boldsymbol{Z} + \boldsymbol{Z}_2$. By Lemma 1 (c), M would be closed, a contradiction.

Conversely, suppose $\pi_1(M)$ is an extension of a nontrivial free product G by $\boldsymbol{Z}_2$. Again, let $N \xrightarrow{P} M$ be the covering of M corresponding to G and let T be the covering transformation. By Kneser's conjecture for bounded 3-manifolds [6] there exists a 2-sphere S^2 in N that separates N into N_1, N_2, both not π-trivial. By Lemma 4, there exists a 2-sphere S that does not separate N into two components one of which is π-trivial and such that $T(S) = S$ (the case $TS \cap S = \phi$ cannot occur). By Lemma 2, S covers a 2-sided P^2 in M. If P^2 were pseudo parallel to a component of ∂M, then lifting the corresponding homotopy $(P^2 \times I)$ we see that S would separate N into two components, one of which would be π-trivial, a contradiction.

PROPOSITION. *Let M be irreducible and suppose $\pi_1(M)$ is not $\boldsymbol{Z}_2$, and not an extension of $\boldsymbol{Z}$ or of a nontrivial free product by $\boldsymbol{Z}_2$. Then if ∂M contains no P^2 (in particular, if M is closed) it follows that M contains no P^2.*

Proof. If M is orientable and contains a P^2, then $M = P^3$, by Lemma 2. If M is nonorientable, let M' be the 2-fold orientable cover of M. If $\pi_2(M') \neq 0$, then the sphere theorem [14] gives us an essential 2-sphere in M' and as in the proof of the preceding theorems, we see that $\pi_1(M') = \boldsymbol{Z}$ or a nontrivial free product. Therefore, $\pi_2(M') = 0$ and hence $\pi_2(M) = 0$. (In fact, M is aspherical.) But any 2-sided $P^2 \subset M$ would be essential [1, Lemma 6.3].

REMARK. A 2-sided P^2 in M is incompressible in M. This follows from the loop theorem and Dehn's lemma [10]. In particular $\pi_1(P^2) \to \pi_1(M)$ is an injection.

4. A counterexample to Theorem 2 if M is not incompressible. Let K be a solid Kleinbottle, T a solid torus. Choose $n \geqq 1$ disjoint discs $D_1, \cdots, D_n$ on ∂K and a disc D on ∂T. Let M be the manifold obtained from K by attaching n copies of T to K at D_i and D ($i = 1, \cdots, n$). Then M is irreducible and does not contain 2-sided projective planes (otherwise by the preceding remark, $\pi_1(M)$ would have an element of order 2, but $\pi_1(M) \cong (n+1)\boldsymbol{Z}$). However, the twofold orientable cover M' of M has fundamental group $\pi_1(M') \cong (2n+1)\boldsymbol{Z}$, the free product of $2n+1$ copies of $\boldsymbol{Z}$, and therefore $\pi_1(M)$ is an extension of the nontrivial free product $(2n+1)\boldsymbol{Z}$ by $\boldsymbol{Z}_2$.

5. The general case. Suppose M is a compact 3-manifold such that ∂M contains no 2-spheres. As in [8, Lemma 1] it follows that if M is prime but not irreducible then M is a S^2-bundle over S^1. If M is not prime, then there exists a decomposition of M into a finite number of prime manifolds

$$(\#) \qquad M \approx M_1 \# M_2 \# \cdots \# M_n ,$$

(if M is nonorientable or with boundary see e.g. [3]). If K denotes the nonorientable S^2-bundle over S^1 then since $K \# K \approx K \# (S^2 \times S^1)$, we say that the decomposition (#) is in *normal form* if at most one $M_i \approx K$. Then Milnor's proof in [8] can be generalized to yield the following:

PROPOSITION. *Any compact 3-manifold M whose boundary contains no 2-spheres has a unique normal decomposition (#) into prime manifolds. Each summand M_i is irreducible or $S^1 \times S^2$ and at most one $M_i \approx K$.*

In the decomposition (#) let m denote the number of prime manifolds which are not π-trivial ($m \leqq n$).

THEOREM 3. *Let M be a closed 3-manifold.*

(a) *If M contains a 2-sided P^2, then $\pi_1(M)$ is an extension of a free product of $2m$ nontrivial factors or of a free product of $2m-1$ nontrivial factors one of which is $\boldsymbol{Z}$, by $\boldsymbol{Z}_2$.*

(b) *If $\pi_1(M)$ is an extension of a free product of $2m$ nontrivial factors by $\boldsymbol{Z}_2$ then M contains a 2-sided P^2.*

Proof. Consider the decomposition (#). Let $S_i \subset M$ be the 2-sphere at which M_i and M_{i+1} are amalgamated and let M_i' be obtained from M_i by removing the interiors of the 3-balls which are used in the construction of the connected sum. We can assume that $M'_i \cap M'_{i+1} = S_i (i = 1, \cdots, n-1)$.

