EXISTENCE OF SPECIAL $K$-SETS IN CERTAIN LOCALLY COMPACT ABELIAN GROUPS

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In all that follows, $G$ is an infinite, nondiscrete, locally compact $T_0$ abelian group with character group $X$ and $\Delta$ is a nonempty subset of $X$. In a standard proof of the existence of infinite (in fact, perfect) Helson sets (see for example Hewitt and Ross) it is shown that each nonvoid open subset of an arbitrary $G$ contains a $K$-set (terminology of Hewitt and Ross) homeomorphic to Cantor’s ternary set (or, in the terminology of Rudin, a Kronecker set or a set of type $K_a$ homeomorphic to the Cantor set). In this paper, it is shown that $K_{0,r}$-sets or $K_{a,r}$-sets homeomorphic to the Cantor set exist in profusion in a large class of infinite nondiscrete locally compact $T_0$ abelian groups $G$, provided that $\Delta$ is not compact. (A nonvoid subset $E$ of $G$ is called a $K_0,r$-set if for every continuous function from $E$ to $T$, the circle group, and every $\varepsilon > 0$, there is a $\gamma \in \Delta$ such that $|\gamma(x) - f(x)| < \varepsilon$ for all $x \in E$. Let $a$ be an integer greater than one. A nonvoid subset $E$ of $G$ is called a $K_{a,r}$-set if it is totally disconnected and every continuous function on $E$ with values in the set of $a$th roots of unity is the restriction to $E$ of some $\gamma \in \Delta$.)

The following theorems will be proved.

**Theorem I.** Let $G$ be compact. Let $\Delta$ be infinite. Suppose that, except for the character which is identically 1, $\Delta\Delta^{-1}$ consists solely of elements of infinite order. (This condition is satisfied automatically if $G$ is connected, for then $X$ is torsion-free.) Then every nonvoid open set in $G$ contains a $K_0,r$-set homeomorphic to the Cantor set.

**Theorem II.** Let $G$ be locally connected. Suppose that $\Delta$ is not compact. Then every nonvoid open set in $G$ contains a $K_0,r$-set homeomorphic to the Cantor set.

**Theorem III.** Let $G$ be a compact torsion group. Let $\Delta$ be infinite. Then there is an integer $a \geq 2$ such that every nonvoid open set in $G$ contains a translate of a $K_{a,r}$-set homeomorphic to the Cantor set.

1. Preliminaries.

**Notation 1.1.** We denote Haar measure on $G$ by $m$, with $m(G) = 1$ when $G$ is compact. When $H$ is a subgroup of $G$, we write
\( \Delta |_{\mathcal{H}} \) for \( \{\gamma |_{\mathcal{H}} : \gamma \in \Delta\} \). \( M(P) \) denotes the set of all (finite) regular Borel measures on the compact subset \( P \) of \( G \).

\( C(A, B) \) denotes the set of all continuous functions from \( A \) to \( B \), where \( A \) and \( B \) are topological spaces. If \( B = \mathbb{C} \), the set of complex numbers, we write \( C(A) \) instead of \( C(A, \mathbb{C}) \).

\( \mathbb{Z} \) is the group of integers. \( \mathbb{R} \) is the group of real numbers. \( \mathbb{Q} \) is the (discrete) group of rational numbers. \( \mathbb{N} \) is the set of positive integers. When \( a \) is an integer greater than one, \( \mathbb{Z}_a \) is the additive group of integers modulo \( a \) and \( \mathbb{T}_{(a)} \) is the multiplicative group of \( a \)th roots of unity.

\( \mathbb{I} \) is the identity element of \( X \).

\( \prod_{i=1}^{n} G_i \) is the weak direct product of the groups \( G_i \).

**Remarks 1.2.**

(a) In § 5, we give examples which show some of the limitations of Theorems I, II, and III.

(b) The hypothesis on \( \Delta \mathcal{A}^{-1} \) in Theorem I is related to connectedness, as will be shown in Theorem 2.1.

(c) When \( G \) is compact, a \( K_{0,1} \)-set (or \( K_{a,r} \)-set) \( E \) is a \( \Delta \)-Helson set—i.e., a set with the property that every \( f \in C(E) \) has the form \( f = \hat{g} |_E \) for some \( g \in L_1(X) \) which vanishes off \( \Delta \). When \( G \) is not compact, a \( K_{0,1} \)-set need not be a \( \Delta \)-Helson set as the example \( G = X = \mathbb{R} \) and \( \Delta = \mathbb{Q} \) shows.

(d) Our proof of Theorem II for the case where \( G \) is metrizable uses a technique due to Kaufman, [6, p. 184–185 and 7]. The general case follows from the case where \( G \) is metrizable and from Theorem I. Our proofs of Theorems I and III depend on the notion of an equidistributed sequence in a compact group. This notion for the case \( G = \mathbb{T} \) is due to Weyl [9]. The notion has been generalized by Eckmann [2] and Hlawka [5]. Eckmann’s work offers more than enough generality for our purposes; relevant parts are given below in 1.3 and 1.4.

**Definition 1.3.** Let \( H \) be a compact abelian group with Haar measure \( \mu \) and \( \mu(H) = 1 \). Let \( \{\alpha_j\}_{j=1}^{\infty} \) be a sequence in \( H \). For \( F \subset H \), let \( n(F) \) be the number of \( \alpha_j \) with index \( j \leq n \) which are in \( F \). The sequence \( \{\alpha_j\}_{j=1}^{\infty} \) is said to be equidistributed in \( H \) if \( \lim_{n \to \infty} n(F)/n = \mu(F) \) for all closed \( F \) with the property that \( \mu(\text{boundary } F) = 0 \).

**Theorem 1.4.** Let \( H \) be a compact abelian group with Haar measure \( \mu \) and \( \mu(H) = 1 \). Let \( \{\alpha_j\}_{j=1}^{\infty} \) be a sequence in \( H \). The following are equivalent:

(i) \( \{\alpha_j\}_{j=1}^{\infty} \) is equidistributed in \( H \);
(ii) for every continuous character \( \gamma \) of \( H \) such that \( \gamma \equiv 1 \), we have \( \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} \gamma(\alpha_j) = 0 \).

