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**THE EVALUATION MAP AND EHP SEQUENCES**

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# THE EVALUATION MAP AND *EHP* SEQUENCES

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Let  $L(\Sigma B, X)$  be the space of maps from  $\Sigma B$  (the reduced suspension of  $B$ ) to  $X$  with the compact-open topology, let  $\mathcal{L}: \Sigma B \rightarrow X$  and  $L(\Sigma B, X; \mathcal{L})$  the path component of  $L(\Sigma B, X)$  containing  $\mathcal{L}$ . For nice spaces the evaluation map  $\omega: L(\Sigma B, X, \mathcal{L}) \rightarrow X$  defined by  $\omega(f) = f(*)$  is a fibration and gives rise to a long exact sequence in homotopy. The purpose of this paper is to show that the boundary map in that long exact sequence can be given by a generalized Whitehead product and that the sequence generalizes the *EHP* sequence of G. W. Whitehead.

1. Preliminary definitions. All spaces are assumed to be *CW* complexes with base point at a vertex. Maps are base point preserving. The cartesian product  $A \times B$  is assumed to be based at  $(a_0, b_0)$ , the unit interval,  $I$ , is based at 0, and quotient spaces are based at the image of the base point under the natural quotient map. Where the space is clear  $*$  will denote the base point as well as the constant map with image at the base point.

We use the following notations.  $L(A, B)$  will denote the space of maps from  $A$  to  $B$  with the compact-open topology and  $L(A, B; \mathcal{L})$  the path component of  $L(A, B)$  containing  $\mathcal{L}: A \rightarrow B$ .  $L_0(A, B)$  and  $L_0(A, B; \mathcal{L})$  will denote the space of base point preserving maps in  $L(A, B)$  and  $L(A, B; \mathcal{L})$  respectively. Let  $A \vee B$  and  $A \# B$  denote the one point union and smash product respectively.

Since spaces are assumed to be *CW* complexes the smash product can be taken as  $A \times B$  with  $A \vee B$  identified with  $(a_0, b_0)$ .  $q: A \times B \rightarrow A \# B$  will denote the quotient map. Note that  $S^{p+q} = \Sigma^p S^q = S^p \# S^q$ ,  $\Sigma^p A = S^p \# A$ , and  $\Sigma(A \vee B) = \Sigma A \vee \Sigma B$ .

Let  $p_1, p_2: A \times B \rightarrow A \vee B$  be defined by  $p_1(a, b) = a \vee b_0$  and  $p_2(a, b) = a_0 \vee b$ . Define  $k: \Sigma(A \times B) \rightarrow \Sigma A \vee \Sigma B$  by  $k = \Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2$ . Since  $k|_{\Sigma(A \vee B)}$  homotopically trivial, by the homotopy extension property there is a map  $k': \Sigma(A \times B) \rightarrow \Sigma A \vee \Sigma B$ , homotopic to  $k$ , such that  $k'|_{\Sigma(A \vee B)} = *$ .  $k'$  then induces a map  $\tilde{k}: \Sigma(A \# B) \rightarrow \Sigma A \vee \Sigma B$ . Arkowitz [1] has shown that  $[\tilde{k}]$  is uniquely determined by the requirement  $k \cong \tilde{k} \circ \Sigma q$ . The following definition is due to Arkowitz [1].

**DEFINITION 1.1.** For  $\alpha = [f] \in [\Sigma A, X]$  and  $\beta = [g] \in [\Sigma B, X]$ , the *generalized Whitehead product*  $[\alpha, \beta]$  is defined by  $[\alpha, \beta] = [(f \vee g) \circ \tilde{k}] \in [\Sigma(A \# B), X]$ .

Hardie shows (Theorem 2.3 in [2]) that the map  $\Sigma p_1 + \Sigma p_2 + \Sigma q: \Sigma(A \times B) \rightarrow \Sigma A \vee \Sigma B \vee \Sigma(A \# B)$  is a homotopy equivalence for  $A$  and  $B$   $CW$  complexes with a single vertex. Then there is a map  $\phi: \Sigma(A \# B) \rightarrow \Sigma(A \times B)$  such that  $\Sigma q \circ \phi \cong 1_{\Sigma(A \# B)}$ .

**DEFINITION 1.2.** If  $f: A \times B \rightarrow X$ , where  $A$  and  $B$  have a single vertex, the element obtained from  $f$  by the *generalized Hopf construction* is defined to be the map  $\Sigma f \circ \phi: \Sigma(A \# B) \rightarrow \Sigma X$ .