We first note that M contains a 2-sided P^2 if and only if one of the M'_i contains a 2-sided P^2. For, by general position we can assume that $P^2 \cap \cup S_i$ is a system of simple closed curves. If $P^2 \cap S_i \neq \emptyset$ then an innermost intersection curve on S_i bounds a disk on P^2 (since P^2 is incompressible) and on S_i. Replacing the disk on P^2 by the disk on S_i and pushing it slightly off S_i, we reduce the number of intersection curves of $P^2 \cap \cup S_i$.

Second, we note that we can assume that in the decomposition (#) no M_i has trivial fundamental group i.e. that $n = m$. For otherwise we consider the manifold M_* obtained from M by deleting all the homotopy spheres M_i which occur in (#). Clearly, $\pi_1(M_*) = \pi_1(M)$ and M_* contains a 2-sided P^2 if and only if M does.

Now assume M contains a 2-sided P^2. Let $p: N \rightarrow M$ be the 2-fold orientable covering and let $N_i = p^{-1}(M'_i)$. If N_i is connected then $\pi_1(N_i) \neq 1$, because otherwise $\pi_1(M'_i) = \boldsymbol{Z}_2$, and since $\partial M'_i$ consists of 2-spheres only, M'_i is orientable (Lemma 1(b)). But then M'_i lifts to two copies, hence N_i would not be connected. Similarily, if N_i is not connected then no component of N_i is π-trivial, because otherwise M_i would be π-trivial. Now each $S_i \subset M$ lifts to two 2-spheres S'_i, S''_i in N, and N is obtained from the N_i by identifying N_i and N_{i+1} along S'_i and S''_i $(i = 1, \cdots, m-1)$.

Construct a manifold N' as follows. If both N_1 and N_2 are connected, identify N_1 and N_2 along one 2-sphere only, say S'_1. Otherwise identify N_1 and N_2 along both S'_1 and S''_1. The result is a manifold $N^{(1)}$. If N_3 is connected, identify $N^{(1)}$ and N_3 along S'_2 only, otherwise identify along S'_2 and S''_2, etc. In this way we obtain a maximal connected manifold N' such that N is obtained from N' by identifying pairs of 2-spheres in $\partial N'$. Then $\pi_1(N') = G_1 * \cdots * G_k$ $(0 \leqq k \leqq 2m-1)$, where each G_j is the fundamental group of a component of some N_i. We obtain N from N' by adding $(2m-1)-k$ handles $S^1 \times S^2$ or K, hence $\pi_1(N) = G_1 * \cdots * G_k * \boldsymbol{Z} * \cdots * \boldsymbol{Z}$ is a free product of $2m-1$ nontrivial factors.

Now $P^2 \subset M'_j$, say $(1 \leqq j \leqq m-1)$. Then M'_j is nonorientable and N_j is connected. Therefore by the above construction, $\pi_1(N_j)$, is one of the groups G_i in the above decomposition of $\pi_1(N)$. Closing the boundary spheres of N_j with 3-balls we get a 2-fold covering $\hat{N}_j \rightarrow M_j$, and it follows from the proof of Theorem 1 that $\pi_1(\hat{N}_j)$ and hence $\pi_1(N_j)$ is $\boldsymbol{Z}$ or a nontrivial free product. This proves part (a) of Theorem 3.

Now suppose $\pi_1(M)$ is an extension of a product G of $2m$ nontrivial groups by $\boldsymbol{Z}_2$. Let $p: \tilde{M} \rightarrow M$ be the covering associated to G. Then as above $\pi_1(\tilde{M}) = \pi_1(\tilde{M}_1) * \cdots * \pi_1(\tilde{M}_k) * \boldsymbol{Z} * \cdots * \boldsymbol{Z}$ is a product of $2m - 1$ groups, where each $\tilde{M}_i$ is a component of $p^{-1}(M_j')$, for some j. (It is possible that some $\pi_1(\tilde{M}_i) = 1$.) It follows from Kurosh's Theorem [7] that at least one factor, $\pi_1(\tilde{M}_1)$ say, is a nontrivial free product. If $\tilde{M}_1$ covers M_j', then either $\pi_1(M_j) \approx \pi_1(\tilde{M}_1)$ or $\pi_1(M_j)$ is an extension of $\pi_1(\tilde{M}_1)$ by $\boldsymbol{Z}_2$. In the first case M_1 can not be a handle and by Kneser's conjecture can not be irreducible, therefore this case can not occur. In the second case we apply Theorem 1 to obtain a P^2 in M_1 and hence in M.

It should be noted that the hypothesis in case (a) of Theorem 3 can not be weakened: If $M = (P^2 \times S^1) \# (S^2 \times S^1)$, then $\pi_1(M)$ is not an extension of a free product of 4 factors by $\boldsymbol{Z}_2$.

It is now easy to see how to obtain an analogous result for 3-manifolds with incompressible boundary.

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Received August 11, 1971 and in revised form April 7, 1972. Partially supported by NFS grant GP 19964.

THE FLORIDA STATE UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 44, No. 1 May, 1973