ON CLOSE-TO-CONVEX FUNCTIONS OF ORDER $\beta$

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For $\beta \geq 0$, denote by $K(\beta)$ the class of normalized functions $f$, regular and locally schlicht in the unit disc, which satisfy the condition that for each $r < 1$, the tangent to the curve $C(r) = \{f(re^{i\theta}): 0 \leq \theta < 2\pi\}$ never turns back on itself as much as $\beta \pi$ radians. $K(\beta)$ is called the class of close-to-convex functions of order $\beta$. The purpose of this paper is to investigate the asymptotic behavior of the integral means and Taylor coefficients of $f \in K(\beta)$. It is shown that the function $F_\beta$, given by $F_\beta(z) = (1/(2(\beta + 1)))((1 + z)/(1 - z))^{\beta + 1} - 1$), is in some sense extremal for each of these problems. In addition, the class $B(a)$ of Bazilevic functions of type $a > 0$ is related to the class $K(1/a)$. This leads to a simple geometric interpretation of the class $B(a)$ as well as a geometric proof that $B(a)$ contains only schlicht functions.

Let $f$ be regular in $U = \{z: |z| < 1\}$ and be given by

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots .$$

Following an argument due to Kaplan [9], we see that $f \in K(\beta)$ iff, for some normalized convex function $\varphi$ and some constant $c$ with $|c| = 1$, we have for all $z \in U$ that

$$\left| \arg \frac{cf'(z)}{\varphi'(z)} \right| \leq \beta \pi/2 .$$

Equivalently,

$$cf'(z) = p(z)^\delta \varphi'(z) ,$$

where $p(z) = \sum_{n=0}^{\infty} p_n z^n$, $|p_0| = 1$, has positive real part in $U$.

It is geometrically clear that for $0 \leq \beta \leq 1$, $K(\beta)$ contains only schlicht functions. However, for any $\beta > 1$, Goodman [3] has shown that $K(\beta)$ contains functions of arbitrarily high valence. $K(0)$ is the class of convex functions, and $K(1)$ is the class of close-to-convex functions introduced by Kaplan [9]. For $0 \leq \alpha \leq 1$, Pommerenke [13, 14] has studied $m$-fold symmetric functions of class $K(\alpha)$. The following theorem shows that the study of these functions is closely related to the study of $K(\beta)$ for arbitrary $\beta \geq 0$.

**Theorem 1.1.** Let $\beta \geq 0$ and $m$ be a positive integer. Then $f \in K(\beta)$ iff there exists an $m$-fold symmetric function $g \in K(\beta/m)$ such that $f''(z^m) = g'(z)^m$.
Proof. Suppose $f \in K(\beta)$, and define $g$ by $g'(z) = f'(z^m)^{1/m}$. From (1.3) it follows that
\[ g'(z) = c^{-1/m} p(z^m)^{\beta/m} \psi'(z) \]
where the convex function $\psi$ is defined by $\psi'(z) = \varphi'(z^m)^{1/m}$. Hence $g \in K(\beta/m)$, and $g$ is clearly $m$-fold symmetric. To prove the converse implication, we merely reverse the above procedure.

Finally, for $k \geq 2$ denote by $V_k$ the class of normalized functions with boundary rotation at most $k\pi$. From the proof of [2, Theorem 2.2], it follows that $V_k \subset K(k/2 - 1)$. However, $f \in V_k$ implies that $f$ is at most $k/2$ valent [2], so $K(k/2 - 1)$ is in general a much larger class than $V_k$. The results in § 2 and 3 of this paper are extensions to $K(\beta)$ of results of the author [10] for the class $V_k$. These results also generalize and improve some of the results of Pommerenke [13] for $K(\alpha)$, $0 \leq \alpha \leq 1$.

2. Behavior of the coefficients. We begin by studying $M(r, f') = \max_{|z|=r} |f'(z)|$.

**Theorem 2.1.** Let $f \in K(\beta)$. Then $((1 - r)/(1 + r))^{\beta + 2} M(r, f')$ is a decreasing function of $r$, and hence $\omega = \lim_{r \to 1} (1 - r)^{\beta + 2} M(r, f')$ exists and is finite. If $\omega > 0$ and $f$ is given by (1.3), then there exists $\theta_0$ such that $\varphi'(z) = (1 - ze^{-i\theta_0})^{-2}$ and $\omega = \lim_{r \to 1} (1 - r)^{\beta + 2} |f'(re^{i\theta_0})|$.

**Proof.** Since for each $\beta \geq 0$, $K(\beta)$ is a linear-invariant family of order $\beta + 1$ in the sense of Pommerenke [12] (See [4, Theorem 3] for a proof.), the first two statements of the theorem follow. Also, if $\varphi'$ is not of the stated form, then $\varphi'(z) = O(1)(1 - r)^{-\delta}$ for some $0 < \delta < 2$, and hence from (1.3) we see $\omega = 0$. Finally, if $\omega > 0$, then $\varphi'(z) = (1 - ze^{-i\theta_0})^{-2}$, and just as in the proof of [10, Theorem 3.1] we see that $\omega = \lim_{r \to 1} (1 - r)^{\beta + 2} |f'(re^{i\theta_0})|$.