Hardie shows in [2] that if  $A$  and  $B$  are spheres, Definition 1.2 reduces to the classical definition of the Hopf construction.

Let  $\phi_r: S^r \rightarrow S^r \vee S^r$  be the map which identifies the equator of  $S^r$ . G. W. Whitehead (Theorem 1.17 in [6]) shows for  $n < p + q + \min(p, q) - 3$  that  $\pi_n(S^p \vee S^q) = \pi_n(S^p) \oplus \pi_n(S^q) \oplus \pi_n(S^{p+q-1})$ . Let  $Q: \pi_n(S^p \vee S^q) \rightarrow \pi_n(S^{p+q-1})$  be the natural projection onto the direct summand  $\pi_n(S^{p+q-1})$ .

**DEFINITION 1.3.** For  $n < 3r - 3$  the *generalized Hopf invariant*  $\tilde{H}: \pi_n(S^r) \rightarrow \pi_n(S^{2r-1})$  is defined by  $\tilde{H} = Q \circ \phi_{r*}$ .

**DEFINITION 1.4.** For  $\lambda = [\lambda] \in [\Sigma B, X]$  the  $\lambda$ -Whitehead homomorphism  $P_\lambda: [\Sigma A, X] \rightarrow [\Sigma(A \# B), X]$  is defined by  $P_\lambda(\alpha) = [\alpha, \lambda]$ .

**DEFINITION 1.5.** If  $F: A \rightarrow L(B, X)$  the map  $G: A \times B \rightarrow X$  given by  $G(a, b) = F(a)(b)$  is said to be an associated map for  $F$ .

**2. The  $\lambda$ -component  $EHP$  sequence.** The purpose of this section is to show that the map  $P_\lambda$  of Definition 1.4 is embedded in a long exact sequence resulting from the fibration  $\omega: L(\Sigma B, X; \lambda) \rightarrow X$ . Each  $\lambda \in [\Sigma B, X]$  determines a path component of  $L(\Sigma B, X)$  and  $\omega$  restricted to each path component determine a fibration and a long exact sequence. In §3 the relationship between these sequences and the James suspension sequence is explored and it is shown that G. W. Whitehead's  $EHP$  sequence [7] is a special case of an  $\iota_n$ -component  $EHP$  sequence where  $\iota_n = [1_{S^n}]$  in  $\pi_n(S^n)$ .

**LEMMA 2.1.** For  $\lambda \in L_0(\Sigma B, X)$ ,  $L_0(\Sigma B, X; *)$  is homotopy equivalent to  $L_0(\Sigma B, X; \lambda)$ .

*Proof.* Let  $\hat{\lambda}: L_0(\Sigma B, X; *) \rightarrow L_0(\Sigma B, X; \lambda)$  be defined by  $\hat{\lambda}(g) = g + \lambda$  and  $\hat{\lambda}^{-1}: L_0(\Sigma B, X; \lambda) \rightarrow L_0(\Sigma B, X; *)$  by  $\hat{\lambda}^{-1}(g) = g - \lambda$ . Then it is clear that  $\hat{\lambda}^{-1}$  is a two sided homotopy inverse of  $\hat{\lambda}$ .

In remaining parts of this section the map  $\hat{\lambda}$  will be taken to be given by

$$\hat{\angle}(g)(b, t) = \begin{cases} g\left(b, \frac{5}{4}t\right) & 0 \leq t \leq \frac{4}{5} \\ \angle(b, 5t - 4) & \frac{4}{5} \leq t \leq 1 \end{cases}.$$

LEMMA 2.2.  $[\Sigma(A \# B), X]$  is isomorphic to  $[A, L_0(\Sigma B, X; *)]$ .

This fact is well known. For the remainder of this section the isomorphism will be denoted by  $\theta: [\Sigma(A \# B), X] \rightarrow [A, L_0(\Sigma B, X; *)]$  defined as follows. If  $f: \Sigma(A \# B) \rightarrow X$ ,  $\theta(f)(a)$  is the map taking  $(b, t)$  to  $f((a, b), t)$  in  $X$ .

DEFINITION 2.3.  $A @ B$  is defined as  $A \times B$  with  $A \times \{b_0\}$  identified with  $(a_0, b_0)$ .

