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In all that follows, G is an infinite, nondiscrete, locally compact T_0 abelian group with character group X and Δ is a nonempty subset of X. In a standard proof of the existence of infinite (in fact, perfect) Helson sets (see for example Hewitt and Ross) it is shown that each nonvoid open subset of an arbitrary G contains a K-set (terminology of Hewitt and Ross) homeomorphic to Cantor's ternary set (or, in the terminology of Rudin, a Kronecker set or a set of type K_a homeomorphic to the Cantor set). In this paper, it is shown that $K_{0,\Delta}$-sets or $K_{a,\Delta}$-sets homeomorphic to the Cantor set exist in profusion in a large class of infinite nondiscrete locally compact T_0 abelian groups G, provided that $\bar{\Delta}$ is not compact. (A nonvoid subset E of G is called a $K_{0,\Delta}$-set if for every continuous function from E to T, the circle group, and every $\varepsilon > 0$, there is a $\gamma \in \Delta$ such that $| \gamma(x) - f(x) | < \varepsilon$ for all $x \in E$. Let a be an integer greater than one. A nonvoid subset E of G is called a $K_{a,\Delta}$-set if it is totally disconnected and every continuous function on E with values in the set of a th roots of unity is the restriction to E of some $\gamma \in \Delta$.)

The following theorems will be proved.

THEOREM I. *Let G be compact. Let Δ be infinite. Suppose that, except for the character which is identically* 1, *$\Delta\Delta^{-1}$ consists solely of elements of infinite order. (This condition is satisfied automatically if G is connected, for then X is torsion-free.) Then every nonvoid open set in G contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set.*

THEOREM II. *Let G be locally connected. Suppose that $\bar{\Delta}$ is not compact. Then every nonvoid open set in G contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set.*

THEOREM III. *Let G be a compact torsion group. Let Δ be infinite. Then there is an integer $a \geqq 2$ such that every nonvoid open set in G contains a translate of a $K_{a,\Delta}$-set homeomorphic to the Cantor set.*

1. Preliminaries.

NOTATION 1.1. We denote Haar measure on G by m, with $m(G) = 1$ when G is compact. When H is a subgroup of G, we write

$\Delta|_H$ for $\{\gamma|_H : \gamma \in \Delta\}$. $M(P)$ denotes the set of all (finite) regular Borel measures on the compact subset P of G.

$C(A, B)$ denotes the set of all continuous functions from A to B, where A and B are topological spaces. If $B = \mathbf{C}$, the set of complex numbers, we write $C(A)$ instead of $C(A, \mathbf{C})$.

$\mathbf{Z}$ is the group of integers. $\mathbf{R}$ is the group of real numbers. $\mathbf{Q}$ is the (discrete) group of rational numbers. $\mathbf{N}$ is the set of positive integers. When a is an integer greater than one, $\mathbf{Z}_a$ is the additive group of integers modulo a and $\mathbf{T}_{(a)}$ is the multiplicative group of ath roots of unity.

1 is the identity element of X.

$\prod^*_{\iota \in I} G_\iota$ is the weak direct product of the groups G_ι.

REMARKS 1.2.

(a) In §5, we give examples which show some of the limitations of Theorems I, II, and III.

(b) The hypothesis on $\Delta\Delta^{-1}$ in Theorem I is related to connectedness, as will be shown in Theorem 2.1.

(c) When G is compact, a $K_{0,\Delta}$-set (or $K_{a,\Delta}$-set) E is a Δ-Helson set—i.e., a set with the property that every $f \in C(E)$ has the form $f = \check{g}|_E$ for some $g \in L_1(X)$ which vanishes off Δ. When G is not compact, a $K_{0,\Delta}$-set need not be a Δ-Helson set as the example $G = X = \mathbf{R}$ and $\Delta = \mathbf{Q}$ shows.

(d) Our proof of Theorem II for the case where G is metrizable uses a technique due to Kaufman, [6, p. 184–185 and 7]. The general case follows from the case where G is metrizable and from Theorem I. Our proofs of Theorems I and III depend on the notion of an equidistributed sequence in a compact group. This notion for the case $G = \mathbf{T}$ is due to Weyl [9]. The notion has been generalized by Eckmann [2] and Hlawka [5]. Eckmann's work offers more than enough generality for our purposes; relevant parts are given below in 1.3 and 1.4.

DEFINITION 1.3. Let H be a compact abelian group with Haar measure μ and $\mu(H) = 1$. Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence in H. For $F \subset H$, let $n(F)$ be the number of α_j with index $j \leqq n$ which are in F. The sequence $\{\alpha_j\}_{j=1}^\infty$ is said to be equidistributed in H if $\lim_{n\to\infty} n(F)/n = \mu(F)$ for all closed F with the property that μ(boundary F) $= 0$.

THEOREM 1.4. *Let H be a compact abelian group with Haar measure μ and $\mu(H) = 1$. Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence in H. The following are equivalent*:

(i) *$\{\alpha_j\}_{j=1}^\infty$ is equidistributed in H*;

(ii) *for every continuous character* γ *of* H *such that* $\gamma \not\equiv 1$, *we have* $\lim_{n\to\infty} n^{-1} \Sigma_{j=1}^{n} \gamma(\alpha_j) = 0$.

REMARKS 1.5.

(a) In the proofs of Theorems I and III we will use the equivalence of (i) and (ii) in Theorem 1.4 for the cases $H = \mathbf{T}$ and $H = \mathbf{T}_{(a)}$ respectively. If $H = \mathbf{T}$ we have Weyl's original result: The sequence $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbf{T}$ is equidistributed in $\mathbf{T}$ if and only if $\lim_{n\to\infty} n^{-1} \Sigma_{j=1}^{n} \alpha_j^r = 0$ for all nonzero integers r (or, equivalently, for all $r \in \mathbf{N}$) [9]. If $H = \mathbf{T}_{(a)}$, we have: The sequence $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbf{T}_{(a)}$ is equidistributed in $\mathbf{T}_{(a)}$ if and only if for every integer $r \in \{1, 2, \cdots, a-1\}$ we have $\lim_{n\to\infty} n^{-1} \Sigma_{j=1}^{n} \alpha_j^r = 0$.

