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TESTING 3-MANIFOLDS FOR PROJECTIVE PLANES

WOLFGANG H. HEIL

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It is well known that a closed 3-manifold M contains a (piecewise linearly embedded) essential separating 2-sphere if and only if $\pi_1(M)$ is a nontrivial free product. In this paper necessary and sufficient conditions, in terms of $\pi_1(M)$, are given for the existence of a projective plane in M . If M is irreducible this condition is that $\pi_1(M)$ be an extension of Z or a nontrivial free product by Z_2 . In particular this provides a criterion for deciding which irreducible closed 3-manifolds are not P^2 -irreducible.

P^2 -irreducible 3-manifolds have been studied in [2], [4]; if they are sufficiently large then their covering spaces are also P^2 -irreducible. This property is not shared by irreducible but not P^2 -irreducible manifolds; in [9] such manifolds are constructed having non prime covering spaces. This leads to the question as to which 3-manifolds are irreducible but not P^2 -irreducible.

0. Notation and definitions. We work in the piecewise linear category. A 3-manifold M is a compact, connected 3-manifold. A surface F in M is a compact 2-manifold embedded in M .

We denote by $U(X)$ a small regular neighborhood of X in M .

$F \subset \text{Int}(M)$ is 2-sided in M if $U(F)$ is homeomorphic to $F \times I$. M is irreducible if every 2-sphere in M bounds a 3-cell in M . M is P^2 -irreducible if M is irreducible and contains no 2-sided projective planes. M is prime if it is not the connected sum of two manifolds each different from the 3-sphere. (Here the connected sum $M_1 \# M_2$ is obtained by removing a 3-ball in the interior of M_1 and M_2 and identifying the boundary spheres under an orientation reversing homeomorphism.) F in M is incompressible if the following holds:

(a) if D is a disc in M such that $D \cap F = \partial D$, then ∂D bounds a disc in F , and

(b) if F is a 2-sphere, then S does not bound a 3-ball in M .

A homotopy N is a manifold that is homotopy equivalent to the manifold N .

Disjoint surfaces F and G in M are pseudo parallel if there exists an embedding of a homotopy ($F \times I$) into M that has two boundary components, one of which is mapped onto F , the other one onto G . Finally, M is called π -trivial, if $\pi_1(M) = 1$.

REMARK. If the Poincaré conjecture is true, then pseudo parallel

is the same as parallel.

1. Preliminaries. Let S^2, P^2 denote the 2-sphere and projective plane, resp.

LEMMA 1. *Let F be a closed surface, let M be an irreducible 3-manifold.*

(a) *If $F \neq S^2, P^2$ then M is a homotopy ($F \times I$) if and only if M is homeomorphic to a line bundle over F .*

(b) *If M is nonorientable and $\pi_1(M) = \mathbf{Z}_2$, then ∂M consists of two projective planes and M is a homotopy ($P^2 \times I$).*

(c) *If $\pi_1(M) = \mathbf{Z} + \mathbf{Z}_2$, then $\partial M = \emptyset$ and M is a homotopy ($P^2 \times S^1$).*

Proof. Part (a) follows from [5, Proposition 1]. Part (b) follows from [1, Theorem 5.1]. Part (c) follows from [11]: We map M onto a circle such that the inverse image of a point is a projective plane P^2 in M . Then, by (b), $\text{cl}(M - U(P^2))$ has as boundary two copies of P^2 and is a homotopy ($P^2 \times I$).

LEMMA 2. *If M is irreducible and contains a 1-sided projective plane, then M is P^3 (the 3-dim. projective space).*

Proof. $U(P^2)$ is the twisted line bundle over P^2 , with boundary a 2-sphere. Since this 2-sphere bounds a 3-cell in M , the result follows. The next lemma is due to J. Tollefson [13, Lemma 1]:

LEMMA 3. *A non-irreducible closed 3-manifold M admitting a fixed point free involution T contains a 2-sphere S not bounding a 3-cell in M such that either $T(S) = S$ or $T(S) \cap S = \emptyset$.*

We will also need the following generalization of Tollefson's lemma.

LEMMA 4. *Let M be a 3-manifold (with or without boundary) admitting a fixed point free involution T . Suppose there exists a 2-sphere in M that does not separate M into two components one of which is π -trivial. Then there exists a 2-sphere S in M having the same property and such that either $T(S) \cap S = \emptyset$ or $T(S) = S$.*

Proof. Take a 2-sphere S in M with the following properties: S does not separate M into two components one of which is π -trivial, $T(S) \cap S$ is a system of disjoint simple closed curves at which the intersection is transversal, and the number $n(T(S) \cap S)$ of components $T(S) \cap S$ is minimal. We show that either $n = 0$ or there exists an S' with

the desired properties such that $T(S') = S'$.