Let  $m: A \times \Sigma B \rightarrow (A \# \Sigma B) \vee (A @ \Sigma B)$  be defined by

$$(m(a, (b, t))) = \begin{cases} \left(a, \left(b, \frac{5}{4}t\right)\right) \vee * & 0 \leq t \leq \frac{4}{5} \\ * \vee (a, (b, 5t - 4)) & \frac{4}{5} \leq t \leq 1 \end{cases}.$$

Now let  $G: A \# \Sigma B \rightarrow X$  be a map associated with  $[g] \in [A, L_0(\Sigma B, X; *)]$ ,  $\angle \in L_0(\Sigma B, X)$ , and  $p_2: A @ \Sigma B \rightarrow \Sigma B$  the natural projection.

The following lemma can be easily verified.

LEMMA 2.4.  $(G \vee (\angle \circ p_2)) \circ m: A \times \Sigma B \rightarrow X$  is an associated map for  $\hat{\angle}_*([g]) \in [A, L_0(\Sigma B, X; \angle)]$ .

Let  $h_1: A \times SB \rightarrow \Sigma(A \times B)$  be defined by  $h_1(a, (b, t)) = ((a, b), t)$ , where  $SA$  is the unreduced suspension.

By the homotopy extension property the quotient map  $q_1: SB \rightarrow \Sigma B$  is a homotopy equivalence. Its homotopy inverse will be denoted  $q_1^{-1}: \Sigma B \rightarrow SB$ .

LEMMA 2.5. Let  $\alpha = [f] \in [\Sigma A, X]$  and  $\lambda = [\angle] \in [\Sigma B, X]$ , then  $(f \vee \angle) \circ (\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2) \circ h_1 \circ (1_A \times q_1^{-1}): A \times \Sigma B \rightarrow X$  is a map associated with  $\hat{\angle}_* \circ \theta([\alpha, \lambda]) \in [A, L_0(\Sigma B, X; \angle)]$ .

*Proof.* Let  $m_1: \Sigma(A \times B) \rightarrow \Sigma(A \times B) \vee \Sigma(A \times B)$  be given by

$$m_1((a, b), t) = \begin{cases} \left((a, b), \frac{5}{4}t\right) \vee * & 0 \leq t \leq \frac{4}{5} \\ * \vee ((a, b), 5t - 4) & \frac{4}{5} \leq t \leq 1 \end{cases}.$$

Consider the following diagram:

$$\begin{array}{ccccc}
 & & & & X \\
 & & & & \nearrow f \vee \ell \\
 \Sigma(A \times B) & \xrightarrow{\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2} & \Sigma A \vee \Sigma B & & \\
 \nearrow m_1 & & \nearrow \Sigma p_2 & & \uparrow \tilde{k} \\
 & \Sigma(A \times B) \vee \Sigma(A \times B) & & & \uparrow \ell \circ p_2 \\
 & \downarrow \Sigma q & \Sigma(A \# B) & & \\
 & & \uparrow q' & & \\
 A \times \Sigma B & \xrightarrow{1_A \times q_1^{-1}} A \times SB & \xrightarrow{1_A \times q_1} A \times \Sigma B & \xrightarrow{m} & A \# \Sigma B \vee A @ \Sigma B
 \end{array}$$

$q': A \# \Sigma B \rightarrow \Sigma(A \# B)$  is the homomorphism defined by  $q'(a, (b, t)) = ((a, b), t)$  and  $k$  is as in Definition 1.1. It is easiest to check the homotopy commutativity of this diagram by looking first at the lower four fifths of the  $t$  coordinate in  $SB$  and then at the upper fifth.

*Part 1.*  $\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2 \cong ((\tilde{k} \circ \Sigma q) \vee \Sigma p_2) \circ m_1$ .

The lower four fifths of  $\Sigma(A \times B)$  is mapped in one case by  $\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2$  and in the other by  $\tilde{k} \circ \Sigma q$ . But these are homotopic by the definition of  $\tilde{k}$ . The upper fifth is mapped by  $\Sigma p_2$  in either case.

*Part 2.*  $(f \vee \ell) \circ (\tilde{k} \circ \Sigma q \vee \Sigma p_2) \circ m_1 \circ h_1 = (((f \vee \ell) \circ \tilde{k} \circ q') \vee \ell \circ p_2) \circ m \circ (1_A \times q_1)$ .

On the lower four fifths the maps "meet" at  $\Sigma(A \# B)$ . In either case the point  $(a, (b, t)) \in A \times SB$  is mapped to  $((a, b), (5/4)t) \in \Sigma(A \# B)$ . On the upper fifth both maps are given by taking  $(a, (b, t))$  to  $\ell((b, 5t - 4))$  in  $X$ .