We now begin to study the coefficient behavior. Our method is the major-minor arc technique used by Hayman [5], and the proofs are similar to the proofs of the corresponding results for the class $V_k$ [10]. Hence we omit details wherever possible. We first require two lemmas.

**Lemma 2.1** Let $f \in K(\beta)$ and $\omega = \lim_{r \to 1} (1 - r)^{\beta + 2} |f'(re^{i\theta_0})| > 0$. Then given $\delta > 0$, we may choose $C = C(\delta) > 0$ and $r_0 = r_0(\delta) < 1$ such that for $r_0 \leq r < 1$ we have
\[ \int_0^1 |f'(re^{i\theta})| d\theta < \frac{\delta}{(1 - r)^{\beta + 2}}. \]
where \( E = \{ \theta : C(\delta)(1 - r) \leq |\theta - \theta_0| \leq \pi \} \).

**Proof.** Without loss of generality we may assume \( \theta_0 = 0 \), so from Theorem 2.1 and (1.3) we find, with \( z = re^{i\theta} \),

\[
|f'(z)| = |p(z)|^\beta |1 - z|^{-2}.
\]

Hence, with \( C > 0 \) and \( E \) as above, we find

\[
\int_E |f'(z)| d\theta = \frac{O(1)}{(1 - r)^\beta} \int_{c(1 - r)} \theta^{-2} d\theta = O(1) \frac{1}{C} \frac{1}{(1 - r)^{\beta + 1}},
\]

and the lemma now follows upon choosing \( C \) sufficiently large.

**Lemma 2.2.** Let \( f \in K(\beta) \), \( \omega = \lim_{r \to 1} (1 - r)^{\beta + 2} |f'(re^{i\theta_0})| > 0 \), \( r_n = 1 - 1/n \), \( \omega_n = (1 - r_n)^{\beta + 2} f'(r_ne^{i\theta_0}) \), and

\[
f_n'(z) = \frac{\omega_n}{(1 - ze^{-i\theta_0})^{\beta + 2}}.
\]

Let \( S \) be a fixed but arbitrary Stolz angle with vertex \( e^{i\theta_0} \), and let \( D_n = \{ z \in S : |e^{i\theta_0} - z| < 2/n \} \). Then as \( n \to \infty \), \( f_n' \sim f' \) uniformly for \( z \in D_n \).

**Proof.** Again assuming \( \theta_0 = 0 \), we have from (1.3) \( cf'(z) = p(z)^\beta (1 - z)^{-2} \), and so

\[
f_n'(z) = \frac{[(1 - r_n)p(r_n)]^\beta}{c(1 - z)^{\beta + 2}}.
\]

Thus, to prove the lemma it suffices to show that as \( n \to \infty \),

\[
(1 - r_n)p(r_n)
\]

uniformly for \( z \in D_n \).

By a theorem of Hayman [6, Theorem 2], \( \lim_{r \to 1} (1 - r)p(r) = L \) exists, and it is clear that \( (1 - z)p(z) \) is bounded as \( |z| \to 1 \), providing \( z \in S \). By a theorem of Lindelöf [8, p. 260], we have for \( z \in S \) that \( \lim_{z \to 1} (1 - z)p(z) = L \) where the limit is approached uniformly as \( |z| \to 1 \). But \( 0 < \omega = \lim_{r \to 1} (1 - r)^{\beta + 2} |f'(r)| = \lim_{r \to 1} [(1 - r)p(r)]^\beta \), so \( L \neq 0 \). Combining these remarks with the inequality

\[
\left| \frac{(1 - z)p(z)}{(1 - r_n)p(r_n)} - 1 \right| \leq \frac{1}{|(1 - r_n)p(r_n)|} \left\{ |(1 - z)p(z) - L| + |L - (1 - r_n)p(r_n)| \right\},
\]
we see that (2.1) holds, so the proof is complete.

We can now determine the asymptotic behavior of $a_n$ as $n \to \infty$.

**Theorem 2.2.** Let $f \in K(\beta)$ be given by (1.1), and let $\omega = \lim_{r \to 1} (1 - r)^{\beta+2} M(r, f')$. Let $\Gamma$ denote the gamma function. Then

$$
\lim_{n \to \infty} \frac{|a_n|}{n^\beta} = \frac{\omega}{\Gamma(\beta + 2)}.
$$

Also, if $\omega = \lim_{r \to 1} (1 - r)^{\beta+2} |f'(re^{i\theta})| > 0$, then as $n \to \infty$

$$
a_n \sim \frac{f'(r_ne^{i\theta})e^{-i(n-1)\theta}}{n^2 \Gamma(\beta + 2)}
$$

where $r_n = 1 - 1/n$.