**REMARKS 1.5.**

(a) In the proofs of Theorems I and III we will use the equivalence of (i) and (ii) in Theorem 1.4 for the cases \( H = T \) and \( H = T_{(a)} \), respectively. If \( H = T \) we have Weyl’s original result: The sequence \( \{a_d\}_{d=1}^{\infty} \subset T \) is equidistributed in \( T \) if and only if \( \lim_{n \to \infty} n^{-1} \sum_{d=1}^{n} r_d = 0 \) for all nonzero integers \( r \) (or, equivalently, for all \( r \in \mathbb{N} \)) [9]. If \( H = T_{(a)} \), we have: The sequence \( \{a_d\}_{d=1}^{\infty} \subset T_{(a)} \) is equidistributed in \( T_{(a)} \) if and only if for every integer \( r \in \{1, 2, \ldots, a-1\} \) we have \( \lim_{n \to \infty} n^{-1} \sum_{d=1}^{n} r_d = 0 \).

(b) Eckmann’s definition differs from Definition 1.3 in that he omits the restriction \( \mu(\text{boundary } F) = 0 \). This restriction is necessary, as has been pointed out [3].

2. Proof of Theorem I.

2.1. We first investigate the hypothesis on \( \Delta^{\sim} \) in the statement of Theorem I and find that it is related to connectedness.

**THEOREM.** Let \( G \) be compact. Let \( \Delta \) be a countably infinite subset of \( X \). The following are equivalent:

(i) \( \Delta^{\sim} \setminus \{1\} \) consists solely of elements of infinite order;

(ii) \( G \) contains a compact connected metrizable subgroup \( H \) with the property that \( \delta \to \delta |_H \) is a one-to-one map from \( \Delta \) to the character group of \( H \).

**Proof.** (ii) implies (i): Let \( \delta_1 \) and \( \delta_2 \) be distinct elements of \( \Delta \). Then \( \delta_1 |_H \neq \delta_2 |_H \), so \( \delta_1 \delta_2^{-1} |_H \neq 1 \). Since \( H \) is connected, its character group is torsion-free. Hence, \( \delta_1 \delta_2^{-1} |_H \) has infinite order and therefore so does \( \delta_1 \delta_2^{-1} \).

(i) implies (ii): Let \( \Gamma' \) be a maximal torsion-free independent subset of \( \Delta \). (Clearly, \( \Delta \) contains at most one element of finite order, so \( \Gamma' \) is nonvoid.) We have \( \Gamma' = \{\gamma_1, \ldots, \gamma_p\} \) for some positive integer \( p \) or \( \Gamma' = \{\gamma_1, \gamma_2, \ldots\} \). If \( \Gamma' \) is finite, let \( P = \mathbb{Q}^p \). If not, let \( P \) be the weak direct product of countably many copies of \( \mathbb{Q} \). (In either case, \( P \) is countable.) For \( n \in \mathbb{N} \) (and \( n \leq p \) if \( \Gamma' \) is finite) let \( e_n \) be that element of \( P \) with \( n \)th coordinate equal to 1 and all other coordinates equal to zero. Let \( Y \) be the subgroup of \( X \) generated by \( \Gamma' \). Since \( \Gamma' \) is independent, the map \( \gamma_n \to e_n \) extends to a (one-to-one) homomorphism from \( Y \) to \( P \). Since \( P \) is divisible, this homomorphism extends to a homomorphism \( \phi: X \to P \). Hence \( W = X/\ker \phi \) is isomorphic to a subgroup of \( P \). Let \( H \) be the annihilator of \( \ker \phi \) in \( G \). Then \( H \)
is a closed subgroup of $G$ and has character group $W$, which is torsion-free and countable. Hence, $H$ is connected and metrizable. Now $\delta_1|_H = \delta_2|_H$ if and only if $\delta_1 \delta_2^{-1} \in \ker \phi$. Let $\delta_1$ and $\delta_2$ be distinct elements of $\Delta$. It is sufficient to show that $\delta_1 \delta_2^{-1} \in \ker \phi$. Since $\Gamma'$ is a maximal torsion-free independent subset of $\Delta$, there exist nonzero integers $r_1$ and $r_2$ such that $\delta_1^{r_1}$ and $\delta_2^{r_2}$ are in $Y$. Therefore there is a nonzero integer $r$ such that $(\delta_1 \delta_2^{-1})^r \in Y$. By the hypothesis on $\Delta^{-1}$, we have $(\delta_1 \delta_2^{-1})^r \neq 1$. Since $\phi$ is one-to-one on $Y$, $r \phi(\delta_1 \delta_2^{-1}) = \phi((\delta_1 \delta_2^{-1})^r)$ is not the identity of $P$. Hence $\delta_1 \delta_2^{-1} \in \ker \phi$ and the proof is complete.

**Lemma 2.2.** Let $G$ be compact. Let $\Delta = \{\gamma_1, \gamma_2, \cdots\}$ be a countably infinite set of distinct elements of $X$ arranged in any fixed order. Suppose that $\Delta^{-1}\{1\}$ consists solely of elements of infinite order. Then for $m$-almost all $x \in G$, the sequence $\{\gamma_j(x)\}_{j=1}^\infty$ is equidistributed in $T$.

**Proof.** Our proof follows Weyl [9]. For $x \in G$, $n \in \mathbb{N}$, and $r \in \mathbb{N}$, define $f_{nr}(x) = n^{-1} \sum_{j=1}^n \gamma_j(x)$. From our hypothesis on $\Delta^{-1}$ we find that $\gamma_j^r = 1$ implies that $\gamma_j = \gamma_k$. Since $G$ is compact, $\int_G \gamma(x) dm(x) = 0$ when $\gamma \neq 1$. Thus, we have

$$\int_G |f_{nr}|^2 dm = n^{-2} \int_G \sum_{j,k=1}^n \gamma_j^r \gamma_k^{\bar{r}} dm(x) = n^{-1}.$$

Therefore we have $\sum_{n=1}^{\infty} ||f_{nr}||_2 < \infty$ and hence $f_{nr}(x) \to 0$ as $n \to \infty$ for $m$-almost all $x \in G$. Suppose that $f_{nr}(x) \to 0$ as $n \to \infty$ for all $x \in A_r$ where $m(A_r) = 0$.

For $n \in \mathbb{N}$, let $\lambda(n)$ be the positive integer such that $\lambda^2 \leq n < (\lambda + 1)^2$. Then we have $|nf_{nr}(x) - \lambda^2 f_{\lambda^2 r}(x)| \leq 2\lambda$ and hence

$$|f_{nr}(x) - \frac{\lambda^2}{n} f_{\lambda^2 r}(x)| \leq 2/\sqrt{n}.$$

Let $\varepsilon > 0$. Fix $x \in A_r$. Then there is a positive integer $M$ such that $|f_{\lambda^2 r}(x)| < \varepsilon/2$ whenever $\lambda \geq M$. Let $n \geq M^2$ and $n > 16/\varepsilon^2$. Let $\lambda$ be such that $\lambda^2 \leq n < (\lambda + 1)^2$. Then $\lambda^2/n \leq 1, 2/\sqrt{n} < \varepsilon/2$, and $\lambda^2 \geq M^2$, so we have

$$|f_{nr}(x)| \leq |f_{nr}(x) - \frac{\lambda^2}{n} f_{\lambda^2 r}(x)| + \frac{\lambda^2}{n} |f_{\lambda^2 r}(x)| < 2/\sqrt{n} + \varepsilon/2 < \varepsilon.$$

Hence, $f_{nr}(x) \to 0$ as $n \to \infty$ for all $x \in A_r$.