Suppose $n > 0$. Let D be an innermost disc on $T(S)$, with ∂D a component of $T(S) \cap S$, (that is, $\text{int}(D) \cap S = \emptyset$). D separates S into two discs D_1, D_2 . Let $S_1 = D \cup D_1, S_2 = D \cup D_2$. It is easy to see that at least one of S_1 or S_2 does not separate M into two components one of which is π -trivial. Suppose S_1 has this property. If $T(S_1) = S_1$, we are done. If $T(S_1) \neq S_1$, then a component S' of $\partial U(S_1)$ (U is small wrt T) has the same property as S_1 , but $n(T(S') \cap S') < n(T(S) \cap S)$ (since the component ∂D has vanished), a contradiction.

LEMMA 5. *If M is closed and $\pi_1(M) \approx \mathbf{Z}$, then M is a connected sum of a homotopy 3-sphere and a S^2 -bundle over S^1 .*

Proof. Write $M \approx M_1 \# M_2$, where M_1 is prime and $\pi_1(M_1) \approx \mathbf{Z}$, $\pi_1(M_2) = 1$ (see §5). An irreducible manifold with fundamental group \mathbf{Z} is bounded (see e.g. [11]). Hence M_1 is not irreducible. Therefore M_1 is an S^2 -bundle over S^1 (see §5).

2. The closed case.

THEOREM 1. *A closed irreducible 3-manifold M contains a 2-sided projective plane if and only if $\pi_1(M)$ is an extension of \mathbf{Z} or a nontrivial free product by \mathbf{Z}_2 .*

Proof. Suppose M contains a 2-sided P^2 . Thus M is nonorientable and we let $p: M' \rightarrow M$ be the 2-fold orientable covering of M . Then $P^2 \subset M$ lifts to an essential 2-sphere $S^2 \subset M'$. If S^2 separates M' into M_1, M_2 then $\pi_1(M') \cong \pi_1(M_1) * \pi_1(M_2)$, a nontrivial free product. (Otherwise, if $\pi_1(M_1) = 1$, say, from $\partial M_1 = S^2$ it would follow that S^2 is contractible in M_1). If S^2 does not separate M' , let k be a simple closed curve that intersects S^2 in exactly one point and let $U = U(S^2 \cup k)$. Then $\pi_1(M') = \mathbf{Z} * \pi_1(\text{cl}(M - U))$.

Conversely, assume $\pi_1(M)$ is an extension of \mathbf{Z} or of a nontrivial free product G by \mathbf{Z}_2 . Let $p: N \rightarrow M$ be the covering of M associated with \mathbf{Z} or G , respectively, and let $T: N \rightarrow N$ be the covering transformation. By Lemma 5 and Kneser's conjecture [12] there exists an essential 2-sphere S^2 in N . Therefore, by Lemma 3 we can find a 2-sphere $S \subset N$ not bounding a 3-cell, such that either $T(S) \cap S = \emptyset$ or $T(S) = S$. The first case cannot occur, since M is irreducible. In the second case, $p(S)$ is a projective plane in M that is 2-sided, by Lemma 2.

3. The bounded case.

THEOREM 2. *Let M be an irreducible 3-manifold with (nonempty) incompressible boundary. M contains a 2-sided P^2 that is not pseudo parallel to a component of ∂M if and only if $\pi_1(M)$ is an extension of a nontrivial free product by \mathbf{Z}_2 .*

Proof. Suppose M contains a 2-sided P^2 that is not pseudo parallel to a component of ∂M . Lift P^2 to S^2 in the 2-fold orientable cover M' of M , let $T: M' \rightarrow M'$ be the covering transformation. If S^2 separates M into M_1, M_2 , we have that $T(M_1) = M_1, T(M_2) = M_2$, since P^2 is 2-sided in M . If $\pi_1(M_1) = 1$, say, then M_1 covers a submanifold M_{1*} having fundamental group \mathbf{Z}_2 . By Lemma 1 (b), M_{1*} is a homotopy ($P^2 \times I$), hence P^2 would be pseudo parallel to a component of ∂M , a contradiction. Therefore, in this case, $\pi_1(M') = \pi_1(M_1) * \pi_1(M_2)$, a nontrivial free product.

If S^2 does not separate M' , then as in the proof of Theorem 1, $\pi_1(M') \cong \mathbf{Z} * \pi_1(\text{cl}(M' - U))$. If $\pi_1(\text{cl}(M' - U))$ would be trivial, then $\pi_1(M) = \mathbf{Z} + \mathbf{Z}_2$. By Lemma 1 (c), M would be closed, a contradiction.