(b) Eckmann's definition differs from Definition 1.3 in that he omits the restriction $\mu(\text{boundary } F) = 0$. This restriction is necessary, as has been pointed out [3].

2. Proof of Theorem I.

2.1. We first investigate the hypothesis on $\Delta\Delta^{-1}$ in the statement of Theorem I and find that it is related to connectedness.

THEOREM. *Let* G *be compact. Let* Δ *be a countably infinite subset of* X. *The following are equivalent*:

(i) $\Delta\Delta^{-1}\backslash\{1\}$ *consists solely of elements of infinite order*;

(ii) G *contains a compact connected metrizable subgroup* H *with the property that* $\delta \to \delta|_H$ *is a one-to-one map from* Δ *to the character group of* H.

Proof. (ii) implies (i): Let δ_1 and δ_2 be distinct elements of Δ. Then $\delta_1|_H \neq \delta_2|_H$, so $\delta_1\delta_2^{-1}|_H \not\equiv 1$. Since H is connected, its character group is torsion-free. Hence, $\delta_1\delta_2^{-1}|_H$ has infinite order and therefore so does $\delta_1\delta_2^{-1}$.

(i) implies (ii): Let Γ be a maximal torsion-free independent subset of Δ. (Clearly, Δ contains at most one element of finite order, so Γ is nonvoid.) We have $\Gamma = \{\gamma_1, \cdots, \gamma_p\}$ for some positive integer p or $\Gamma = \{\gamma_1, \gamma_2, \cdots\}$. If Γ is finite, let $P = \mathbf{Q}^p$. If not, let P be the weak direct product of countably many copies of $\mathbf{Q}$. (In either case, P is countable.) For $n \in \mathbf{N}$ (and $n \leqq p$ if Γ is finite) let e_n be that element of P with nth coordinate equal to 1 and all other coordinates equal to zero. Let Y be the subgroup of X generated by Γ. Since Γ is independent, the map $\gamma_n \to e_n$ extends to a (one-to-one) homomorphism from Y to P. Since P is divisible, this homomorphism extends to a homomorphism $\phi: X \to P$. Hence $W = X/\ker\phi$ is isomorphic to a subgroup of P. Let H be the annihilator of $\ker\phi$ in G. Then H

is a closed subgroup of G and has character group W, which is torsion-free and countable. Hence, H is connected and metrizable. Now $\delta_1|_H = \delta_2|_H$ if and only if $\delta_1\delta_2^{-1} \in \ker \phi$. Let δ_1 and δ_2 be distinct elements of Δ. It is sufficient to show that $\delta_1\delta_2^{-1} \notin \ker \phi$. Since Γ is a maximal torsion-free independent subset of Δ, there exist nonzero integers r_1 and r_2 such that $\delta_1^{r_1}$ and $\delta_2^{r_2}$ are in Y. Therefore there is a nonzero integer r such that $(\delta_1\delta_2^{-1})^r \in Y$. By the hypothesis on $\Delta\Delta^{-1}$, we have $(\delta_1\delta_2^{-1})^r \neq 1$. Since ϕ is one-to-one on Y, $r\phi(\delta_1\delta_2^{-1}) = \phi((\delta_1\delta_2^{-1})^r)$ is not the identity of P. Hence $\delta_1\delta_2^{-1} \notin \ker \phi$ and the proof is complete.

LEMMA 2.2. *Let G be compact. Let $\Delta = \{\gamma_1, \gamma_2, \cdots\}$ be a countably infinite set of distinct elements of X arranged in any fixed order. Suppose that $\Delta\Delta^{-1}\backslash\{1\}$ consists solely of elements of infinite order. Then for m-almost all $x \in G$, the sequence $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in* $\mathbf{T}$.

Proof. Our proof follows Weyl [9]. For $x \in G$, $n \in \mathbf{N}$, and $r \in \mathbf{N}$, define $f_{nr}(x) = n^{-1}\Sigma_{j=1}^{n}\gamma_j^r(x)$. From our hypothesis on $\Delta\Delta^{-1}$ we find that $\gamma_j^r\overline{\gamma_k^r} = 1$ implies that $\gamma_j = \gamma_k$. Since G is compact, $\int_G \gamma(x)dm(x) = 0$ when $\gamma \neq 1$. Thus, we have

$$\int_G |f_{nr}|^2 dm = n^{-2}\int_G \Sigma_{j,k=1}^{n}\gamma_j^r(x)\overline{\gamma_k^r(x)}dm(x) = n^{-1} .$$

Therefore we have $\Sigma_{n=1}^{\infty}\|f_{n^2,r}\|_2^2 < \infty$ and hence $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for m-almost all $x \in G$. Suppose that $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for all $x \notin A_r$ where $m(A_r) = 0$.

For $n \in \mathbf{N}$, let $\lambda(n)$ be the positive integer such that $\lambda^2 \leqq n < (\lambda + 1)^2$. Then we have $|nf_{nr}(x) - \lambda^2 f_{\lambda^2,r}(x)| \leqq 2\lambda$ and hence

$$\left| f_{nr}(x) - \frac{\lambda^2}{n} f_{\lambda^2,r}(x)\right| \leqq 2/\sqrt{n} .$$

Let $\varepsilon > 0$. Fix $x \notin A_r$. Then there is a positive integer M such that $|f_{\lambda^2,r}(x)| < \varepsilon/2$ whenever $\lambda \geqq M$. Let $n \geqq M^2$ and $n > 16/\varepsilon^2$. Let λ be such that $\lambda^2 \leqq n < (\lambda + 1)^2$. Then $\lambda^2/n \leqq 1$, $2/\sqrt{n} < \varepsilon/2$, and $\lambda^2 \geqq M^2$, so we have

$$|f_{nr}(x)| \leqq \left| f_{nr}(x) - \frac{\lambda^2}{n} f_{\lambda^2,r}(x) \right| + \frac{\lambda^2}{n}|f_{\lambda^2,r}(x)| < 2/\sqrt{n} + \varepsilon/2 < \varepsilon .$$

Hence, $f_{nr}(x) \to 0$ as $n \to \infty$ for all $x \notin A_r$.