Let $A = \bigcup A_r$. Then $m(A) = 0$ and for $x \in A$ we have for all $r \in \mathbb{N}$ that $f_{nr}(x) \to 0$ as $n \to \infty$. Therefore, by 1.5(a), $\{\gamma_j(x)\}_{j=1}^\infty$ is equidistributed in $T$ for all $x \in A$. 

LEMMA 2.3. Let G and A be as in Theorem 1. Let $V_1, \ldots, V_k$ be nonvoid open subsets of G. Then there exist $x_j \in V_j (1 \leq j \leq k)$ with the property that for every $\varepsilon > 0$ and for all $z_1, \ldots, z_k \in T$ there is a $\gamma \in A$ such that $|\gamma(x_j) - z_j| < \varepsilon (1 \leq j \leq k)$, i.e., there exist $x_j \in V_j (1 \leq j \leq k)$ such that $\{x_1, \ldots, x_k\}$ is a $K_{\alpha, \varepsilon}$-set.

Proof. We may suppose that $A$ is countable. Let $q \in \{1, 2, \ldots, k\}$. Let “$P(q)$ holds for $x_1, \ldots, x_q$” mean “$x_j \in V_j (1 \leq j \leq q)$ and $\{x_1, \ldots, x_q\}$ is a $K_{\alpha, \varepsilon}$ set.” By Lemma 2.2, there is an $x_1 \in V_1$ such that $P(1)$ holds for $x_1$. Suppose that $1 \leq r \leq k - 1$ and that $P(r)$ holds for $x_1, \ldots, x_r$. It is sufficient to show there is an $x_{r+1} \in V_{r+1}$ such that $P(r + 1)$ holds for $x_1, \ldots, x_{r+1}$. Let $A = \{w \in V_{r+1} | P(r + 1) \text{ does not hold for } x_1, \ldots, x_r, w\}$. It is sufficient to show that $m(A) = 0$. Let $S$ be a countable dense subset of $T$. Then $w \in A$ if and only if $w \in V_{r+1}$ and there exist $p \in \mathbb{N}$ and $s_1, s_{r+1} \in S$ such that for all $\gamma \in J$ either $|\gamma(x_j) - s_j| < p^{-1}$ for some $j (1 \leq j \leq r)$ or $|\gamma(w) - s_{r+1}| < p^{-1}$, i.e., we have

$$A = \bigcup_{p \in \mathbb{N}} \bigcup_{s_1 \in S} \cdots \bigcup_{s_{r+1} \in S} A(p, s_1, \ldots, s_{r+1})$$

where $A(p, s_1, \ldots, s_{r+1}) = \bigcap_{\gamma \in J} \{y \in V_{r+1} : |\gamma(y) - s_{r+1}| \geq p^{-1} \text{ or at least one } |\gamma(x_j) - s_j| \geq p^{-1}\}$.

Let

$$\tilde{A}(p, s_1, \ldots, s_r) = \{\gamma \in J : |\gamma(x_j) - s_j| < p^{-1}, 1 \leq j \leq r\}.$$ 

Then we have

$$A(p, s_1, \ldots, s_{r+1}) = \{y \in V_{r+1} : |\gamma(y) - s_{r+1}| \geq p^{-1} \text{ for all } \gamma \in \tilde{A}(p, s_1, \ldots, s_r)\}.$$ 

Hence, it is sufficient to show that each $\tilde{A}(p, s_1, \ldots, s_r)$ is infinite (for then, by Lemma 2.2, each $A(p, s_1, \ldots, s_{r+1})$ is $m$-null and therefore so is $A$).

We assume that for some $p \in \mathbb{N}$ and $s_1, \ldots, s_r \in S$ the set $\tilde{A} = \tilde{A}(p, s_1, \ldots, s_r)$ is finite and use this to obtain a contradiction. A basic neighborhood of the point $z = (z_1, \ldots, z_r) \in T^r$ has the form $B(z, \varepsilon) = \{w = (w_1, \ldots, w_r) : |z_j - w_j| < \varepsilon, 1 \leq j \leq r\}$ for some $\varepsilon > 0$. Let $s = (s_1, \ldots, s_r)$ and $x = (x_1, \ldots, x_r)$. For $\gamma \in A$, let $\gamma(x) = (\gamma(x_1), \ldots, \gamma(x_r))$. If $\tilde{A}$ is finite, then $\{\gamma \in A : |\gamma(x) \in B(s, p^{-1})|\}$ is finite. Then there exist $z \in B(s, p^{-1})$ and $\varepsilon > 0$ be such that $B(z, \varepsilon) \subset B(s, p^{-1})$ and $B(z, \varepsilon)$ is disjoint from $\{\gamma(x) : \gamma \in A\}$. This contradicts the induction hypothesis that $P(r)$ holds for $x_1, \ldots, x_r$.

THEOREM 2.4. Theorem I holds when $G$ is metrizable.

Proof. Repeat the proof of [4, (41.5), part I] choosing all charac-
TERS in $A$ and using Lemma 2.3 whenever [4] uses [4, (41.3)].

**Theorem 2.5.** Let $G$ and $A$ be as in Theorem I. Let $U$ be a neighborhood of the identity in $G$. Then $U$ contains a $K_{0,*}$-set homeomorphic to the Cantor set.

**Proof.** By Theorem 2.1, $G$ contains a compact connected metrizable subgroup $H$ with the property that $\Gamma = \Delta|_H$ is infinite. Let $V = U \cap H$. Since $H$ is connected, its character group is torsion-free. Hence, by Theorem 2.4, $V$ contains a $K_{0,*}$-set $P$ homeomorphic to the Cantor set. Clearly, $P$ is a $K_{0,*}$-set contained in $U$.

**Theorem 2.6.** Let $P$ be a compact metrizable $K_{0,*}$-set in $G$, where $G$ is compact and $\Delta \Delta^{-1}\{1\}$ consists solely of elements of infinite order. Then for almost all $x \in G$, $xP$ is a $K_{0,*}$-set.