Conversely, suppose $\pi_1(M)$ is an extension of a nontrivial free product G by \mathbf{Z}_2 . Again, let $N \xrightarrow{P} M$ be the covering of M corresponding to G and let T be the covering transformation. By Kneser's conjecture for bounded 3-manifolds [6] there exists a 2-sphere S^2 in N that separates N into N_1, N_2 , both not π -trivial. By Lemma 4, there exists a 2-sphere S that does not separate N into two components one of which is π -trivial and such that $T(S) = S$ (the case $TS \cap S = \emptyset$ cannot occur). By Lemma 2, S covers a 2-sided P^2 in M . If P^2 were pseudo parallel to a component of ∂M , then lifting the corresponding homotopy ($P^2 \times I$) we see that S would separate N into two components, one of which would be π -trivial, a contradiction.

PROPOSITION. *Let M be irreducible and suppose $\pi_1(M)$ is not \mathbf{Z}_2 , and not an extension of \mathbf{Z} or of a nontrivial free product by \mathbf{Z}_2 . Then if ∂M contains no P^2 (in particular, if M is closed) it follows that M contains no P^2 .*

Proof. If M is orientable and contains a P^2 , then $M = P^3$, by Lemma 2. If M is nonorientable, let M' be the 2-fold orientable cover of M . If $\pi_2(M') \neq 0$, then the sphere theorem [14] gives us an essential 2-sphere in M' and as in the proof of the preceding theorems, we see that $\pi_1(M') = \mathbf{Z}$ or a nontrivial free product. Therefore, $\pi_2(M') = 0$ and hence $\pi_2(M) = 0$. (In fact, M is aspherical.) But any 2-sided $P^2 \subset M$ would be essential [1, Lemma 6.3].

REMARK. A 2-sided P^2 in M is incompressible in M . This follows

from the loop theorem and Dehn's lemma [10]. In particular $\pi_1(P^2) \rightarrow \pi_1(M)$ is an injection.

4. A counterexample to Theorem 2 if M is not incompressible. Let K be a solid Kleinbottle, T a solid torus. Choose $n \geq 1$ disjoint discs D_1, \dots, D_n on ∂K and a disc D on ∂T . Let M be the manifold obtained from K by attaching n copies of T to K at D_i and D ($i = 1, \dots, n$). Then M is irreducible and does not contain 2-sided projective planes (otherwise by the preceding remark, $\pi_1(M)$ would have an element of order 2, but $\pi_1(M) \cong (n+1)\mathbf{Z}$). However, the two-fold orientable cover M' of M has fundamental group $\pi_1(M') \cong (2n+1)\mathbf{Z}$, the free product of $2n+1$ copies of \mathbf{Z} , and therefore $\pi_1(M)$ is an extension of the nontrivial free product $(2n+1)\mathbf{Z}$ by \mathbf{Z}_2 .

5. The general case. Suppose M is a compact 3-manifold such that ∂M contains no 2-spheres. As in [8, Lemma 1] it follows that if M is prime but not irreducible then M is a S^2 -bundle over S^1 . If M is not prime, then there exists a decomposition of M into a finite number of prime manifolds

$$(\#) \quad M \approx M_1 \# M_2 \# \dots \# M_n,$$

(if M is nonorientable or with boundary see e.g. [3]). If K denotes the nonorientable S^2 -bundle over S^1 then since $K \# K \approx K \# (S^2 \times S^1)$, we say that the decomposition (#) is in *normal form* if at most one $M_i \approx K$. Then Milnor's proof in [8] can be generalized to yield the following:

PROPOSITION. *Any compact 3-manifold M whose boundary contains no 2-spheres has a unique normal decomposition (#) into prime manifolds. Each summand M_i is irreducible or $S^1 \times S^2$ and at most one $M_i \approx K$.*

In the decomposition (#) let m denote the number of prime manifolds which are not π -trivial ($m \leq n$).

THEOREM 3. *Let M be a closed 3-manifold.*

(a) *If M contains a 2-sided P^2 , then $\pi_1(M)$ is an extension of a free product of $2m$ nontrivial factors or of a free product of $2m-1$ nontrivial factors one of which is \mathbf{Z} , by \mathbf{Z}_2 .*

(b) *If $\pi_1(M)$ is an extension of a free product of $2m$ nontrivial factors by \mathbf{Z}_2 then M contains a 2-sided P^2 .*

Proof. Consider the decomposition (#). Let $S_i \subset M$ be the 2-sphere at which M_i and M_{i+1} are amalgamated and let M'_i be obtained

from M_i by removing the interiors of the 3-balls which are used in the construction of the connected sum. We can assume that $M'_i \cap M'_{i+1} = S_i (i = 1, \dots, n - 1)$.