**Proof.** Suppose first that $\omega > 0$, and define

$$
f_n'(z) = \omega \sum_{m=0}^{\infty} d_m e^{-im\theta_0} z^m
$$
as in Lemma 2.2. We note that

$$
d_m = \frac{\Gamma(m + \beta + 2)}{\Gamma(m + 1)\Gamma(\beta + 2)},
$$
so $d_m \sim m^{\beta+1}/\Gamma(\beta + 2)$ as $m \to \infty$. Computation shows that

$$
n\alpha_n - \omega_n d_{n-1} e^{-i(n-1)\theta_0} = \frac{1}{2\pi r_n^{-1}} \int_{-\pi}^{\pi} \{f'(re^{i\theta}) - f_n'(re^{i\theta})\} e^{-i(n-1)\theta} d\theta.
$$

Given $\delta > 0$, we choose $C = C(\delta)$ and $E$ as in Lemma 2.1, and we let $r_n = 1 - 1/n$. With $n$ sufficiently large, Lemma 2.1 gives

$$
\int_E |f'(r_ne^{i\theta})| d\theta < \delta n^{\beta+1},
$$
and clearly this inequality is also true for $f_n'$. Hence, we see that

$$
\left| \int_E \{f'(r_ne^{i\theta}) - f_n'(r_ne^{i\theta})\} e^{-i(n-1)\theta} d\theta \right| < 2\delta n^{\beta+1}
$$

for $n$ sufficiently large. We now choose a Stolz angle $S$, depending on $\delta$, such that $\{r_n e^{i\theta}: \theta \in E'\} \subset S$ for large $n$, where $E' = [-\pi, \pi] \setminus E$. By Lemma 2.2, we have as $n \to \infty$ and with $\theta \in E'$,

$$
f'(r_ne^{i\theta}) - f_n'(r_ne^{i\theta}) = o(1)\{f_n'(r_ne^{i\theta})\}
$$

$$
= o(1)n^{\beta+2},
$$
where $o(1)$ is uniform for $\theta \in E'$, and hence as $n \to \infty$, we have
\[(2.5) \quad \left| \int_{\theta} \{f'(r_ne^{i\theta}) - f'_n(r_ne^{i\theta})\} e^{-i(n-1)\theta} d\theta \right| \leq o(1)2C(\delta)(1 - r_n)n^{\beta + 2}
\quad = o(1)n^{\beta + 1}.
\]

Note that although \(o(1)\) depends on \(\delta\), \(o(1) \to 0\) as \(n \to \infty\) once \(\delta\) has been fixed.

Combining (2.3), (2.4), and (2.5), we find

\[
\forall a_n - \omega_n \frac{d_{n-1}}{n} e^{-i(n-1)\theta_0} < \{2\delta + o(1)\}n^{\beta + 1}
\]

for sufficiently large \(n\). Since \(\delta > 0\) is arbitrary and since \(o(1) \to 0\) once \(\delta\) has been fixed, we have

\[
a_n = \omega_n \frac{d_{n-1}}{n} e^{-i(n-1)\theta_0} + o(1)n^{\beta}.
\]

From (2.2) and the definition of \(\omega_n\) we see that as \(n \to \infty\),

\[
a_n \sim \omega_n e^{-i(n-1)\theta_0} n^{\beta}/\Gamma(\beta + 2)
\quad \sim \frac{f'(r_ne^{i\theta})e^{-i(n-1)\theta_0}}{n^2\Gamma(\beta + 2)}.
\]

In particular,

\[
\lim_{n \to \infty} \frac{|a_n|}{n^\beta} = \frac{\omega}{\Gamma(\beta + 2)}.
\]

We now suppose \(\omega = 0\). We shall subsequently prove (Theorem 3.1 with \(\lambda = 1\)) that if \(\omega = 0\), then

\[
\lim_{r \to 1} (1 - r)^{\beta + 1} \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta = 0.
\]

Using a standard inequality relating coefficients and integral means [7, p. 11] we have \(\lim_{n \to \infty} |a_n|/n^\beta = 0\). This completes the proof of the theorem. Note that if \(\omega > 0\), then it follows easily from the theorem that \(\lim_{n \to \infty} a_{n+1}/a_n = e^{-i\theta_0}\), and so the radius of maximal growth can be determined from the coefficients.

We now consider the problem of determining

\[
\max \{|a_n|: f \in K(\beta)\}.
\]

It is natural to conjecture that for each \(n \geq 2\) this problem is solved by the function

\[
F_\beta(z) = \frac{1}{2(\beta + 1)} \left\{ \left( \frac{1 + z}{1 - z} \right)^{\beta + 1} - 1 \right\} = z + \sum_{j=\frac{1}{2}}^\infty A_j(\beta)z^j.
\]

Toward this end we have the following theorem.
THEOREM 2.3. Let \( f \in K(\beta) \) be given by (1.1) and let \( F_\beta \) be as above.