By Lemma 2.4 and the definition of  $[\alpha, \lambda]$ ,  $((f \vee \ell) \circ \tilde{k} \circ q') \vee (\ell \circ p_2) \circ m$  is an associated map for  $\ell_* \theta([\alpha, \lambda])$ . This is the lower route in the above diagram. Since  $q_1$  and  $q_1^{-1}$  are homotopy inverses

$$\begin{aligned}
 & (((f \vee \ell) \circ \tilde{k} \circ q') \vee \ell \circ p_2) \circ m \\
 & \cong (((f \vee \ell) \circ \tilde{k} \circ q') \vee \ell \circ p_2) \circ m \circ (1_A \times q_1) \circ (1_A \times q_1^{-1}) \\
 & \cong (f \vee \ell) \circ (\tilde{k} \circ \Sigma q \vee \Sigma p_2) \circ m_1 \circ h_1 \circ (1_A \times q_1^{-1}) \\
 & \cong (f \vee \ell) \circ (\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2) \circ h_1 \circ (1_A \times q_1^{-1}).
 \end{aligned}$$

The last two homotopies follow from Part 2 and Part 1 respectively. But the last map is the one claimed to be an associated map for

$\hat{\mathcal{L}}_*\theta([\alpha, \lambda])$  and the lemma is proven.

For  $\lambda = [\mathcal{L}] \in [\Sigma B, X]$  the evaluation map  $\omega: L(\Sigma B, X; \mathcal{L}) \rightarrow X$  is a fibration with fiber  $L_0(\Sigma B, X; \mathcal{L})$ . Then there is a long exact sequence of homotopy groups

$$\begin{aligned} \dots \longrightarrow [\Sigma^{r+1}A, X] &\xrightarrow{\partial} [\Sigma^r A, L_0(\Sigma B, X; \mathcal{L})] \xrightarrow{i_*} \\ [\Sigma^r A, L(\Sigma B, X; \mathcal{L})] &\xrightarrow{\omega_*} [\Sigma^r A, X] \xrightarrow{\partial} \dots \longrightarrow [\Sigma A, X] \xrightarrow{\partial} \\ [A, L_0(\Sigma B, X; \mathcal{L})] &\xrightarrow{i_*} [A, L(\Sigma B, X; \mathcal{L})] \xrightarrow{\omega_*} [A, X], \end{aligned}$$

where exactness at the last two stages is as pointed sets. Recall that Lemma 2.2 shows there is an isomorphism  $\theta: [\Sigma(A \# B), X] \rightarrow [A, L(\Sigma B, X; *)]$ .

**THEOREM 2.6.** For  $\alpha \in [\Sigma^r A, X]$ ,  $\partial(\alpha) = \hat{\mathcal{L}}_* \circ \theta \circ P_\lambda(\alpha)$ .

*Proof.* Let  $\alpha$  be represented by a map  $f: \Sigma^r A \rightarrow X$  and let  $q_2: C(\Sigma^{r-1}A) \rightarrow \Sigma^r A$  be the natural quotient map from the cone to the suspension. Define  $F: C(\Sigma^{r-1}A) \times SB \rightarrow \Sigma(\Sigma^{r-1}A) \vee \Sigma B$  by

$$F((a, r), (b, t)) = \begin{cases} q_2(a, r + 3t) \vee * & 0 \leq t \leq \frac{1}{3} \quad r \leq -3t + 1 \\ * & 0 \leq t \leq \frac{1}{3} \quad r \geq -3t + 1 \\ * \vee q_1(b, 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ * & \frac{2}{3} \leq t \leq 1 \quad r \geq 3t - 2 \\ q_2(a, 3 + r - 3t) \vee * & \frac{2}{3} \leq t \leq 1 \quad r \leq 3t - 2 \end{cases}$$

where  $(a, r) \in C(\Sigma^{r-1}A)$ ,  $r$  being the level on the cone and  $(b, t) \in SB$ ,  $t$  being the level on the suspension. At  $t = 1/3$  or  $2/3$  and on the lines  $r = -3t + 1$  and  $r = 3t - 2$ , the image of  $F$  is at  $*$  and  $F$  is well defined and continuous at these points. Since  $F$  is independent of  $a$  at  $r = 1$  and independent of  $b$  at  $t = 1$  and  $t = 0$ ,  $F$  is well defined. Let  $\Sigma^{r-1}A \times SB \rightarrow C(\Sigma^{r-1}A) \times SB$  be induced by including  $\Sigma^{r-1}A$  at the 0 level of  $C(\Sigma^{r-1}A)$ . Consider the following diagram:

$$\begin{array}{ccc} C(\Sigma^{r-1}A) \vee SB & & \\ \downarrow & \searrow q_2 \vee q_1 & \\ C(\Sigma^{r-1}A) \times SB & \xrightarrow{F} & \Sigma(\Sigma^{r-1}A) \vee \Sigma B \\ \uparrow & & \downarrow \Sigma p_1 + \Sigma p_2 - \Sigma p_1 \\ \Sigma^{r-1}A \times SB & \xrightarrow{h_1} & \Sigma(\Sigma^{r-1}A \times B). \end{array}$$