Let $A = \cup A_r$. Then $m(A) = 0$ and for $x \notin A$ we have for all $r \in \mathbf{N}$ that $f_{nr}(x) \to 0$ as $n \to \infty$. Therefore, by 1.5(*a*), $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $\mathbf{T}$ for all $x \notin A$.

LEMMA 2.3. *Let G and Δ be as in Theorem 1. Let $V_1, \cdots, V_k$ be nonvoid open subsets of G. Then there exist $x_j \in V_j (1 \leqq j \leqq k)$ with the property that for every $\varepsilon > 0$ and for all $z_1, \cdots, z_k \in \mathbf{T}$ there is a $\gamma \in \Delta$ such that $|\gamma(x_j) - z_j| < \varepsilon (1 \leqq j \leqq k)$, i.e., there exist $x_j \in V_j$ $(1 \leqq j \leqq k)$ such that $\{x_1, \cdots, x_k\}$ is a $K_{0,\Delta}$-set.*

Proof. We may suppose that Δ is countable. Let $q \in \{1, 2, \cdots, k\}$. Let "$P(q)$ holds for $x_1, \cdots, x_q$" mean "$x_j \in V_j (1 \leqq j \leqq q)$ and $\{x_1, \cdots, x_q\}$ is a $K_{0,\Delta}$ set." By Lemma 2.2, there is an $x_1 \in V_1$ such that $P(1)$ holds for x_1. Suppose that $1 \leqq r \leqq k - 1$ and that $P(r)$ holds for $x_1, \cdots, x_r$. It is sufficient to show there is an $x_{r+1} \in V_{r+1}$ such that $P(r + 1)$ holds for $x_1, \cdots, x_{r+1}$. Let $A = \{w \in V_{r+1} | P(r + 1)$ does not hold for $x_1, \cdots, x_r, w\}$. It is sufficient to show that $m(A) = 0$. Let S be a countable dense subset of $\mathbf{T}$. Then $w \in A$ if and only if $w \in V_{r+1}$ and there exist $p \in \mathbf{N}$ and $s_1, \cdots, s_{r+1} \in S$ such that for all $\gamma \in \Delta$ either $|\gamma(x_j) - s_j| \geqq p^{-1}$ for some $j (1 \leqq j \leqq r)$ or $|\gamma(w) - s_{r+1}| \geqq p^{-1}$, i.e., we have

$$A = \bigcup_{p \in \mathbf{N}} \bigcup_{s_1 \in S} \cdots \bigcup_{s_{r+1} \in S} A(p, s_1, \cdots, s_{r+1})$$

where $A(p, s_1, \cdots, s_{r+1}) = \bigcap_{\gamma \in \Delta} \{y \in V_{r+1}: |\gamma(y) - s_{r+1}| \geqq p^{-1}$ or at least one $|\gamma(x_j) - s_j| \geqq p^{-1}\}$.

Let

$$\tilde{\Delta}(p, s_1, \cdots, s_r) = \{\gamma \in \Delta: |\gamma(x_j) - s_j| < p^{-1}, 1 \leqq j \leqq r\} .$$

Then we have

$$\begin{aligned} &A(p, s_1, \cdots, s_{r+1}) \\ &= \{y \in V_{r+1}: |\gamma(y) - s_{r+1}| \geqq p^{-1} \text{ for all } \gamma \in \tilde{\Delta}(p, s_1, \cdots, s_r)\} . \end{aligned}$$

Hence, it is sufficient to show that each $\tilde{\Delta}(p, s_1, \cdots, s_r)$ is infinite (for then, by Lemma 2.2, each $A(p, s_1, \cdots, s_{r+1})$ is m-null and therefore so is A).

We assume that for some $p \in \mathbf{N}$ and $s_1, \cdots, s_r \in S$ the set $\tilde{\Delta} = \tilde{\Delta}(p, s_1, \cdots, s_r)$ is finite and use this to obtain a contradiction. A basic neighborhood of the point $\mathbf{z} = (z_1, \cdots, z_r) \in \mathbf{T}^r$ has the form $B(\mathbf{z}, \varepsilon) = \{\mathbf{w} = (w_1, \cdots, w_r): |z_j - w_j| < \varepsilon, 1 \leqq j \leqq r\}$ for some $\varepsilon > 0$. Let $\mathbf{s} = (s_1, \cdots, s_r)$ and $\mathbf{x} = (x_1, \cdots, x_r)$. For $\gamma \in \Delta$, let $\gamma(\mathbf{x}) = (\gamma(x_1), \cdots, \gamma(x_r))$. If $\tilde{\Delta}$ is finite, then $\{\gamma \in \Delta | \gamma(\mathbf{x}) \in B(\mathbf{s}, p^{-1})\}$ is finite. Then there exist $\mathbf{z} \in B(\mathbf{s}, p^{-1})$ and $\varepsilon > 0$ be such that $B(\mathbf{z}, \varepsilon) \subset B(\mathbf{s}, p^{-1})$ and $B(\mathbf{z}, \varepsilon)$ is disjoint from $\{\gamma(\mathbf{x}) | \gamma \in \Delta\}$. This contradicts the induction hypothesis that $P(r)$ holds for $x_1, \cdots, x_r$.

THEOREM 2.4. *Theorem I holds when G is metrizable.*

Proof. Repeat the proof of [4, (41.5), part I] choosing all characters in Δ and using Lemma 2.3 whenever [4] uses [4, (41.3)].

THEOREM 2.5. *Let G and Δ be as in Theorem I. Let U be a neighborhood of the identity in G. Then U contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set.*

Proof. By Theorem 2.1, G contains a compact connected metrizable subgroup H with the property that $\Gamma = \Delta|_H$ is infinite. Let $V = U \cap H$. Since H is connected, its character group is torsion-free. Hence, by Theorem 2.4, V contains a $K_{0,\Gamma}$-set P homeomorphic to the Cantor set. Clearly, P is a $K_{0,\Delta}$-set contained in U.