(i) There exists an integer \( n_0 \) depending on \( f \) such that \( |a_n| \leq A_n(\beta) \) for \( n \geq n_0 \).

(ii) If \( n \leq \beta + 2 \), then \( |a_n| \leq A_n(\beta) \).

(iii) If \( \beta \) is an integer, then \( |a_n| \leq A_n(\beta) \) for all \( n \).

Note that since \( V_k \subset K(\beta) \) with \( \beta = k/2 - 1 \), we have from (ii) that \( |a_n| \leq A_n(\beta) \) for \( n \leq k/2 + 1 \) and from (iii) that \( |a_n| \leq A_n(\beta) \) for all \( n \) whenever \( k \) is an even integer.

Proof. We have from (1.3), with \( |c| = 1 \),

\[
f'(z) = p(z)\beta \varphi'(z),
\]

where \( p \) has positive real part and \( \varphi \) is convex. Suppose that \( p(z) = \sum_{n=0}^{\infty} p_n z^n \), \( |p_0| = 1 \), and \( p(z)^{\beta} = \sum_{n=0}^{\infty} q_n z^n \). Then it is easily verified by induction that for \( m \geq 1 \),

\[
q_m = \frac{1}{m!} \sum_{j=1}^{m} \beta(\beta - 1) \cdots (\beta - (j - 1)) p_0^{\beta - j} D_j(p)
\]

where \( D_j(p) \) is a polynomial, with nonnegative coefficients, in the variables \( p_0, p_1, \cdots, p_m \).

Therefore, if \( \beta \) is an integer, \( |q_m| \) is maximal for all \( m \geq 1 \) when \( p_0 = 1 \) and \( p_j = 2 \) for \( j \geq 1 \), which implies \( p(z) = (1 + z)/(1 - z) \). Also, for any \( \beta \geq 0 \), we see as above that if \( n \leq \beta + 2 \), then \( |q_m| \) is maximal for \( 1 \leq m \leq n - 1 \) when \( p(z) = (1 + z)/(1 - z) \). In addition, if \( \varphi'(z) = 1 + \sum_{j=2}^{\infty} u_j z^{j-1} \), it is well-known that \( |u_j| \leq j \) for all \( j \), with equality for \( \varphi'(z) = (1 - z)^{-2} \). But when \( p(z) = (1 + z)/(1 - z) \) and \( \varphi'(z) = (1 - z)^{-2} \), we have \( cf'(z) = F'(z) \). Hence, since

\[
cn a_n = \sum_{j=0}^{n-1} q_j u_{n-j}
\]

where we define \( u_1 = 1 \), we see that (ii) and (iii) are proved.

We now prove (i). We first note that as \( n \to \infty \),

\[
A_n(\beta) \sim \frac{2^\beta n^\beta}{\Gamma(\beta + 2)}.
\]

Let \( \omega = \lim_{r \to 1} (1 - r)^{\beta + 2} M(r, f') \). If \( \omega = 0 \), then Theorem 2.2 shows \( a_n = o(1)n^\beta \), and so it is clear from (2.6) that (i) holds. We now suppose \( \omega = \lim_{r \to 1} (1 - r)^{\beta + 2} |f'(re^{i\theta})| > 0 \), and we recall that in this case \( \omega = \lim_{r \to 1} [(1 - r) |p(re^{i\theta})|]^{\beta} \). Hence, from [6, Theorem 2], it follows easily that \( \omega \leq 2^\beta \) with equality only if
\( p(z) = \frac{1 + ze^{-i\theta_0}}{1 - ze^{-i\theta_0}}. \)

But \( \omega > 0 \) implies also that \( \varphi'(z) = (1 - ze^{-i\theta_0})^{-2} \), and thus we have \( \omega \leq 2^\beta \) with equality only if \( c f'(z) = F'_\beta(e^{-i\theta}z) \), in which case \( |c_n| = A_n(\beta) \) for all \( n \), since \( |c| = 1 \). Thus we may suppose \( \omega < 2^\beta \), and using Theorem 2.2 and (2.6) we see that (i) holds. This completes the proof of Theorem 2.3.

To conclude this section we examine the asymptotic behavior of the quantity \( |a_{n+1} - a_n| \) for \( f \in K(\beta) \).

**Theorem 2.4.** Let \( f \in K(\beta) \) be given by (1.1). If \( \omega > 0 \), then

\[
\lim_{n \to \infty} \frac{|a_{n+1} - a_n|}{n^{\beta-1}} = \frac{\beta \omega}{\Gamma(\beta + 2)}.
\]

The theorem is in general false when \( \omega = 0 \).

**Proof.** If \( \beta = 0 \) and \( \omega > 0 \), then from (1.3) it follows that \( c f'(z) = (1 - ze^{-i\theta_0})^{-2} \), so \( |c_n| = 1 \) for all \( n \), and the theorem is trivially true. Thus, we may assume without loss of generality that \( \beta > 0 \). The proof will be divided into three parts.