The map  $(\Sigma p_1 + \Sigma p_2 - \Sigma p_1) \circ h_1$  is given by

$$(a, (b, t)) \longrightarrow \begin{cases} (a, 3t) \vee * & 0 \leq t \leq \frac{1}{3} \\ * \vee q_1(b, 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ (a, 3 - 3t) \vee * & \frac{2}{3} \leq t \leq 1. \end{cases}$$

But this is the same as  $F((a, 0), (b, t))$ , that is  $F|_{\Sigma^{r-1}A \times SB}$ . Therefore the lower square commutes. In the upper triangle, when  $t = 0$  (the base point of  $SB$ ),  $F((a, r), (b, 0)) = q_2(a, r)$  by definition. At the base point of  $C(\Sigma^{r-1}A)$  consider

$$F((a_0, 0), (b, t)) = \begin{cases} q_2(a_0, 3t) \vee * = * & 0 \leq t \leq \frac{1}{3} \\ * \vee q_1(b, 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ q_2(a_0, 3 - 3t) \vee * = * & \frac{2}{3} \leq t \leq 1. \end{cases}$$

But this is clearly homotopic to  $* \vee q_1$ , thus the upper triangle commutes up to homotopy. Now consider the map  $\tilde{F}: C(\Sigma^{r-1}A) \times \Sigma B \rightarrow X$  given by  $\tilde{F} = (f \vee \wr) \circ F \circ (1_{C(\Sigma^{r-1}A)} \times q_1^{-1})$ .  $\tilde{F}$  is then an associated map for an element of  $[(C(\Sigma^{r-1}A), \Sigma^{r-1}A), (L(\Sigma B, X; \wr), L_0(\Sigma B, X; \wr))]$ . Since  $\tilde{F}|_{C(\Sigma^{r-1}A) \times *}$  is given by  $f \circ q_2$ ,  $\tilde{F}$  is associated to the class  $[f] \in [\Sigma^r A, X]$  under the bijection (see p. 104 in [5])  $\omega_*: [(C(\Sigma^{r-1}A), \Sigma^{r-1}A), (L(\Sigma B, X; \wr), L_0(\Sigma B, X; \wr))] \rightarrow [\Sigma^r A, X]$ . Then by definition of the boundary homomorphism,  $\partial([f]) = \partial(\alpha)$  has associated map  $\tilde{F}|_{\Sigma^{r-1}A \times \Sigma B}$ . But by commutativity of the above diagram  $\tilde{F}|_{\Sigma^{r-1}A \times \Sigma B} \cong (f \vee \wr) \circ (\Sigma p_1 + \Sigma p_2 - \Sigma p_1) \circ h_1 \circ (1_{\Sigma^{r-1}A} \times q_1^{-1})$  and by Lemma 2.5 this is an associated map for  $\hat{\wr}_* \circ \theta \circ ([\alpha, \lambda]) = \hat{\wr}_* \circ \theta \circ P_\lambda(\alpha)$ .

The existence of the  $\lambda$ -component *EHP* sequence now can be shown. Let  $i'_*: [\Sigma(\Sigma^{r-1}A \# B), X] \rightarrow [\Sigma^{r-1}A, L(\Sigma B, X; \wr)]$  be given by  $i'_* = i_* \circ \hat{\wr}_* \circ \theta$ .

**THEOREM 2.7.** *There is a long exact sequence*

$$\begin{aligned} \dots \longrightarrow [\Sigma^r A, L(\Sigma B, X; \wr)] &\xrightarrow{\omega_*} [\Sigma^r A, X] \xrightarrow{P_\lambda} \\ &[\Sigma(\Sigma^{r-1}A \# B); X] \xrightarrow{i'_*} [\Sigma^{r-1}A, L(\Sigma B, X; \wr)] \longrightarrow \dots \end{aligned}$$

*Proof.* Since  $\hat{\wr}_*$  and  $\theta$  are isomorphisms, the exactness of this sequence is immediate from the exactness of the homotopy exact sequence of the fibration  $\omega_*: L(\Sigma B, X; \wr) \rightarrow X$  and Theorem 2.6.