THEOREM[1] 2.6. *Let P be a compact metrizable $K_{0,\Delta}$-set in G, where G is compact and $\Delta\Delta^{-1}\backslash\{1\}$ consists solely of elements of infinite order. Then for almost all $x \in G$, xP is a $K_{0,\Delta}$-set.*

Proof. Let $\{f_1, f_2, \cdots\}$ be a (uniformly) dense subset of $C(P, \mathbf{T})$. For each j, there is a sequence $\{\gamma_{ij}\}_{i=1}^{\infty}$ of elements of Δ such that $\gamma_{ij} \to f_j$ uniformly on P. By Lemma 2.2, there is an m-null set A_j such that $\{\gamma_{ij}(x)\}_{i=1}^{\infty}$ is equidistributed in $\mathbf{T}$ whenever $x \in G\backslash A_j$. Let $A = \cup A_j$. Then A is m-null. Let $x \in G\backslash A$. For each j, let $g_j(xy) = f_j(y)$. To show that xP is a $K_{0,\Delta}$-set, it is sufficient to show that each g_j is uniformly approximable by $\{\gamma_{ij}: i, j = 1, 2, \cdots\}$. Let $\varepsilon > 0$. Fix j. Then for some i_0, we have $|\gamma_{ij}(y) - f_j(y)| < \varepsilon/2$ for all $y \in P$ whenever $i > i_0$ and, since $\{\gamma_{ij}(x)\}_{i=1}^{\infty}$ is equidistributed in $\mathbf{T}$, there is an $i > i_0$ such that $|\gamma_{ij}(x) - 1| < \varepsilon/2$. For this i we have $|\gamma_{ij}(xy) - g_j(xy)| < \varepsilon$ for all $y \in P$.

Proof of Theorem I. 2.7. Immediate from Theorems 2.5 and Theorem 2.6.

3. Proof of Theorem II.

THEOREM 3.1. *Let G be locally connected. Let Δ be such that $\bar{\Delta}$ is not compact. Let U be a neighborhood of the identity in G. Then there is a γ in Δ such that $\gamma(U) = \mathbf{T}$.*

Proof. The topology on X is the restriction of the compact-open topology on $C(G)$ to the (closed) subspace X of $C(G)$. Hence, $\bar{\Delta}$ is

[1] In the original version of this paper, the conclusion of Theorem I was as follows: Every open set in G containing an element of finite order contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set and, if G is metrizable, every nonvoid open set in G contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set. Theorem 2.6 and the stronger version of Theorem I which it yields are due to Robert Kaufman [private communication, December, 1971].

compact as a subspace of X if and only if it is compact as a subspace of $C(G)$ with the compact-open topology. Since by hypothesis $\bar{\Delta}$ is not compact, it follows from Ascoli's Theorem that Δ is not equicontinuous [1, p. 267] and, hence, that Δ is not equicontinuous at the identity of G. Therefore, there exists $\varepsilon > 0$ such that for every neighborhood W of the identity in G, there is an $x \in W$ and a $\gamma \in \Delta$ such that $|\gamma(x) - 1| \geqq \varepsilon$. Let $S = \{e^{it} | 0 \leqq t \leqq \varepsilon/2\}$. Let M be a positive integer with the property that $S^M = \mathbf{T}$. Let V be a connected neighborhood of the identity in G such that $V^M \subset U$. Then there exist $x \in V$ and $\gamma \in \Delta$ such that $|\gamma(x) - 1| \geqq \varepsilon$. Hence, $\gamma(V)$ contains an arc of length at least ε. Therefore we have $\mathbf{T} = \gamma(V)^M \subset \gamma(U) \subset \mathbf{T}$.

THEOREM 3.2. *Let G be locally connected and metrizable. Let Δ be such that $\bar{\Delta}$ is not compact. Let E be a compact totally disconnected subset of* $\mathbf{R}$ *or* $\mathbf{T}$. *Then there is a first category set $H \subset C(E, G)$ such that each $f \in C(E, G)\backslash H$ maps E homeomorphically onto a $K_{0,\Delta}$-set in G.*

Proof. Our proof follows the ideas of Kaufman [7] as given by Katznelson [6, p. 184–185].

For $h \in C(E, \mathbf{T}), f \in C(E, G)$, and $\varepsilon > 0$, let "(*) holds for h, f and ε" mean "there is a $\gamma \in \Delta$ such that $|\gamma(f(y)) - h(y)| < \varepsilon$ for all $y \in E$." Let $f \in C(E, G)$. Clearly, f is a homeomorphism of E onto $f(E)$ if and only if f is one-to-one. Also, if f is not one-to-one, it is clear that there exist $h \in C(E, \mathbf{T})$ and $\varepsilon > 0$ such that (*) fails for h, f, and ε. Hence, f is a homeomorphism of E onto $f(E)$ and $f(E)$ is a $K_{0,\Delta}$-set if and only if for every $h \in C(E, \mathbf{T})$ and every $\varepsilon > 0$, (*) holds for h, f, and ε.

Let d be an invariant metric on G compatible with the topology of G. For f and g in $C(E, G)$, let $D(f, g) = \sup\{d(f(y), g(y)) | y \in E\}$. Observe that $D(f, g) < \infty$ since E is compact.

Let $h \in C(E, \mathbf{T}), g \in C(E, G), \varepsilon > 0$, and $\eta > 0$. We now show that there exist an $f \in C(E, G)$ such that (*) holds for h, f, and ε and $D(f, g) < \eta$. Let U be the open η-ball about the identity e of G. By Theorem 3.1, there is a $\gamma \in \Delta$ such that $\gamma(U) = \mathbf{T}$. Write $E = \bigcup_{j=1}^{n} E_j$, where the E_j are disjoint nonvoid open-closed subsets of E and $\gamma \circ g$ and h both vary by less than $\varepsilon/3$ on each E_j. (The E_j exist since E is totally disconnected.) Let $y_j \in E_j$ and suppose that $\gamma(g(y_j)) = \alpha_j$ and $h(y_j) = \beta_j$, $1 \leqq j \leqq n$. Let $x_j \in U$ be such that $\gamma(x_j) = \overline{\alpha_j}\beta_j$. Define $f \in C(E, G)$ by $f(y) = x_j g(y)$ when $y \in E_j$. We see that $D(f, g) = \max\{d(x_j, e)\} < \eta$ and for $y \in E_j$ we have

$$|\gamma(f(y)) - h(y)| \leqq |\gamma(g(y))\gamma(x_j) - \gamma(g(y_j))\gamma(x_j)| + |\gamma(g(y_j))\gamma(x_j) - h(y_j)| + |h(y_j) - h(y)| < \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} < \varepsilon .$$

Hence, (*) holds for h, f, and ε.