We first claim that given \( \delta > 0 \), there exists \( C(\delta) > 0 \) such that

\[
(2.7) \quad \left| \frac{1}{2\pi} \int_{E} (1 - re^{i(\theta - \theta_o)}) f'(re^{i\theta}) d\theta \right| < \frac{\delta}{(1 - r)^{\delta}}
\]

where \( \theta_o \) is as in Theorem 2.1 and \( E = \{ \theta: C(\delta)(1 - r) \leq |\theta - \theta_o| \leq \pi \} \). To prove (2.7), we note that \( \omega > 0 \) implies that

\[
 cf'(z) = p(z)^\delta (1 - z)^{-\frac{1}{\lambda}}.
\]

where we have assumed without loss of generality that \( \theta_o = 0 \). Also, for notational ease, we assume \( c = 1 \) and \( p(0) = 1 \), so

\[
 (1 - z)f'(z) = p(z)^\delta/(1 - z) .
\]

Choose \( \lambda > 1 \) such that \( \lambda \beta > 1 \), and let \( 1/\lambda + 1/\lambda' = 1 \). If \( C \) is an arbitrary positive constant, we have from Hölder's inequality that

\[
(2.8) \quad \int_{E} |(1 - z)f'(z)| d\theta \leq \left\{ \int_{E} |p(z)|^{2\lambda} d\theta \right\}^{1/2} \left\{ \int_{E} \left| 1 - z \right|^{2\lambda'} d\theta \right\}^{1/2}.
\]

Since \( p \) is subordinate to \( (1 + z)/(1 - z) \), and since \( \lambda \beta > 1 \),

\[
(2.9) \quad \left\{ \int_{0}^{2\pi} |p(z)|^{2\lambda} d\theta \right\}^{1/2} = O(1) \frac{1}{(1 - r)^{\delta - 1/\lambda}} .
\]

Also, as in the proof of Lemma 2.1, we have (since \( \lambda' > 1 \)
Hence, combining (2.8), (2.9), and (2.10), we find
\[ \int_E (1-z)f'(z)d\theta = O(1) \frac{1}{C^{2^{\beta-1}}} \frac{1}{(1-r)^{2^{\beta-1}}} , \]
which gives (2.7) if we choose C sufficiently large.

From this point on we proceed essentially as in the proof of [11, Theorem 2], and thus we merely sketch the proof. We define \( \omega_n \) as in Lemma 2.2, \( \lambda_n = \arg \omega_n \), and
\[ f_n'(z) = \frac{\omega e^{i\lambda_n}}{(1-ze^{-i\theta_0})^{\beta+2}} = \omega e^{i\lambda_n} \sum_{m=0}^{\infty} d_m e^{-i\theta_0 z^m} . \]
Since \( \omega_n = [(1-r_n)p(r_n e^{i\theta_0})]^\beta \), lim \( n \to \infty \lambda_n \) exists by [6, Theorem 2]. As in [11, Lemma 3] we find that as \( n \to \infty \),
\[ a_n - e^{-i\theta_0}a_{n-1} = -\frac{e^{-i\theta_0}a_{n-1}}{n} + \frac{\omega e^{i\lambda_n-(n-1)\theta_0}}{\Gamma(\beta + 1)} n^{\beta-1} + o(1)n^{\beta-1} , \]
and hence as \( n \to \infty \),
\[ \frac{a_n - e^{-i\theta_0}a_{n-1}}{n^{\beta-1}} = \frac{\omega e^{i\lambda_n-(n-1)\theta_0}}{\Gamma(\beta + 1)} \left[ 1 - \frac{1}{\beta + 1} (1 + o(1)) \right] + o(1) , \]
where we have used (2.11) and Theorem 2.2. Theorem 2.2 also implies that as \( n \to \infty \),
\[ \arg e^{-i\theta_0}a_n = \arg \omega e^{i\lambda_n-n\theta_0} + o(1) , \]
and since \( \lim_{n \to \infty} \lambda_n \) exists we have as \( n \to \infty \) that
\[ \arg e^{-i\theta_0}a_{n-1} = \arg we^{i\lambda_n-(n-1)\theta_0} + o(1) . \]
Combining (2.12) with (2.13), we find
\[ \frac{||a_n|| - |a_{n-1}|}{n^{\beta-1}} = \frac{\beta \omega}{\Gamma(\beta + 2)} + o(1) \]
as \( n \to \infty \), which proves the theorem.

We now show that the theorem is false when \( \omega = 0 \). Let \( \beta \geq 0 \) be given, and define \( f \in K(\beta) \) by
\[ f'(z) = \frac{1}{(1-z^{\beta})^{\beta+1}} . \]
Clearly \( f \) is an odd function, and it is easily verified that \( a_{2n+1} \sim n^{\beta-1}/2\Gamma(\beta + 1) \) as \( n \to \infty \), so
\[
\lim_{n \to \infty} \left| \frac{a_{2n+1}}{n^{\beta-1}} - \frac{a_{2n}}{n^{\beta-1}} \right| = \lim_{n \to \infty} \frac{|a_{2n+1}|}{n^{\beta-1}} = \frac{1}{2\Gamma(\beta+1)}.
\]

However, \( \omega = \lim_{r \to 1} (1 - r)^{\beta+2} M(r, f') = \lim_{r \to 1} (1 - r)/(1 + r)^{\beta+1} = 0 \), so the theorem is false when \( \omega = 0 \). This is in sharp contrast to the corresponding result [11] for \( V_k \), where the result is true for all \( k > 2 \) even if \( \omega = 0 \).