**Proof.** Let $\{f_1, f_2, \ldots \}$ be a (uniformly) dense subset of $C(P, T)$. For each $j$, there is a sequence $\{r_{ij}\}_{i=1}^\infty$ of elements of $A$ such that $r_{ij} \to f_j$ uniformly on $P$. By Lemma 2.2, there is an $m$-null set $A_j$ such that $\{r_{ij}(x)\}_{i=1}^\infty$ is equidistributed in $T$ whenever $x \in G \setminus A_j$. Let $A = \bigcup A_j$. Then $A$ is $m$-null. Let $x \in G \setminus A$. For each $j$, let $g_j(xy) = f_j(y)$. To show that $xP$ is a $K_{0,*}$-set, it is sufficient to show that each $g_j$ is uniformly approximable by $\{r_{ij}; i, j = 1, 2, \ldots \}$. Let $\varepsilon > 0$. Fix $j$. Then for some $i_0$, we have $|r_{ij}(y) - f_j(y)| < \varepsilon/2$ for all $y \in P$ whenever $i > i_0$ and, since $\{r_{ij}(x)\}_{i=1}^\infty$ is equidistributed in $T$, there is an $i > i_0$ such that $|r_{ij}(x) - 1| < \varepsilon/2$. For this $i$ we have $|r_{ij}(xy) - g_j(xy)| < \varepsilon$ for all $y \in P$.

**Proof of Theorem I.** 2.7. Immediate from Theorems 2.5 and Theorem 2.6.

3. **Proof of Theorem II.**

**Theorem 3.1.** Let $G$ be locally connected. Let $A$ be such that $\Delta$ is not compact. Let $U$ be a neighborhood of the identity in $G$. Then there is a $\gamma$ in $\Delta$ such that $\gamma(U) = T$.

**Proof.** The topology on $X$ is the restriction of the compact-open topology on $C(G)$ to the (closed) subspace $X$ of $C(G)$. Hence, $\tilde{\Delta}$ is

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In the original version of this paper, the conclusion of Theorem I was as follows: Every open set in $G$ containing an element of finite order contains a $K_{0,*}$-set homeomorphic to the Cantor set and, if $G$ is metrizable, every nonvoid open set in $G$ contains a $K_{0,*}$-set homeomorphic to the Cantor set. Theorem 2.6 and the stronger version of Theorem I which it yields are due to Robert Kaufman [private communication, December, 1971].
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compact as a subspace of \( X \) if and only if it is compact as a subspace of \( C(G) \) with the compact-open topology. Since by hypothesis \( \Delta \) is not compact, it follows from Ascoli’s Theorem that \( \Delta \) is not equicontinuous \([1, p. 267]\) and, hence, that \( \Delta \) is not equicontinuous at the identity of \( G \). Therefore, there exists \( \varepsilon > 0 \) such that for every neighborhood \( W \) of the identity in \( G \), there is an \( x \in W \) and a \( \gamma \in \Delta \) such that \( |\gamma(x) - 1| \geq \varepsilon \). Let \( S = \{ \varepsilon^t | 0 \leq t \leq \varepsilon/2 \} \). Let \( M \) be a positive integer with the property that \( S^M = T \). Let \( V \) be a connected neighborhood of the identity in \( G \) such that \( V^M \subset U \). Then there exist \( x \in V \) and \( \gamma \in \Delta \) such that \( |\gamma(x) - 1| \geq \varepsilon \). Hence, \( \gamma(V) \) contains an arc of length at least \( \varepsilon \). Therefore we have \( T = \gamma(V)^M \subset \gamma(U) \subset T \).

**Theorem 3.2.** Let \( G \) be locally connected and metrizable. Let \( \Delta \) be such that \( \Delta \) is not compact. Let \( E \) be a compact totally disconnected subset of \( \mathbb{R} \) or \( \mathbb{T} \). Then there is a first category set \( H \subset C(E, G) \) such that each \( f \in C(E, G) \backslash H \) maps \( E \) homeomorphically onto a \( K_{0,\varepsilon} \)-set in \( G \).

**Proof.** Our proof follows the ideas of Kaufman \([7]\) as given by Katznelson \([6, p. 184-185]\).

For \( h \in C(E, T), f \in C(E, G), \) and \( \varepsilon > 0 \), let “\((*)\) holds for \( h, f \) and \( \varepsilon \)” mean “there is a \( \gamma \in \Delta \) such that \( |\gamma(f(y)) - h(y)| < \varepsilon \) for all \( y \in E \)”.

Let \( f \in C(E, G) \). Clearly, \( f \) is a homeomorphism of \( E \) onto \( f(E) \) if and only if \( f \) is one-to-one. Also, if \( f \) is not one-to-one, it is clear that there exist \( h \in C(E, T) \) and \( \varepsilon > 0 \) such that \((*)\) fails for \( h, f, \) and \( \varepsilon \). Hence, \( f \) is a homeomorphism of \( E \) onto \( f(E) \) and \( f(E) \) is a \( K_{0,\varepsilon} \)-set if and only if for every \( h \in C(E, T) \) and every \( \varepsilon > 0 \), \((*)\) holds for \( h, f, \) and \( \varepsilon \).

Let \( d \) be an invariant metric on \( G \) compatible with the topology of \( G \). For \( f \) and \( g \) in \( C(E, G) \), let \( D(f, g) = \sup \{ d(f(y), g(y)) | y \in E \} \). Observe that \( D(f, g) < \infty \) since \( E \) is compact.

Let \( h \in C(E, T), g \in C(E, G), \) and \( \varepsilon > 0 \), and \( \eta > 0 \). We now show that there exist an \( f \in C(E, G) \) such that \((*)\) holds for \( h, f, \) and \( \varepsilon \) and \( D(f, g) < \eta \). Let \( U \) be the open \( \eta \)-ball about the identity \( e \) of \( G \). By Theorem 3.1, there is a \( \gamma \in \Delta \) such that \( \gamma(U) = T \). Write \( E = \bigcup_{j=1}^{n} E_j \), where the \( E_j \) are disjoint nonvoid open-closed subsets of \( E \) and \( \gamma \circ g \) and \( h \) both vary by less than \( \varepsilon/3 \) on each \( E_j \). (The \( E_j \) exist since \( E \) is totally disconnected.) Let \( y_j \in E_j \) and suppose that \( \gamma(g(y_j)) = \alpha_j \) and \( h(y_j) = \beta_j, 1 \leq j \leq n \). Let \( x_j \in U \) be such that \( \gamma(x_j) = \overline{\alpha_j} \beta_j \). Define \( f \in C(E, G) \) by \( f(y) = x_j g(y) \) when \( y \in E_j \). We see that \( D(f, g) = \max \{ d(x_j, e) \} < \eta \) and for \( y \in E_j \) we have

\[
\begin{align*}
|\gamma(f(y)) - h(y)| & \leq |\gamma(g(y))\gamma(x_j) - \gamma(g(y_j))\gamma(x_j)| \\
+ |\gamma(g(y_j))\gamma(x_j) - h(y_j)| + |h(y_j) - h(y)| < \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} < \varepsilon.
\end{align*}
\]


Hence, (*) holds for $h$, $f$, and $\varepsilon$.