For $h \in C(E, \mathbf{T})$ and $\varepsilon > 0$, let $H(h, \varepsilon) = \{f \in C(E, G) \mid (*) \text{ fails for } h, f, \text{ and } \varepsilon\}$. It is easy to show that $H(h, \varepsilon)$ is closed. By the preceding paragraph, $H(h, \varepsilon)$ is nowhere dense in $C(E, G)$. Let $\{h_n\}_{n=1}^{\infty}$ be dense in $C(E, \mathbf{T})$. Let $H = \bigcup_{n,k=1}^{\infty} H(h_n, 1/k)$. Then H is a first category set in the complete metric space $C(E, G)$. Also, we have $f \in C(E, G)\backslash H$ if and only if every $h \in C(E, \mathbf{T})$ can be uniformly approximated by $\gamma \circ f$'s $(\gamma \in \Delta)$, which by the second paragraph of the proof is true if and only if f is a homeomorphism and $f(E)$ is a $K_{0,\Delta}$-set.

THEOREM 3.3. *Theorem II holds when G is metrizable.*

Proof. Let U be a nonvoid open subset of G. Let E be the Cantor set. Let H be as in Theorem 3.2. The result follows from Theorem 3.2 since $C(E, U)$ is open in $C(E, G)$ and $C(E, G)\backslash H$ is dense in $C(E, G)$.

THEOREM 3.4. *Let G be locally connected. Then G is topologically isomorphic with $D \times \mathbf{R}^n \times K$, where D is discrete, n is a nonnegative integer, and K is a compact, connected, locally connected abelian group.*

Proof. Let C be the component of the identity in G. Then G is topologically isomorphic with $(G/C) \times C$. Since G/C is totally disconnected and locally connected, it is discrete. Since C is connected and locally connected, it is topologically isomorphic with $\mathbf{R}^n \times K$, where n is a nonnegative integer and K is compact, connected, and locally connected.

Proof of Theorem II. 3.5. By Theorem 3.4, we may suppose that $G = H \times K$, where H is locally connected and metrizable and K is compact, connected, and locally connected. We then have $X = Y \times F$, where Y and F are the character groups of H and K, respectively. Let U be a nonvoid open subset of G. We may suppose that $U = V \times W$, where V and W are nonvoid open subsets of H and K, respectively. We denote elements of X by (α, β), where $\alpha \in Y$ and $\beta \in F$. Let $\Gamma = \{\beta \in F \mid (\alpha, \beta) \in \Delta\}$.

Case 1. Γ is finite: There is a $\beta_0 \in \Gamma$ such that $\{(\alpha, \beta_0) \in \Delta\}^-$ is not compact in X. Let $\Delta_0 = \{\alpha \in Y \mid (\alpha, \beta_0) \in \Delta\}$. Then Δ_0^- is not compact in Y. Hence, by Theorem 3.3, V contains a K_{0,Δ_0}-set P homeomorphic to the Cantor set. Let $z \in W$. Then $P \times \{z\}$ is a $K_{0,\Delta}$-set in U homeomorphic to the Cantor set.

Case 2. Γ is infinite: Let $x \in V$. Let $\{(\alpha_m, \beta_m)\}_{m=1}^{\infty}$ be a sequence in Δ such that the β_m are distinct and such that $\alpha_m(x) \to s \in \mathbf{T}$ as $m \to \infty$. Let $\Delta_0 = \{\beta_m\}_{m=1}^{\infty}$. Since Δ_0 is infinite and K is compact and connected, W contains a K_{0,Δ_0}-set P homeomorphic to the Cantor set by Theorem I. Then $\{x\} \times P$ is a $K_{0,\Delta}$-set in U homeomorphic to the Cantor set.

4. Proof of Theorem III.

LEMMA 4.1. *Let k be an integer greater than one. Let G be the product of infinitely many copies of $\mathbf{T}_{(k)}$. Let Δ be an infinite subset of X and suppose there is an integer a greater than one such that all elements of Δ have order a and that whenever γ_1 and γ_2 are distinct elements of Δ, then $\gamma_1\gamma_2^{-1}$ has order a. Then for every sequence $\Delta_0 = \{\gamma_1, \gamma_2, \cdots\}$ of distinct elements of Δ, the sequence $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $\mathbf{T}_{(a)}$ for m-almost all $x \in G$.*

Proof. For $r \in \{1, 2, \cdots, a-1\}$ and $n \in \mathbf{N}$, let $f_{nr}(x) = 1/n\, \Sigma_{j=1}^{n} \gamma_j^r(x)$. By our hypothesis on Δ, $\gamma_j \neq \gamma_l$ implies that $(\gamma_j\gamma_l^{-1})^r \neq 1$. Also, since G is compact, $\int_G \gamma(x)dm(x) = 0$ when $\gamma \neq 1$. Hence we have

$$\int_G |f_{nr}|^2 dm = n^{-2}\int_G \Sigma_{j,l=1}^{n} \gamma_j^r(x)\overline{\gamma_l^r(x)}dm(x) = n^{-1}\ .$$

We thus have $\Sigma_{n=1}^{\infty} ||f_{n^2,r}||_2^2 < \infty$ and hence $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for m-almost all $x \in G$. Suppose that $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for all $x \notin A_r$, where $m(A_r) = 0$. The device used in the proof of Lemma 2.2 yields $f_{nr}(x) \to 0$ as $n \to \infty$ for $x \notin A_r$. Let $A = \cup_{r=1}^{a-1} A_r$. Then $m(A) = 0$ and for $x \notin A$ we have for all $r \in \{1, 2, \cdots, a-1\}$ that $f_{nr}(x) \to 0$ as $n \to \infty$. Therefore, by 1.5(a), $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $\mathbf{T}_{(a)}$ for all $x \notin A$.