**3. The Whitehead and James sequences.** The purpose of this section is to compare the  $\lambda$ -component *EHP* sequence with the classical *EHP* sequence of George W. Whitehead [7] and the suspension sequence of I. M. James [4]. The spaces  $A$  and  $B$  will be assumed to be  $CW$  complexes with a single vertex. For  $\alpha \in [A, L(\Sigma B, X; \wr)]$  the element  $H(\alpha) \in [\Sigma(A \# \Sigma B), \Sigma X]$  is defined by the element obtained from a map associated with  $\alpha$  by the Hopf construction of Definition 1.2. The homomorphism  $E: [A \# \Sigma B, X] \rightarrow [\Sigma(A \# \Sigma B), \Sigma X]$  is defined by  $E([f]) = [\Sigma f]$ .

LEMMA 3.1. *The following diagram commutes:*

$$\begin{array}{ccc}
 [A \# \Sigma B, X] & \xrightarrow{E} & [\Sigma(A \# \Sigma B), \Sigma X] \\
 \cong \downarrow \theta & & \downarrow H \\
 [A, L_0(\Sigma B, X; *)] & & \\
 \cong \downarrow \wr_* & & \\
 [A, L_0(\Sigma B, X; \wr)] & \xrightarrow{i_*} & [A, L(\Sigma B, X; \wr)] .
 \end{array}$$

*Proof.* Let  $f: A \# \Sigma B \rightarrow X$  represent an element of  $[A \# \Sigma B, X]$ . Then  $\wr_* \circ \theta([f])$  has an associated map  $F: A \times \Sigma B \rightarrow X$  given by

$$F(a, (b, t)) = \begin{cases} f(a, (b, 2t)) & 0 \leq t \leq \frac{1}{2} \\ \wr(b, 2t - 1) & \frac{1}{2} \leq t \leq 1 . \end{cases}$$

Let  $\phi: \Sigma(A \# \Sigma B) \rightarrow \Sigma(A \times \Sigma B)$  and  $q: A \times \Sigma B \rightarrow A \# \Sigma B$  be as in the comments preceding Definition 1.2. Consider the following diagram:

$$\begin{array}{ccc}
 & & \Sigma X \\
 & \nearrow \Sigma F & \uparrow \Sigma f \vee \Sigma l \\
 \Sigma(A \times \Sigma B) & \xrightarrow{\Sigma q + \Sigma p_2} & \Sigma(A \# \Sigma B) \vee \Sigma \Sigma B \\
 \uparrow \phi + * & \nearrow i_1 & \\
 \Sigma(A \# \Sigma B) & & 
 \end{array}$$

where  $i_1$  is the inclusion of  $\Sigma(A \# \Sigma B)$  in  $\Sigma(A \# \Sigma B) \vee \Sigma \Sigma B$ . The homotopy commutativity of this diagram will establish the result since  $\Sigma F \circ (\phi + *) \cong \Sigma F \circ \phi$  which by definition is the element obtained from



$\mathcal{L}_* \circ \theta([f])$  by the Hopf construction and  $(\Sigma f \vee \Sigma \mathcal{L}) \circ i_1 = \Sigma f$ , a representative of  $E([f])$ .

In the lower triangle of the diagram  $(\Sigma q + \Sigma p_2) \circ (\phi + *) \cong \Sigma q \circ \phi$  which is homotopic to  $i_1$  by the definition of  $\phi$ .

In the upper triangle

$$\Sigma F(a, (b, t), r) = \begin{cases} (f(a, (b, 2t)), r) & 0 \leq t \leq \frac{1}{2} \\ (\mathcal{L}(b, 2t - 1), r) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$(\Sigma f \vee \Sigma \mathcal{L}) \circ (\Sigma q + \Sigma p_2)(a, (b, t), r) = \begin{cases} (f(a, (b, t)), 2r) & 0 \leq r \leq \frac{1}{2} \\ (\mathcal{L}(b, t), 2r - 1) & \frac{1}{2} \leq r \leq 1. \end{cases}$$

The usual homotopy to interchange the roles of  $t$  and  $r$  for homotopy will work in this case since  $f$  is defined on  $A \# \Sigma B$  and  $\mathcal{L}$  is independent of  $a$ . Thus the upper triangle is homotopy commutative and the lemma is established.