3. Behavior of the integral means. In this section we shall investigate the behavior of \( I_\lambda(r, f') \) and \( I_\lambda(r, f) \), where for \( \lambda > 0 \) we define

\[
I_\lambda(r, g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^\lambda d\theta.
\]

Our results again include as special cases previous results of the author [10] for the class \( V_k \) as well as generalizing results of Pommerenke [13] for the classes \( K(\alpha) \), \( 0 \leq \alpha \leq 1 \). Although the details of the proofs given here are slightly more involved than those for \( V_k \), we refer to [10] whenever possible. We first need two lemmas, the first of which is proved in exactly the same way as [10, Lemma 4.1].

**Lemma 3.1.** Let \( f \in K(\beta) \), \( \omega = \lim_{r \to 1} (1 - r)^{\beta+2} |f'(re^{i\theta_0})| > 0 \). Let \( C > 0 \) and \( \lambda > 0 \) be fixed, and for \( 0 < R < 1 \) define \( E = \{ \theta : C(1 - R) \leq |\theta - \theta_0| \leq \pi \} \), \( E' = [-\pi, \pi] \setminus E \). Define \( \omega(R) = (1 - R)^{\beta+2} |f'(Re^{i\theta_0})| \) and

\[
f'_R(z) = \frac{\omega(R)}{(1 - ze^{-i\theta_0})^{\beta+2}}.
\]

Then as \( R \to 1 \),

\[
\int_{E'} |f'_R(Re^{i\theta})|^\lambda d\theta \sim \int_{E'} |f'(Re^{i\theta})|^\lambda d\theta.
\]

**Lemma 3.2.** Let \( f \in K(\beta) \), \( \omega > 0 \), and \( f'_R \) be as above. If \( \lambda(\beta + 2) > 1 \), then as \( r \to 1 \),

\[
I_\lambda(r, f') = I_\lambda(r, f'_R) + o(1)(1 - r)^{-2(\beta+2)}.
\]

**Proof.** By definition, with \( z = re^{i\theta} \), we have

\[
2\pi |I_\lambda(r, f') - I_\lambda(r, f'_R)| \leq \int_E |f''(z)|^\lambda d\theta + \int_E |f'_R(z)|^\lambda d\theta
\]

\[
+ \int_{E'} \left( |f'(z)|^\lambda - |f'_R(z)|^\lambda \right) d\theta,
\]

where \( E \) and \( E' \) are as in Lemma 3.1. If \( \beta = 0 \), then \( \omega > 0 \) implies
\[ f'(z) = (1 - z)^{-2}, \] and so the lemma is trivial. With \( \beta > 0 \), let \( \gamma = 1 + 2/\beta \) and \( \gamma' = 1 + \beta/2 \), so \( 1/\gamma + 1/\gamma' = 1 \). Recalling that in (1.3) we have \( \varphi'(z) = (1 - z)^{-2} \) since \( \omega > 0 \), we have from Hölder’s inequality that
\[
\int_E |f'(z)|^2 d\theta \leq \left( \int_E |p(z)|^{2(\beta+2)} d\theta \right)^{\beta/(\beta+2)} \left( \int_E |1 - z|^{-(\beta+2)} d\theta \right)^{2/(\beta+2)}.
\]
As in the proof of (2.9) and (2.10) it follows that
\[
\int_E |p(z)|^{\lambda(\beta+2)} d\theta = O(1)(1 - r)^{1 - \lambda(\beta+2)}.
\]
Also, with \( \delta > 0 \), it follows that
\[
\int_E |1 - z|^{-\lambda(\beta+2)} d\theta < \frac{\delta}{(1 - r)^{2(\beta+2) - 1}}
\]
for \( C(\delta) \) depending on \( \delta \) and for \( \lambda(\beta + 2) > 1 \), and therefore
\[
\int_E |f'(z)|^2 d\theta < \frac{\delta}{(1 - r)^{2(\beta+2) - 1}}
\]
for \( r \) sufficiently close to 1. Clearly this inequality also holds for \( f'_r \), and so using Lemma 3.1 we have for \( r \) sufficiently close to 1 that
\[
2\pi |I_\lambda(r, f') - I_\lambda(r, f'_r)| < \frac{2\delta}{(1 - r)^{2(\beta+2) - 1}} + o(1) \int_{E'} |f'_r(z)|^2 d\theta
\]
\[
< \frac{2\delta}{(1 - r)^{2(\beta+2) - 1}} + o(1)\omega(r)^{1/2(\beta+2)} \int_0^{(1-r)^{\lambda(\beta)}} d\theta
\]
\[
< \frac{2\delta}{(1 - r)^{2(\beta+2) - 1}} + o(1)\omega(r)^{1/2(\beta+2)} C(\delta).
\]
Since \( \delta > 0 \) was arbitrary and since \( o(1) \) approaches zero once \( \delta \) has been fixed, the lemma follows.