LEMMA 4.2. *Let k, G, Δ, and a be as in Lemma 4.1. Let $V_1, \cdots, V_n$ be nonempty open subsets of G. Then there are $x_j \in V_j (1 \leqq j \leqq n)$ such that $\{x_1, \cdots, x_n\}$ is a $K_{a,\Delta}$-set.*

Proof. For a positive integer q, $y_1, \cdots, y_q \in \mathbf{T}_{(a)}$, and $w_j \in V_j (1 \leqq j \leqq q)$, let $\Delta(y_1, \cdots, y_q, w_1, \cdots, w_q) = \{\gamma \in \Delta \mid \gamma(w_j) = y_j, 1 \leqq j \leqq q\}$. By Lemma 4.1, there is an $x_1 \in V_1$ such that for all $y_1 \in \mathbf{T}_{(a)}$, $\Delta(y_1, x_1)$ is infinite.

Let $r \in \{1, 2, \cdots, n-1\}$ and suppose that $x_j \in V_j (1 \leqq j \leqq r)$ have been found with the property that for all $y_1, \cdots, y_r \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_r, x_1, \cdots, x_r)$ is infinite. Fixing $(y_1, \cdots, y_r) \in \mathbf{T}_{(a)}^r$ and applying Lemma 4.1 with $\Delta(y_1, \cdots, y_r, x_1, \cdots, x_r)$ in place of Δ, we find that m-almost all $x \in V_{r+1}$ have the property that for all $y_{r+1} \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_{r+1}, x_1, \cdots, x_r, x)$ is infinite. Hence, m-almost all $x \in V_{r+1}$ have the property that for all $y_1, \cdots, y_{r+1} \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_{r+1}, x_1, \cdots, x_r, x)$ is infinite. In particular, an $x_{r+1} \in V_{r+1}$ with this property exists.

Hence, by induction, there are $x_j \in V_j (1 \leqq j \leqq n)$ such that for all $y_1, \cdots, y_n \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_n, x_1, \cdots, x_n)$ is infinite and, in particular, nonvoid. Hence, $\{x_1, \cdots, x_n\}$ is a $K_{a,\Delta}$-set.

THEOREM 4.3. *Let k, G, Δ, and a be as in Lemma 4.1. Let G be metrizable. Let U be a nonvoid open subset of G. Then U contains a $K_{a,\Delta}$-set homeomorphic to the Cantor set.*

Proof. Repeat the proof of [4, (41.5), part III], choosing all characters in Δ and using Lemma 4.2 whenever [4] uses [4, (41.4)].

REMARK 4.4. We now proceed to reduce Theorem III to the case described in Theorem 4.3.

LEMMA 4.5. *Let k be an integer greater than one. Let X be the weak direct product of infinitely many copies of $\mathbf{T}_{(k)}$. Let Δ be an infinite subset of X. Then there exist an integer $a \geqq 2$ and an infinite subset Γ of Δ with the property that whenever γ_1 and γ_2 are distinct elements of Γ, then $\gamma_1\gamma_2^{-1}$ has order exactly a.*

Proof. We remark that this result is trivial if k is prime. (Take $a = k$ and $\Gamma = \Delta$.)

Let $b_0 = k$ and $\Delta_0 = \Delta$. Let $\gamma_1 \in \Delta_0$. Let $\Gamma_1 = \{\gamma_1\alpha^{-1} | \alpha \in \Delta_0\}$. Since Γ_1 is infinite, there is an integer b_1, $2 \leqq b_1 \leqq b_0$, such that Γ_1 contains infinitely many elements of order b_1. Let $\Delta_1 = \{\alpha \in \Delta_0 | \gamma_1\alpha^{-1}$ has order $b_1\}$. Suppose that $n \in \mathbf{N}$ and that $\gamma_1, \cdots, \gamma_n, \Gamma_1, \cdots, \Gamma_n, b_1 \cdots, b_n$ and $\Delta_1, \cdots, \Delta_n$ have been found such that for $1 \leqq j \leqq n$ we have (i) $\gamma_j \in \Delta_{j-1}$, $\Gamma_j = \{\gamma_j\alpha^{-1} | \alpha \in \Delta_{j-1}\}$, Γ_j has infinitely many elements of order b_j, $2 \leqq b_j \leqq b_{j-1}$, and $\Delta_j = \{\alpha \in \Delta_{j-1} | \gamma_j\alpha^{-1}$ has order $b_j\}$. Observe that from (i) it follows that (ii) for $1 \leqq j \leqq n$, we have $\gamma_j \notin \Delta_j$ so Δ_j is a proper infinite subset of Δ_{j-1} and the γ_j are distinct.

Let $\gamma_{n+1} \in \Delta_n$. Let $\Gamma_{n+1} = \{\gamma_{n+1}\alpha^{-1} | \alpha \in \Delta_n\}$. Since Γ_{n+1} is infinite, there is an integer b_{n+1} with $2 \leqq b_{n+1} \leqq b_n$ such that Γ_{n+1} contains infinitely many elements of order b_{n+1}. Let $\Delta_{n+1} = \{\alpha \in \Delta_n | \gamma_{n+1}\alpha^{-1}$ has order $b_{n+1}\}$. Thus, we can define γ_n, Γ_n, Δ_n, and b_n for all $n \in \mathbf{N}$ in such a way that properties (i) hold for all n. Since $\{b_n\}$ is a monotone nonincreasing sequence of integers greater than one, there exist positive integers r and a such that $b_n = a$ for all $n > r$. Let $\Gamma = \{\gamma_{r+n} | n \in \mathbf{N}\}$. We show that Γ and a are as demanded. Let n_1 and $n_2 \in \mathbf{N}$ with $n_1 > n_2$. Then, by construction of the Δ_n, we have $\gamma_{r+n_1} \in$

$\Delta_{r+n_1-1} \subset \Delta_{r+n_2}$ so $\gamma_{r+n_2}\gamma_{r+n_1}^{-1}$ has order $b_{r+n_2} = a$.