DEFINITION 3.2. The *classical EHP sequence* is given by:

$$\begin{aligned} \pi_{3n-2}(S^n) &\xrightarrow{E} \pi_{3n-1}(S^{n+1}) \longrightarrow \dots \longrightarrow \pi_{n+p}(S^n) \longrightarrow \\ \pi_{n+p+1}(S^{n+1}) &\xrightarrow{H'} \pi_p(S^n) \xrightarrow{P} \pi_{p+n-1}(S^n) \xrightarrow{E} \dots \end{aligned}$$

where  $E$  is the suspension homomorphism,  $H' = -E^{-n-1} \circ \tilde{H}$  where  $\tilde{H}$  is the Hopf invariant of Definition 1.8,  $P = P_{\iota_n}$ , where  $\iota_n = [1_{S^n}] \in \pi_n(S^n)$ .

This sequence was shown exact in [7]; the form used in Definition 3.2 is that of P. J. Hilton and J. H. C. Whitehead in [3]. The classical *EHP* sequence can now be compared with the  $\iota_n$ -component *EHP* sequence for the fibration  $\omega: L(S^n, S^n; 1_{S^n}) \rightarrow S^n$ .

THEOREM 3.3. For  $q \leq 3n - 2$  the following exact ladder is commutative and  $H$  is an isomorphism:

$$\begin{array}{ccccccc} \longrightarrow & \pi_q(S^n) & \xrightarrow{E} & \pi_{q+1}(S^{n+1}) & \xrightarrow{H'} & \pi_{q-n}(S^n) & \xrightarrow{P} \pi_{q-1}(S^n) \longrightarrow \\ & \uparrow 1 & & \uparrow H & & \uparrow 1 & \uparrow 1 \\ \longrightarrow & \pi_q(S^n) & \xrightarrow{i'_*} & \pi_{q-n}L(S^n, S^n; 1_{S^n}) & \xrightarrow{\omega_*} & \pi_{q-n}(S^n) & \xrightarrow{P_{\iota_n}} \pi_{q-1}(S^n) \longrightarrow \end{array}$$

*Proof.* The left square commutes by Lemma 3.1 since, by defini-

tion,  $i'_* = i_* \circ \hat{\iota}_* \circ \theta$ . The right square commutes by the definition of  $P$ . For the range  $q \leq 3n - 2$ , G. W. Whitehead shows (Corollary 6-4 in [7]) that every element  $\alpha \in \pi_{q+1}(S^{n+1})$  is obtainable from a map  $F: S^{q-n} \times S^n \rightarrow S^n$  of type  $(H'(\alpha), \iota_n)$  by a Hopf construction. Thus if  $F$  is considered as an associated map for an element  $\beta \in \pi_{q-n}(L(S^n, S^n; 1_{S^n}))$ ,  $\beta$  has type  $(\omega_*(\beta), \iota_n)$  and  $H(\beta)$  is obtainable by a Hopf construction from a map of type  $(\omega_*(\beta), \iota_n)$  as well as a map of type  $(H'(\alpha), \iota_n)$ . But then by 5.1 in [6],  $\omega_*(\beta) * \iota_n = H'(\alpha) * \iota_n$ , where  $*$  is the join operation. Since  $\iota_n$  is the homotopy class of  $1_{S^n}$ ,  $E^{n+1}\omega_*(\beta) = \omega_*(\beta) * \iota_n = H'(\alpha) * \iota_n = E^{n+1}H'(\alpha)$ . Now  $q \leq 3n - 2$  so  $q - n \leq 2n - 2$  and by the Freudenthal suspension theorem  $E^{n+1}$  is an isomorphism, thus  $\omega^*(\beta) = H'(\alpha) = H'(H(\beta))$ . This establishes the commutivity of the ladder. That  $H$  is an isomorphism follows from the five lemma.

Since the bottom line is the  $\iota_n$ -component EHP sequence, the classical EHP sequence can be considered as the  $\iota_n$ -component EHP sequence for spheres in the range  $q \leq 3n - 2$ .

Some definitions will be required before describing the suspension sequence of James. Let  $D^n$  denote the solid  $n$ -ball. Then  $\partial D^n = S^{n-1} = D_+^{n-1} \cup D_-^{n-1}$  where  $D_+^{n-1}$  and  $D_-^{n-1}$  are the northern and southern hemispheres of  $S^{n-1}$  respectively. Note that  $D_+^{n-1} \cap D_-^{n-1} = S^{n-2}$ .