We can now determine the asymptotic behavior of \( I_\lambda(r, f') \) when \( \lambda(\beta + 2) > 1 \). For notational convenience, define
\[
G(\lambda, \beta) = \frac{\Gamma'(\lambda(\beta + 2) - 1)}{2^{2(\beta+2) - 1} \Gamma^2((\lambda(\beta + 2))/2)}.
\]

**Theorem 3.1.** Let \( f \in K(\beta) \) and \( \lambda(\beta + 2) > 1 \). Then
\[
\lim_{r \to 1} (1 - r)^{2(\beta+2) - 1} I_\lambda(r, f') = \omega^2 G(\lambda, \beta).
\]

**Proof.** If \( \omega > 0 \), then the theorem is an immediate consequence of Lemma 3.2 and Pommerenke’s result [13] that as \( r \to 1 \),
whenever $m > 1$. Hence, we now assume $\omega = 0$, and we divide the proof into two cases. We first assume that in (1.3) $\varphi'$ is not of the form $(1 - ze^{-i\theta})^{-\gamma}$. Then, as is well known, $M(r, \varphi') = O(1)(1 - r)^{-\gamma}$ for some $0 < \gamma < 2$. Without loss of generality we assume $\gamma \lambda(\beta + 2)/2 > 1$. As in the proof of Lemma 3.2, we find

\[
\int_0^{2\pi} |f'(z)|^2 \, d\theta \leq \left\{ \int_0^{2\pi} |p(z)|^{(\lambda + 2)/2} \, d\theta \right\} \left\{ \int_0^{2\pi} |\varphi'(z)|^{(2(\lambda + 2))/2} \, d\theta \right\}^{2/(\beta + 2)}
\]

and

\[
\left\{ \int_0^{2\pi} |p(z)|^{(\lambda + 2)/2} \, d\theta \right\}^{\beta/(\beta + 2)} = O(1)(1 - r)^{\beta/(\beta + 2) - 2\beta}.
\]

Also, since $\varphi$ is convex, $z\varphi'$ is starlike and schlicht, so from [7, Theorem 3.2] we have

\[
\left\{ \int_0^{2\pi} |\varphi'(z)|^{(\lambda + 2)/2} \, d\theta \right\}^{2/(\beta + 2)} = O(1)(1 - r)^{2/(\beta + 2) - \gamma}.
\]

Hence

\[
\int_0^{2\pi} |f'(z)|^2 \, d\theta = O(1)(1 - r)^{1 - \lambda(\beta + \gamma)},
\]

and since $\gamma < 2$ we have as $r \to 1$

\[
(1 - r)^{2(\beta + 2) - \lambda} I_s(r, f') \to 0.
\]

It remains only to consider the case $\omega = 0$ and $\varphi'(z) = (1 - ze^{-i\theta})^{-\gamma}$ for some $\theta_0$. Assuming without loss of generality that $\theta_0 = 0$, we find from (1.3) and our hypothesis $\omega = 0$ that

\[
0 = \lim_{r \to 1} (1 - r)p(r).
\]

As in Lemma 2.2, it now follows that for $z$ in a Stolz angle with vertex at 1, we have $\lim_{|z| \to 1} (1 - z)p(z) = 0$ where the limit is approached uniformly as $|z| \to 1$. Hence, since $(1 - r) |p(z)| \leq |1 - z||p(z)|,$

\[
|p(z)| \leq \frac{h(r)}{1 - r}
\]

for $z$ in the Stolz angle, where $h(r) \to 0$ as $r \to 1$. Thus, given $C > 0$,

\[
\left\{ \int_0^{C(1-r)} |f'(z)|^2 \, d\theta \right\}^{\beta/(\beta + 2)} \leq \left\{ \int_0^{C(1-r)} |p(z)|^{(\lambda + 2)/2} \, d\theta \right\}^{\beta/(\beta + 2)} \left\{ \int_0^{C(1-r)} |1 - z|^{-(\lambda + 2)} \, d\theta \right\}^{2/(\beta + 2)}
\]

(3.2)
where we have used (3.1). Exactly as in the proof of Lemma 3.2 we also have, given \( \delta > 0 \),

\[
\int_{C(1-r)} |f'(z)|^2 \, d\theta < \frac{\delta}{(1 - r)^{2(\beta+2)+1 - \delta}}
\]

for an appropriate choice of \( C = C(\delta) \), and hence from (3.2) and (3.3)

\[
\lim_{r \to 1} (1 - r)^{2(\beta+2)+1} I_1(r, f') = 0 ,
\]

which completes the proof of Theorem 3.1.