LEMMA 4.6. *Let k be an integer greater than one. Let I be an infinite index set and let $X = \prod^*_{\iota \in I} G_\iota$, where each G_ι is a copy of $\mathbf{T}_{(k)}$. Let Δ be an infinite subset of X. Then there exist an integer $a \geqq 2$ and an infinite subset Δ_0 of Δ and a finite (possibly empty) subset I_0 of I such that projection of Δ_0 onto $Y = \prod^*_{\iota \in I \setminus I_0} G_\iota$ gives an infinite subset $\tilde{\Delta}_0$ of Y consisting solely of elements of order a and such that whenever γ_1 and γ_2 are distinct elements of $\tilde{\Delta}_0$, $\gamma_1\gamma_2^{-1}$ has order a.*

Proof. By Lemma 4.5, there exist an integer $a_1 \geqq 2$ and an infinite subset Γ_1 of Δ such that whenever γ_1 and γ_2 are distinct elements of Γ_1, $\gamma_1\gamma_2^{-1}$ has order a_1. Let $\tilde{\Gamma}_1$ be an infinite subset of Γ_1 consisting of elements all of the same order b_1. It is clear that $b_1 \geqq a_1$. (If γ_1 and γ_2 are distinct elements of $\tilde{\Gamma}_1$, then $\gamma_1\gamma_2^{-1}$ has order at most b_1. But $\gamma_1\gamma_2^{-1}$ has order a_1.) If $b_1 = a_1$, we are done. (Take $I_0 = \emptyset$, $\Delta_0 = \tilde{\Gamma}_1$, and $a = a_1$.) Suppose $b_1 > a_1$. Let $\tilde{\gamma}_1 \in \tilde{\Gamma}_1$. There is a finite subset I_1 of I such that the ιth coordinate of $\tilde{\gamma}_1$ is the identity of G_ι for $\iota \notin I_1$. Let $X_1 = \prod^*_{\iota \in I \setminus I_1} G_\iota$. Since I_1 is finite and $\tilde{\Gamma}_1$ is infinite, projection of $\tilde{\Gamma}_1$ onto X_1 (denoted by π_1) gives an infinite subset Δ_1 of X_1 consisting of elements of order at most a_1. (For $\alpha \in \tilde{\Gamma}_1$, order of $\pi_1(\alpha)$ in X_1 = order of $\pi_1(\alpha\tilde{\gamma}_1^{-1})$ in $X_1 \leqq a_1$.) Applying Lemma 4.5 to X_1 and Δ_1 we get an integer a_2 with $2 \leqq a_2 \leqq a_1$ and an infinite subset Γ_2 of Δ_1 such that whenever γ_1 and γ_2 are distinct elements of Γ_2, then $\gamma_1\gamma_2^{-1}$ has order a_2. Let $\tilde{\Gamma}_2$ be an infinite subset of Γ_2 consisting of elements all of the same order b_2. Then we have $a_2 \leqq b_2 \leqq a_1 < b_1$. If $a_2 = b_2$, we are done. (Take $I_0 = I_1$, $a = a_2$, $Y = X_1$, and $\Delta_0 = \{\alpha \in \Delta \mid \pi_1(\alpha) \in \tilde{\Gamma}_2\}$) Suppose $a_2 < b_2 \leqq a_1 < b_1$. Pick $\tilde{\gamma}_2 \in \tilde{\Gamma}_2$; let $I_2 = \{\iota \in I \setminus I_1 \mid \iota\text{th coordinate of } \tilde{\gamma}_2 \text{ is not the identity of } G_\iota\}$; project $\tilde{\Gamma}_2$ onto $X_2 = \prod^*_{\iota \in I \setminus (I_1 \cup I_2)} G_\iota$; $\cdots$ etc. We must eventually have $b_n = a_n$ for some n (otherwise, $\{b_n\}$ would be an infinite strictly decreasing sequence of positive integers). For that n, we have a finite subset $I_0 = I_1 \cup \cdots \cup I_{n-1}$ of I and an infinite subset $\tilde{\Gamma}_n$ of $Y = \prod^*_{\iota \in I \setminus I_0} G_\iota$ such that all elements of $\tilde{\Gamma}_n$ have order $a_n = b_n$ and such that whenever γ_1 and γ_2 are distinct elements of $\tilde{\Gamma}_n$, $\gamma_1\gamma_2^{-1}$ has order a_n. Let $\Delta_0 = \{\alpha \in \Delta \mid \pi(\alpha) \in \tilde{\Gamma}_n\}$, where π is the projection of X onto Y.

THEOREM 4.7. *Let k be an integer greater than one. Let $G = \prod_{\iota \in I} G_\iota$, where each G_ι is a copy of $\mathbf{T}_{(k)}$ and I is infinite. Let Δ be an infinite subset of X. Then there is an integer a greater than one such that every neighborhood of the identity of G contains a $K_{a,\Delta}$-set homeomorphic to the Cantor set.*

Proof. We may suppose that Δ is countable. We identify X with $\prod^*_{\iota \in I} G_\iota$. Let a, I_0, Y, and $\tilde{\Delta}_0$ be as in Lemma 4.6. Let $I_1 = \{\iota \in I \backslash I_0 |$ some $\gamma \in \tilde{\Delta}_0$ has ιth coordinate different from the identity of $G_\iota\}$. Plainly I_1 is countably infinite. Let $I_2 = I \backslash (I_0 \cup I_1)$. Let $G_j = \prod_{\iota \in I_j} G_\iota$, and let G_j have character group X_j, $j = 0, 1, 2$. Since I_1 is countable, G_1 is metrizable. Since I_0 is finite, G_0 is finite. Let Γ_0 be the image of the projection of $\tilde{\Delta}_0$ onto X_1. We may suppose that our neighborhood of the identity of G has the form $U = \{e_0\} \times V_1 \times V_2$, where e_0 is the identity of G_0 and V_j is open in G_j, $j = 1, 2$. Applying Theorem 4.3 to k, G_1, Γ_0, and a, we find a subset P_1 of V_1 homeomorphic to the Cantor set which is a K_{a,Γ_0}-set. Let $P = \{e_0\} \times P_1 \times \{e_2\}$, where e_2 is the identity of G_2. Then P is a $K_{a,\Delta}$-set in U homeomorphic to the Cantor set.