**DEFINITION 3.4.** For  $A$  and  $B$  subspaces of  $X$  such that  $A \cap B \neq \emptyset$  let  $\pi_n(X; A, B)$  be the set of homotopy classes maps of  $f: (D^n, D_+^{n-1}, D_-^{n-1}) \rightarrow (X, A, B)$ .

There are natural boundary operators  $\partial_1: \pi_n(X; A, B) \rightarrow \pi_{n-1}(A, A \cap B)$  and  $\partial_2: \pi_{n-1}(A, A \cap B) \rightarrow \pi_{n-2}(A \cap B)$  defined by restriction to  $(D_+^{n-1}, S_-^{n-2})$  and  $S_-^{n-2}$  respectively.

**DEFINITION 3.5.** The repeated boundary operator  $\Delta: \pi_n(X; A, B) \rightarrow \pi_{n-2}(A \cap B)$  is defined by  $\Delta = \partial_2 \circ \partial_1$ .

The following result of James will be useful.

**THEOREM 3.6.** There is a pairing  $\{\beta, \gamma\} \in \pi_{p+q+1}(\Sigma X; C_+X, C_-X)$  for  $\beta \in \pi_p(X)$  and  $\gamma \in \pi_q(X)$  such that

- (i)  $\Delta\{\beta, \gamma\} = [\beta, \gamma] \in \pi_{p+q+1}(X)$ , the usual Whitehead product and
- (ii) If  $i_*: \pi_{p+q+1}(\Sigma X) \rightarrow \pi_{p+q+1}(\Sigma X; C_+X, C_-X)$  is the natural inclusion, an element  $\alpha \in \pi_{p+q+1}(\Sigma X)$  is obtainable by a Hopf construction of type  $(\beta, \gamma)$  iff  $i_*(\alpha) = \{\beta, \gamma\}$ .

*Proof.* See § 4 and Theorem 2.17 in [4].

**DEFINITION 3.7.** The James suspension sequence is

$$\longrightarrow \pi_{p+q}(X) \xrightarrow{E} \pi_{p+q+1}(\Sigma X) \xrightarrow{i_*} \pi_{p+q+1}(\Sigma X; C_+X, C_-X) \xrightarrow{\Delta} \pi_{p+q-1}(X)$$

where  $E$  is the suspension homomorphism and  $\Delta$  is the repeated boundary operator.

**THEOREM 3.8.** *The following exact ladder is commutative:*

$$\begin{array}{ccccccc} \longrightarrow & \pi_{p+q}(X) & \xrightarrow{i'_*} & \pi_p(L(S^q, X; \wr)) & \xrightarrow{\omega_*} & \pi_p(X) & \xrightarrow{P_\lambda} \longrightarrow \pi_{p+q-1}(X) \longrightarrow \\ & \downarrow 1 & & \downarrow H & & \downarrow \tilde{P} & \downarrow 1 \\ \longrightarrow & \pi_{p+q}(X) & \xrightarrow{E} & \pi_{p+q+1}(\Sigma X) & \xrightarrow{i_*} & \pi_{p+q+1}(\Sigma X; C_+X, C_-X) & \xrightarrow{\Delta} \pi_{p+q-1}(X) \longrightarrow \end{array}$$

where  $H$  is as in Lemma 3.1,  $\widetilde{P}(\alpha) = \{\alpha, \lambda\}$ , and  $\lambda = [\wr]$  for any  $[\wr] \in \pi_q(X)$ .

*Proof.* The left square commutes by Lemma 3.1 since  $i'_* = i_* \circ \hat{\wr}_* \circ \theta$  by definition. If  $\alpha \in \pi_p(L(S^q, X; \wr))$  then by definition of  $H$ ,  $H(\alpha)$  is obtainable by a Hopf construction of type  $(\omega_*(\alpha), \lambda)$  and by Theorem 3.6, (ii),  $i_*H(\alpha) = \{\omega_*(\alpha), \lambda\} = \tilde{P} \circ \omega_*(\alpha)$ . Thus the middle square commutes. The right square commutes by Theorem 3.6, (i).

Theorem 3.8 clearly indicates the extent to which the map  $i'_* = i_* \circ \hat{\wr}_* \circ \theta$  of the  $\lambda$ -component  $EHP$  sequence approximates the suspension homomorphism. Indeed,  $E = H \circ i'_*$ . While the James sequence contains the suspension homomorphism in a straight forward form, the  $\lambda$ -component  $EHP$  sequence contains the generalized Whitehead product in a more direct form.

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