For $h \in C(E, T)$ and $\varepsilon > 0$, let $H(h, \varepsilon) = \{f \in C(E, G) \mid (*) \text{ fails for } h, f, \text{ and } \varepsilon\}$. It is easy to show that $H(h, \varepsilon)$ is closed. By the preceding paragraph, $H(h, \varepsilon)$ is nowhere dense in $C(E, G)$. Let $\{h_n\}_{n=1}^{\infty}$ be dense in $C(E, T)$. Let $H = \bigcup_{k=1}^{\infty} H(h_n, 1/k)$. Then $H$ is a first category set in the complete metric space $C(E, G)$. Also, we have $f \in C(E, G) \setminus H$ if and only if every $h \in C(E, T)$ can be uniformly approximated by $\gamma \circ f$'s ($\gamma \in \mathcal{A}$), which by the second paragraph of the proof is true if and only if $f$ is a homeomorphism and $f(E)$ is a $K_{0,r}$-set.

**Theorem 3.3.** Theorem II holds when $G$ is metrizable.

**Proof.** Let $U$ be a nonvoid open subset of $G$. Let $E$ be the Cantor set. Let $H$ be as in Theorem 3.2. The result follows from Theorem 3.2 since $C(E, U)$ is open in $C(E, G)$ and $C(E, G) \setminus H$ is dense in $C(E, G)$.

**Theorem 3.4.** Let $G$ be locally connected. Then $G$ is topologically isomorphic with $D \times \mathbb{R}^n \times K$, where $D$ is discrete, $n$ is a nonnegative integer, and $K$ is a compact, connected, locally connected abelian group.

**Proof.** Let $C$ be the component of the identity in $G$. Then $G$ is topologically isomorphic with $(G/C) \times C$. Since $G/C$ is totally disconnected and locally connected, it is discrete. Since $C$ is connected and locally connected, it is topologically isomorphic with $\mathbb{R}^n \times K$, where $n$ is a nonnegative integer and $K$ is compact, connected, and locally connected.

**Proof of Theorem II.** 3.5. By Theorem 3.4, we may suppose that $G = H \times K$, where $H$ is locally connected and metrizable and $K$ is compact, connected, and locally connected. We then have $X = Y \times F$, where $Y$ and $F$ are the character groups of $H$ and $K$, respectively. Let $U$ be a nonvoid open subset of $G$. We may suppose that $U = V \times W$, where $V$ and $W$ are nonvoid open subsets of $H$ and $K$, respectively. We denote elements of $X$ by $(\alpha, \beta)$, where $\alpha \in Y$ and $\beta \in F$. Let $\Gamma = \{\beta \in F \mid (\alpha, \beta) \in \mathcal{A}\}$.

**Case 1.** $\Gamma$ is finite: There is a $\beta_0 \in \Gamma$ such that $\{(\alpha, \beta_0) \in \mathcal{A}\}$ is not compact in $X$. Let $\mathcal{A}_0 = \{\alpha \in Y \mid (\alpha, \beta_0) \in \mathcal{A}\}$. Then $\mathcal{A}_0$ is not compact in $Y$. Hence, by Theorem 3.3, $V$ contains a $K_{0,r}$-set $P$ homeomorphic to the Cantor set. Let $z \in W$. Then $P \times \{z\}$ is a $K_{0,r}$-set in $U$ homeomorphic to the Cantor set.
Case 2. $\Gamma$ is infinite: Let $x \in V$. Let $\{(\alpha_m, \beta_m)\}_{m=1}^{\infty}$ be a sequence in $\Delta$ such that the $\beta_m$ are distinct and such that $\alpha_m(x) \to s \in T$ as $m \to \infty$. Let $\Delta_0 = \{\beta_m\}_{m=1}^{\infty}$. Since $\Delta_0$ is infinite and $K$ is compact and connected, $W$ contains a $K_{0,\Delta_0}$-set $P$ homeomorphic to the Cantor set by Theorem I. Then $\{x\} \times P$ is a $K_{0,\Delta}$-set in $U$ homeomorphic to the Cantor set.

4. Proof of Theorem III.

**Lemma 4.1.** Let $k$ be an integer greater than one. Let $G$ be the product of infinitely many copies of $T_k$. Let $\Delta$ be an infinite subset of $X$ and suppose there is an integer $a$ greater than one such that all elements of $\Delta$ have order $a$ and that whenever $\gamma_1$ and $\gamma_2$ are distinct elements of $\Delta$, then $\gamma_1 \gamma_2^{-1}$ has order $a$. Then for every sequence $\Delta_0 = \{\gamma_1, \gamma_2, \cdots\}$ of distinct elements of $\Delta$, the sequence $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $T_k$ for $m$-almost all $x \in G$.

**Proof.** For $r \in \{1, 2, \cdots, a-1\}$ and $n \in \mathbb{N}$, let $f_{n,r}(x) = 1/n \sum_{j=1}^{n} \gamma_j^r(x)$. By our hypothesis on $\Delta$, $\gamma_j \neq \gamma_i$ implies that $(\gamma_j \gamma_i^{-1})^r \neq 1$. Also, since $G$ is compact, $\int_G \gamma(x)dm(x) = 0$ when $\gamma \neq 1$. Hence we have

$$\int_G ||f_{n,r}||^2dm = n^{-2} \int_G \sum_{j=1}^{n} \gamma_j^r(x)\overline{\gamma_j^r(x)}dm(x) = n^{-1}.$$  

We thus have $\Sigma_{n=1}^{\infty} ||f_{n,r}||^2 < \infty$ and hence $f_{n,r}(x) \to 0$ as $n \to \infty$ for $m$-almost all $x \in G$. Suppose that $f_{n,r}(x) \to 0$ as $n \to \infty$ for all $x \in A_r$, where $m(A_r) = 0$. The device used in the proof of Lemma 2.2 yields $f_{n,r}(x) \to 0$ as $n \to \infty$ for $x \in A_r$. Let $A = \bigcup_{r=1}^{a-1} A_r$. Then $m(A) = 0$ and for $x \in A$ we have for all $r \in \{1, 2, \cdots, a-1\}$ that $f_{n,r}(x) \to 0$ as $n \to \infty$. Therefore, by 1.5(a), $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $T_k$ for all $x \in A$.