Proof of Theorem III. 4.8. If G is a compact torsion group, then there are integers $r_1, \cdots, r_q$ greater than one and disjoint infinite index sets $I_1, \cdots, I_q$ and there is a finite abelian group F such that G is topologically isomorphic to $F \times G_1 \times \cdots \times G_q$, where $G_j = \prod_{\iota \in I_j} K_\iota$ and each K_ι is a copy of $\mathbf{T}_{(r_j)}$ when $\iota \in I_j$ $(1 \leqq j \leqq q)$. Let G_j have character group X_j $(1 \leqq j \leqq q)$. Then for some j_0, the image Γ of the projection of Δ onto X_{j_0} is infinite. Let a be as in Theorem 4.7 applied to G_{j_0}, X_{j_0}, and Γ. Let U be a neighborhood of the identity of G. We will prove that U contains a $K_{a,\Delta}$-set homeomorphic to the Cantor set. Clearly, this will establish Theorem III. We may suppose that U has the form $\{e_F\} \times U_1 \times \cdots \times U_q$, where e_F is the identity of F and U_j is a neighborhood of the identity e_j of G_j $(1 \leqq j \leqq q)$. By Theorem 4.7, U_{j_0} contains a $K_{a,\Gamma}$-set P_{j_0} homeomorphic to the Cantor set. Let

$$P = \{e_F\} \times \{e_1\} \times \cdots \times \{e_{j_0-1}\} \times P_{j_0} \times \{e_{j_0+1}\} \times \cdots \times \{e_q\} .$$

Then P is a $K_{a,\Delta}$-set in U homeomorphic to the Cantor set.

5. Examples.

5.1. The hypothesis that $\bar{\Delta}$ is not compact is necessary in Theorem II. If $\bar{\Delta}$ is compact, then there is a nonempty open $U \subset G$ which contains no $K_{0,\Delta}$-set and no $K_{a,\Delta}$-set for any integer $a \geqq 2$. Indeed, let $U = \{x \in G: |\gamma(x) - 1| < 1$ for all $\gamma \in \bar{\Delta}\}$. Then U is an open neighborhood of the identity in G and $\operatorname{Re} \gamma(x) > 0$ for all $x \in U$ and all $\gamma \in \Delta$. Hence, the function -1 cannot be matched within 1 on any nonvoid subset of U by any $\gamma \in \Delta$, nor can the function ω_a (where ω_a is an ath root of unity with $\operatorname{Re} \omega_a < 0$) be matched on any nonvoid subset of U by any $\gamma \in \Delta$ for any integer $a \geqq 2$. Hence, no subset of U is a $K_{0,\Delta}$-set or a $K_{a,\Delta}$-set.

5.2. The phrase "a translate of" is a necessary part of the conclusion of Theorem III, as is shown by the following example. Let $G = \mathbf{T}_{(2)} \times H$, where H is the product of infinitely many copies of $\mathbf{T}_{(3)}$. Write $X = \mathbf{Z}_2 \times Y$, where Y is the character group of H. Let $\Delta = \{1\} \times Y$. Let $U = \{-1\} \times H$. Then U is open in G and $\gamma(x) \in -\mathbf{T}_{(3)}$ for all $x \in U$ and all $\gamma \in \Delta$, so the constant function 1 cannot be matched on any subset of U by any $\gamma \in \Delta$. Hence, no subset of U is a $K_{a,\Delta}$-set for any integer $a \geqq 2$.

5.3. The hypothesis that G is a compact torsion group in Theorem III cannot be weakened to the hypothesis that G is compactly generated and contains a compact open torsion subgroup. For example, let H be an infinite compact torsion group and let $G = \mathbf{Z} \times H$. Take $\Delta = \mathbf{T} \times \{e\}$ (where e is the identity of the character group of H) and $U = \{0\} \times H$. Then $\gamma(x) = 1$ for all $x \in U$ and all $\gamma \in \Delta$. Hence, whenever $P \subset G$ is such that a translate of P is contained in U, we have γ constant on P. Therefore, no such totally disconnected P containing more than one point can be a $K_{a,\Delta}$-set for any integer $a \geqq 2$.

5.4. The hypothesis of local connectedness or something closely related to connectedness (cf. Theorem 2.1) in Theorems II and I respectively cannot be weakened to the hypothesis that G is not a torsion group. Indeed, there exist a compact metrizable group G which is not a torsion group and an infinite subset Δ of X such that G contains no $K_{0,\Delta}$-set. For example, let $G = \prod_{j=2}^{\infty} \mathbf{T}_{(2j)}$. Then, writing $X = \prod_{j=2}^{*\infty} \mathbf{Z}_{2j}$ and letting $\Delta = \{\gamma_2, \gamma_3, \cdots\}$ where γ_j has jth coordinate equal to j and the rest zero, we have $\gamma_j(x) = \pm 1$ for all $x \in G$ and all j, so every nonempty subset of G fails to be a $K_{0,\Delta}$-set.

Also, there exist a compact metrizable group G which is not a torsion group and an infinite subset Δ of X such that no subset of G containing more than one point is a $K_{a,\Delta}$-set for any integer $a \geqq 2$. Let $G = \prod_{j=1}^{\infty} \mathbf{T}_{(p_j)}$ where p_j is the jth prime. Write $X = \prod_{j=2}^{*\infty} \mathbf{Z}_{p_j}$ and let $\Delta = \{\gamma_1, \gamma_2, \cdots\}$ where γ_j has jth coordinate equal to 1 and the rest zero. Let P be a subset of G containing at least two points. Let $a \geqq 2$ be an integer. We will show that P is not a $K_{a,\Delta}$-set. Let p_k be a divisor of a. The open-closed sets in G form a basis for the topology of G, so there are two distinct $\mathbf{T}_{(p_k)}$-valued (and, hence, $\mathbf{T}_{(a)}$-valued) continuous functions, f_1 and f_2, on P both different from 1. If either f_i is matched on P by some γ_j, it must be matched by γ_k since no other γ_j attains values in $\mathbf{T}_{(p_k)}$ different from 1. Thus either f_1 or f_2 is a $\mathbf{T}_{(a)}$-valued continuous function not matched on P by any γ_j. Hence, P is not a $K_{a,\Delta}$-set.

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