We first note that M contains a 2-sided P^2 if and only if one of the M'_i contains a 2-sided P^2 . For, by general position we can assume that $P^2 \cap \cup S_i$ is a system of simple closed curves. If $P^2 \cap S_i \neq \emptyset$ then an innermost intersection curve on S_i bounds a disk on P^2 (since P^2 is incompressible) and on S_i . Replacing the disk on P^2 by the disk on S_i and pushing it slightly off S_i , we reduce the number of intersection curves of $P^2 \cap \cup S_i$.

Second, we note that we can assume that in the decomposition (#) no M_i has trivial fundamental group i.e. that $n = m$. For otherwise we consider the manifold M_* obtained from M by deleting all the homotopy spheres M_i which occur in (#). Clearly, $\pi_1(M_*) = \pi_1(M)$ and M_* contains a 2-sided P^2 if and only if M does.

Now assume M contains a 2-sided P^2 . Let $p: N \rightarrow M$ be the 2-fold orientable covering and let $N_i = p^{-1}(M'_i)$. If N_i is connected then $\pi_1(N_i) \neq 1$, because otherwise $\pi_1(M'_i) = \mathbf{Z}_2$, and since $\partial M'_i$ consists of 2-spheres only, M'_i is orientable (Lemma 1(b)). But then M'_i lifts to two copies, hence N_i would not be connected. Similarly, if N_i is not connected then no component of N_i is π -trivial, because otherwise M_i would be π -trivial. Now each $S_i \subset M$ lifts to two 2-spheres S'_i, S''_i in N , and N is obtained from the N_i by identifying N_i and N_{i+1} along S'_i and S''_i ($i = 1, \dots, m - 1$).

Construct a manifold N' as follows. If both N_1 and N_2 are connected, identify N_1 and N_2 along one 2-sphere only, say S'_1 . Otherwise identify N_1 and N_2 along both S'_1 and S''_1 . The result is a manifold $N^{(1)}$. If N_3 is connected, identify $N^{(1)}$ and N_3 along S'_2 only, otherwise identify along S'_2 and S''_2 , etc. In this way we obtain a maximal connected manifold N' such that N is obtained from N' by identifying pairs of 2-spheres in $\partial N'$. Then $\pi_1(N') = G_1 * \dots * G_k$ ($0 \leq k \leq 2m - 1$), where each G_j is the fundamental group of a component of some N_i . We obtain N from N' by adding $(2m - 1) - k$ handles $S^1 \times S^2$ or K , hence $\pi_1(N) = G_1 * \dots * G_k * \mathbf{Z} * \dots * \mathbf{Z}$ is a free product of $2m - 1$ nontrivial factors.

Now $P^2 \subset M'_j$, say ($1 \leq j \leq m - 1$). Then M'_j is nonorientable and N_j is connected. Therefore by the above construction, $\pi_1(N_j)$, is one of the groups G_i in the above decomposition of $\pi_1(N)$. Closing the boundary spheres of N_j with 3-balls we get a 2-fold covering $\hat{N}_j \rightarrow M_j$, and it follows from the proof of Theorem 1 that $\pi_1(\hat{N}_j)$ and hence $\pi_1(N_j)$ is \mathbf{Z} or a nontrivial free product. This proves part (a) of Theorem 3.

Now suppose $\pi_1(M)$ is an extension of a product G of $2m$ nontrivial groups by \mathbf{Z}_2 . Let $p: \tilde{M} \rightarrow M$ be the covering associated to G . Then

as above $\pi_1(\tilde{M}) = \pi_1(\tilde{M}_1) * \dots * \pi_1(\tilde{M}_k) * \mathbf{Z} * \dots * \mathbf{Z}$ is a product of $2m - 1$ groups, where each \tilde{M}_i is a component of $p^{-1}(M'_j)$, for some j . (It is possible that some $\pi_1(\tilde{M}_i) = 1$.) It follows from Kurosh's Theorem [7] that at least one factor, $\pi_1(\tilde{M}_i)$ say, is a nontrivial free product. If \tilde{M}_i covers M'_j , then either $\pi_1(M_j) \approx \pi_1(\tilde{M}_i)$ or $\pi_1(M_j)$ is an extension of $\pi_1(\tilde{M}_i)$ by \mathbf{Z}_2 . In the first case M_1 can not be a handle and by Kneser's conjecture can not be irreducible, therefore this case can not occur. In the second case we apply Theorem 1 to obtain a P^2 in M_1 and hence in M .

It should be noted that the hypothesis in case (a) of Theorem 3 can not be weakened: If $M = (P^2 \times S^1) \# (S^2 \times S^1)$, then $\pi_1(M)$ is not an extension of a free product of 4 factors by \mathbf{Z}_2 .

It is now easy to see how to obtain an analogous result for 3-manifolds with incompressible boundary.

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