**Lemma 4.2.** Let $k, G, \Delta,$ and $a$ be as in Lemma 4.1. Let $V_1, \cdots, V_n$ be nonempty open subsets of $G$. Then there are $x_j \in V_j (1 \leq j \leq n)$ such that $\{x_1, \cdots, x_n\}$ is a $K_{a,r}$-set.

**Proof.** For a positive integer $q$, $y_1, \cdots, y_q \in T_{(a)}$, and $w_j \in V_j (1 \leq j \leq q)$, let $\Delta(y_1, \cdots, y_q, w_1, \cdots, w_q) = \{\gamma \in \Delta | \gamma(w_i) = y_j, 1 \leq j \leq q\}$. By Lemma 4.1, there is an $x_1 \in V_1$ such that for all $y_1 \in T_{(a)}$, $\Delta(y_1, x_1)$ is infinite.

Let $r \in \{1, 2, \cdots, n-1\}$ and suppose that $x_j \in V_j (1 \leq j \leq r)$ have been found with the property that for all $y_1, \cdots, y_r \in T_{(a)}$, $\Delta(y_1, \cdots, y_r, x_1, \cdots, x_r)$ is infinite. Fixing $(y_1, \cdots, y_r) \in T_{(a)}^r$ and applying Lemma 4.1 with $\Delta(y_1, \cdots, y_r, x_1, \cdots, x_r)$ in place of $\Delta$, we find that $m$-almost
all \( x \in V_{r+1} \) have the property that for all \( y_{r+1} \in T_{\{a\}}, \Delta(y_1, \ldots, y_{r+1}, x_1, \ldots, x_r, x) \) is infinite. Hence, \( m \)-almost all \( x \in V_{r+1} \) have the property that for all \( y_1, \ldots, y_{r+1} \in T_{\{a\}}, \Delta(y_1, \ldots, y_{r+1}, x_1, \ldots, x_r, x) \) is infinite. In particular, an \( x_{r+1} \in V_{r+1} \) with this property exists.

Hence, by induction, there are \( x \in V \) such that for all \( y_1, \ldots, y_n \in T_{\{\alpha\}}, \Delta(y_1, \ldots, y_n, x_1, \ldots, x_n) \) is infinite and, in particular, nonvoid. Hence, \( \{x_1, \ldots, x_n\} \) is a \( K_{a,r} \)-set.

**Theorem 4.3.** Let \( k, G, \Delta, \) and \( a \) be as in Lemma 4.1. Let \( G \) be metrizable. Let \( U \) be a nonvoid open subset of \( G \). Then \( U \) contains a \( K_{a,r} \)-set homeomorphic to the Cantor set.

**Proof.** Repeat the proof of [4, (41.5), part III], choosing all characters in \( \Delta \) and using Lemma 4.2 whenever [4] uses [4, (41.4)].

**Remark 4.4.** We now proceed to reduce Theorem III to the case described in Theorem 4.3.

**Lemma 4.5.** Let \( k \) be an integer greater than one. Let \( X \) be the weak direct product of infinitely many copies of \( T_{\{k\}} \). Let \( A \) be an infinite subset of \( X \). Then there exist an integer \( a \geq 2 \) and an infinite subset \( \Gamma \) of \( A \) with the property that whenever \( \gamma_1 \) and \( \gamma_2 \) are distinct elements of \( \Gamma \), then \( \gamma_1^{-1} \gamma_2 \) has order exactly \( a \).

**Proof.** We remark that this result is trivial if \( k \) is prime. (Take \( a = k \) and \( \Gamma = \Delta \).

Let \( b_0 = k \) and \( \Delta_0 = \Delta \). Let \( \gamma_1 \in \Delta_0 \). Let \( \Gamma_1 = \{\gamma_1 \alpha^{-1} | \alpha \in \Delta_0\} \). Since \( \Gamma_1 \) is infinite, there is an integer \( b_1 \), \( 2 \leq b_1 \leq b_0 \), such that \( \Gamma_1 \) contains infinitely many elements of order \( b_1 \). Let \( \Delta_1 = \{\alpha \in \Delta_0 | \gamma_1 \alpha^{-1} \) has order \( b_1\} \). Suppose that \( n \in \mathbb{N} \) and that \( \gamma_1, \gamma_2, \gamma_3, \Delta_1, \ldots, a_n \) have been found such that for \( 1 \leq j \leq n \) we have (i) \( \gamma_j \in \Delta_{j-1}, \Gamma_j = \{\gamma_j \alpha^{-1} | \alpha \in \Delta_{j-1}\} \), \( \Gamma_j \) has infinitely many elements of order \( b_j \), \( 2 \leq b_j \leq b_{j-1} \), and \( \Delta_j = \{\alpha \in \Delta_{j-1} | \gamma_j \alpha^{-1} \) has order \( b_j\} \). Observe that from (i) it follows that (ii) for \( 1 \leq j \leq n \), we have \( \gamma_j \in \Delta_j \) so \( \Delta_j \) is a proper infinite subset of \( \Delta_{j-1} \) and the \( \gamma_j \) are distinct.

Let \( \gamma_{n+1} \in \Delta_n \). Let \( \Gamma_{n+1} = \{\gamma_{n+1} \alpha^{-1} | \alpha \in \Delta_n\} \). Since \( \Gamma_{n+1} \) is infinite, there is an integer \( b_{n+1} \) with \( 2 \leq b_{n+1} \leq b_n \) such that \( \Gamma_{n+1} \) contains infinitely many elements of order \( b_{n+1} \). Let \( \Delta_{n+1} = \{\alpha \in \Delta_n | \gamma_{n+1} \alpha^{-1} \) has order \( b_{n+1}\} \). Thus, we can define \( \gamma_n, \Gamma_n, \Delta_n, \) and \( b_n \) for all \( n \in \mathbb{N} \) in such a way that properties (i) hold for all \( n \). Since \( \{b_n\} \) is a monotone nonincreasing sequence of integers greater than one, there exist positive integers \( r \) and \( a \) such that \( b_n = a \) for all \( n > r \). Let \( \Gamma = \{\gamma_{r+n} | n \in \mathbb{N}\} \). We show that \( \Gamma \) and \( a \) are as demanded. Let \( n_1 \) and \( n_2 \in \mathbb{N} \) with \( n_1 > n_2 \). Then, by construction of the \( \Delta_n \), we have \( \gamma_{r+n_1} \in \)
$A_{r+n_1-1} \subseteq A_{r+n_2}$ so $\gamma_{r+n_2}\gamma_{r+n_1}^{-1}$ has order $b_{r+n_2} = a$.

**Lemma 4.6.** Let $k$ be an integer greater than one. Let $I$ be an infinite index set and let $X = \prod_{i \in I} G_i$, where each $G_i$ is a copy of $T_{(k)}$. Let $\Delta$ be an infinite subset of $X$. Then there exist an integer $a \geq 2$ and an infinite subset $\Delta_0$ of $\Delta$ and a finite (possibly empty) subset $I_0$ of $I$ such that projection of $\Delta_0$ onto $Y = \prod_{i \in I \setminus I_0} G_i$ gives an infinite subset $\Delta_n$ of $Y$ consisting solely of elements of order $a$ and such that whenever $\gamma_1$ and $\gamma_2$ are distinct elements of $\Delta_n$, $\gamma_1\gamma_2^{-1}$ has order $a$.

**Proof.** By Lemma 4.5, there exist an integer $a_1 \geq 2$ and an infinite subset $I_1$ of $I$ such that whenever $\gamma_1$ and $\gamma_2$ are distinct elements of $I_1$, $\gamma_1\gamma_2^{-1}$ has order $a_1$. Let $\bar{\gamma}_1$ be an infinite subset of $I_1$ consisting of elements all of the same order $b_1$. It is clear that $b_1 \geq a_1$. (If $\gamma_1$ and $\gamma_2$ are distinct elements of $I_1$, then $\gamma_1\gamma_2^{-1}$ has order at most $b_1$. But $\gamma_1\gamma_2^{-1}$ has order $a_1$.) If $b_1 = a_1$, we are done. (Take $I_0 = \emptyset$, $\Delta_0 = \bar{\gamma}_1$, and $a = a_1$.) Suppose $b_1 > a_1$. Let $\gamma_1 \in \bar{\gamma}_1$. There is a finite subset $I_1$ of $I$ such that the $\iota$th coordinate of $\gamma_1$ is the identity of $G_i$ for $\iota \in I_1$. Let $X_i = \prod_{\iota \in I \setminus I_1} G_i$. Since $I_1$ is finite and $\bar{\gamma}_1$ is infinite, projection of $\bar{\gamma}_1$ onto $X_i$ (denoted by $\pi_i$) gives an infinite subset $\Delta_i$ of $X_i$ consisting of elements of order at most $a_i$. (For $\alpha \in \bar{\gamma}_1$, order of $\pi_i(\alpha)$ in $X_i = \text{order of } \pi_i(\alpha\gamma_1^{-1})$ in $X_i \leq a_i$.) Applying Lemma 4.5 to $X_i$ and $\Delta_i$ we get an integer $a_2$ with $2 \leq a_2 \leq a_1$ and an infinite subset $I_2$ of $I_i$ such that whenever $\gamma_1$ and $\gamma_2$ are distinct elements of $I_2$, then $\gamma_1\gamma_2^{-1}$ has order $a_2$. Let $\bar{\gamma}_2$ be an infinite subset of $I_2$ consisting of elements all of the same order $b_2$. Then we have $a_2 \leq b_2 \leq a_1 < b_1$. If $a_2 = b_2$, we are done. (Take $I_0 = I_1$, $a = a_2$, $Y = X_i$, and $\Delta_0 = \{\alpha \in \Delta | \pi_i(\alpha) \in \bar{\gamma}_2\}$ Suppose $a_2 < b_2 \leq a_1 < b_1$. Pick $\gamma_2 \in \bar{\gamma}_2$; let $I_2 = \{\iota \in I \setminus I_1 \mid \iota\text{'th coordinate of } \gamma_2 \text{ is not the identity of } G_i\}$; project $\bar{\gamma}_2$ onto $X_2 = \prod_{\iota \in I \setminus (I_1 \cup I_2)} G_i; \cdots$ etc. We must eventually have $b_n = a_n$ for some $n$ (otherwise, $\{b_n\}$ would be an infinite strictly decreasing sequence of positive integers). For that $n$, we have a finite subset $I_0 = I_1 \cup \cdots \cup I_{n-1}$ of $I$ and an infinite subset $\bar{\gamma}_n$ of $Y = \prod_{i \in I \setminus I_0} G_i$, such that all elements of $\bar{\gamma}_n$ have order $a_n = b_n$ and such that whenever $\gamma_1$ and $\gamma_2$ are distinct elements of $\bar{\gamma}_n$, $\gamma_1\gamma_2^{-1}$ has order $a_n$. Let $\Delta_0 = \{\alpha \in \Delta | \pi(\alpha) \in \bar{\gamma}_n\}$, where $\pi$ is the projection of $X$ onto $Y$.

**Theorem 4.7.** Let $k$ be an integer greater than one. Let $G = \prod_{i \in I} G_i$, where each $G_i$ is a copy of $T_{(k)}$ and $I$ is infinite. Let $\Delta$ be an infinite subset of $X$. Then there is an integer $a$ greater than one such that every neighborhood of the identity of $G$ contains a $K_{a,l}$-set homeomorphic to the Cantor set.
Proof. We may suppose that $\Delta$ is countable. We identify $X$ with $\prod_{e \in I} G_i$. Let $a, I_0, Y, \text{ and } \tilde{A}_0$ be as in Lemma 4.6. Let $I_1 = \{ \epsilon \in I \setminus I_0 \mid \text{some } \gamma \in \tilde{A}_0 \text{ has } \epsilon \text{ th coordinate different from the identity of } G_i \}$. Plainly $I_1$ is countably infinite. Let $I_2 = I \setminus (I_0 \cup I_1)$. Let $G_j = \prod_{e \in I} G_i$, and let $G_j$ have character group $X_j$, $j = 0, 1, 2$. Since $I_1$ is countable, $G_1$ is metrizable. Since $I_0$ is finite, $G_0$ is finite. Let $I_0$ be the image of the projection of $\tilde{A}_0$ onto $X$. We may suppose that our neighborhood of the identity of $G$ has the form $U = \{e_0\} \times V_1 \times V_2$, where $e_0$ is the identity of $G_0$ and $V_j$ is open in $G_j$, $j = 1, 2$. Applying Theorem 4.3 to $k, G_i, I_0, \text{ and } a$, we find a subset $P$ of $V_i$ homeomorphic to the Cantor set which is a $K_{a, r_0}$-set. Let $P = \{e_0\} \times P_1 \times \{e_2\}$, where $e_2$ is the identity of $G_2$. Then $P$ is a $K_{a, r}$-set in $U$ homeomorphic to the Cantor set.

Proof of Theorem III. 4.8. If $G$ is a compact torsion group, then there are integers $r_1, \ldots, r_q$ greater than one and disjoint infinite index sets $I_1, \ldots, I_q$ and there is a finite abelian group $F$ such that $G$ is topologically isomorphic to $F \times G_1 \times \cdots \times G_q$, where $G_j = \prod_{e \in I_j} K_i$, and each $K_i$ is a copy of $T_{(r_j)}$ when $\epsilon \in I_j$ $(1 \leq j \leq q)$. Let $G_j$ have character group $X_j$ $(1 \leq j \leq q)$. Then for some $j_0$, the image $I_j$ of the projection of $\tilde{A}$ onto $X_{j_0}$ is infinite. Let $a$ be as in Theorem 4.7 applied to $G_{j_0}, X_{j_0}, \text{ and } I_j$. Let $U$ be a neighborhood of the identity of $G$. We will prove that $U$ contains a $K_{a, r}$-set homeomorphic to the Cantor set. Clearly, this will establish Theorem III. We may suppose that $U$ has the form $\{e_F\} \times U_1 \times \cdots \times U_q$, where $e_F$ is the identity of $F$ and $U_j$ is a neighborhood of the identity $e_j$ of $G_j$ $(1 \leq j \leq q)$. By Theorem 4.7, $U_{j_0}$ contains a $K_{a, r}$-set $P_{j_0}$ homeomorphic to the Cantor set. Let

$$P = \{e_F\} \times \{e_1\} \times \cdots \times \{e_{j_0-1}\} \times P_{j_0} \times \{e_{j_0+1}\} \times \cdots \times \{e_q\}. $$

Then $P$ is a $K_{a, r}$-set in $U$ homeomorphic to the Cantor set.

5. Examples.

5.1. The hypothesis that $\tilde{A}$ is not compact is necessary in Theorem II. If $\tilde{A}$ is compact, then there is a nonempty open $U \subset G$ which contains no $K_{a, r}$-set and no $K_{a, r}$-set for any integer $a \geq 2$. Indeed, let $U = \{x \in G \mid |\gamma(x) - 1| < 1 \text{ for all } \gamma \in \tilde{A}\}$. Then $U$ is an open neighborhood of the identity in $G$ and $Re \gamma(x) > 0$ for all $x \in U$ and all $\gamma \in \tilde{A}$. Hence, the function $-1$ cannot be matched within 1 on any nonvoid subset of $U$ by any $\gamma \in \tilde{A}$, nor can the function $\omega_a$ (where $\omega_a$ is an $a$th root of unity with $Re \omega_a < 0$) be matched on any nonvoid subset of $U$ by any $\gamma \in \tilde{A}$ for any integer $a \geq 2$. Hence, no subset of $U$ is a $K_{a, r}$-set or a $K_{a, r}$-set.
5.2. The phrase "a translate of" is a necessary part of the conclusion of Theorem III, as is shown by the following example. Let \( G = T_{(3)} \times H \), where \( H \) is the product of infinitely many copies of \( T_{(3)} \). Write \( X = Z_2 \times Y \), where \( Y \) is the character group of \( H \). Let \( \Delta = \{1\} \times Y \). Let \( U = \{-1\} \times H \). Then \( U \) is open in \( G \) and \( \gamma(x) \in -T_{(3)} \) for all \( x \in U \) and all \( \gamma \in \Delta \), so the constant function \( 1 \) cannot be matched on any subset of \( U \) by any \( \gamma \in \Delta \). Hence, no subset of \( U \) is a \( K_{a,s} \)-set for any integer \( a \geq 2 \).

5.3. The hypothesis that \( G \) is a compact torsion group in Theorem III cannot be weakened to the hypothesis that \( G \) is compactly generated and contains a compact open torsion subgroup. For example, let \( H \) be an infinite compact torsion group and let \( G = Z \times H \). Take \( \Delta = T \times \{e\} \) (where \( e \) is the identity of the character group of \( H \)) and \( U = \{0\} \times H \). Then \( \gamma(x) = 1 \) for all \( x \in U \) and all \( \gamma \in \Delta \). Hence, whenever \( P \subset G \) is such that a translate of \( P \) is contained in \( U \), we have \( \gamma \) constant on \( P \). Therefore, no such totally disconnected \( P \) containing more than one point can be a \( K_{a,s} \)-set for any integer \( a \geq 2 \).

5.4. The hypothesis of local connectedness or something closely related to connectedness (cf. Theorem 2.1) in Theorems II and I respectively cannot be weakened to the hypothesis that \( G \) is not a torsion group. Indeed, there exist a compact metrizable group \( G \) which is not a torsion group and an infinite subset \( \Delta \) of \( X \) such that \( G \) contains no \( K_{0,s} \)-set. For example, let \( G = \prod_{j=1}^\infty T_{(p_j)} \). Then, writing \( X = \prod_{j=1}^\infty Z_{p_j} \) and letting \( \Delta = \{\gamma_2, \gamma_3, \cdots\} \) where \( \gamma_j \) has \( j \)th coordinate equal to \( j \) and the rest zero, we have \( \gamma_j(x) = \pm 1 \) for all \( x \in G \) and all \( j \), so every nonempty subset of \( G \) fails to be a \( K_{a,s} \)-set.

Also, there exist a compact metrizable group \( G \) which is not a torsion group and an infinite subset \( \Delta \) of \( X \) such that no subset of \( G \) containing more than one point is a \( K_{a,s} \)-set for any integer \( a \geq 2 \). Let \( G = \prod_{j=1}^\infty T_{(p_j)} \) where \( p_j \) is the \( j \)th prime. Write \( X = \prod_{j=1}^\infty Z_{p_j} \) and let \( \Delta = \{\gamma_1, \gamma_2, \cdots\} \) where \( \gamma_j \) has \( j \)th coordinate equal to \( 1 \) and the rest zero. Let \( P \) be a subset of \( G \) containing at least two points. Let \( a \geq 2 \) be an integer. We will show that \( P \) is not a \( K_{a,s} \)-set. Let \( p_k \) be a divisor of \( a \). The open-closed sets in \( G \) form a basis for the topology of \( G \), so there are two distinct \( T_{(p_k)} \)-valued (and, hence, \( T_{(a)} \)-valued) continuous functions, \( f_1 \) and \( f_2 \), on \( P \) both different from \( 1 \). If either \( f_1 \) is matched on \( P \) by some \( \gamma_j \), it must be matched by \( \gamma_k \) since no other \( \gamma_j \) attains values in \( T_{(p_k)} \) different from \( 1 \). Thus either \( f_1 \) or \( f_2 \) is a \( T_{(a)} \)-valued continuous function not matched on \( P \) by any \( \gamma_j \). Hence, \( P \) is not a \( K_{a,s} \)-set.
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