To complete this section, we examine \( I_\lambda(r, f) \).

**Theorem 3.2.** Let \( f \in K(\beta) \) and let \( G(\lambda, \beta) \) be as in Theorem 3.1.

(i) If \( \lambda \geq 1 \), then

\[
\lim_{r \to 1} (1 - r)^{2(\beta+1)+1} I_1(r, f) \geq \frac{\omega^2 G(\lambda, \beta)}{2^{2(\beta+2)+1}} .
\]

(ii) If \( \lambda \geq 1 \) and \( \lambda(\beta + 1) > 1 \), then

\[
\lim_{r \to 1} (1 - r)^{2(\beta+1)+1} I_1(r, f) \leq \frac{\omega^2 G(\lambda, \beta)}{(\beta + 1 - (1/\lambda))^1} .
\]

Note that when \( \omega = 0 \), \( \lim_{r \to 1} (1 - r)^{2(\beta+1)+1} I_1(r, f) = 0 \), and when \( \omega > 0 \) the growth of \( I_1(r, f) \) is regular in the sense that \( \limsup_{r \to 1} \) and \( \liminf_{r \to 1} \) are either both positive or both zero.

**Proof.** The proof of (i) is very similar to that of [10, Theorem 4.4], and so we omit the details. To prove (ii), we first note that

\[
f(re^{i\theta}) = \int_0^r f'(te^{i\theta}) \, dt .
\]

Since \( \lambda \geq 1 \), a generalization of Minkowski's inequality [15, p. 260] gives

\[
I_\lambda(r, f)^{1/\lambda} \leq \int_0^r I_\lambda(t, f')^{1/\lambda} \, dt .
\]

Since Theorem 3.1 gives us the asymptotic behavior of \( I_\lambda(t, f') \) as \( t \to 1 \), a straightforward argument shows that whenever \( \lambda(\beta + 1) > 1 \),
In conclusion, it should be noted that the basic result underlying the theorems of §§2 and 3 is the existence of \( \omega = \lim_{r \to 1} (1 - r)^{\alpha+1} M(r, f') \), where \( \alpha = \beta + 1 \). Since this limit exists whenever \( f \) belongs to a linear-invariant family of order \( \alpha \), it is interesting to speculate as to whether the results of the previous sections remain true if we assume only that \( f \) belong to such a linear-invariant family. Nothing seems to be known concerning this question. The similarity between the results of the previous sections and results of Hayman [5] on mean \( p \)-valent functions should also be noted. In this direction, W. E. Kirwan has recently shown (unpublished) that given \( f \in V_k \) with \( 2 \leq k \leq 4 \), there exists a constant \( d(f) \) such that \( f - d(f) \) is circumferentially mean-\( k/4 \) valent.

4. Bazilevic functions and \( K(\beta) \). For any \( \alpha > 0 \), define \( B(\alpha) \) to be the class of functions \( g \) which are regular in \( U \) and which are given by

\[
g(z) = \left\{ \alpha \int_0^z \xi^{\alpha-1} p(\xi) \left( \frac{h(\xi)}{\xi} \right)^{\alpha} d\xi \right\}^{1/\alpha},
\]

where \( p \in \mathcal{P} \), the class of functions \( P \) regular in \( U \) satisfying \( \text{Re} \, P(z) > 0 \) and \( P(0) = 1 \), and where \( h \in \mathcal{S}^* \), the class of normalized starlike functions. The powers appearing in (4.1) are meant as principal values. It is known [1] that \( B(\alpha) \) contains only schlicht functions, and it is easy to verify that for various special choices of \( \alpha, p, \) and \( h \), the class \( B(\alpha) \) reduces to the classes of convex, starlike, and close-to-convex functions. However, in general very little seems to be known about the geometry of \( B(\alpha) \). In this section we shall relate \( B(\alpha) \) to \( K(1/\alpha) \). This relationship will allow us to give a simple geometric interpretation of \( B(\alpha) \) as well as a simple geometric proof that \( B(\alpha) \) contains only schlicht functions.

We first need a technical lemma.

**Lemma 4.1.** Let \( g \) be given by (4.1). Then \( g \) is locally schlicht and vanishes only at the origin.

**Proof.** If \( \alpha = 1 \), then it is easily seen that \( g \) is close-to-convex, and hence the lemma is trivial. Thus we assume \( \alpha \neq 1 \). Let \( z_0 \neq 0 \) be given. We claim that \( g(z_0) = 0 \) if \( g'(z_0) = 0 \). If \( g(z_0) \neq 0 \), then \( (g(z)/z)^{\alpha} \) is regular in a neighborhood of \( z_0 \), and from (4.1)

\[
(g(z)/z)^{\alpha-1} g'(z) = p(z)(h(z)/z)^{\alpha}.
\]
Since neither $p$ nor $h$ vanish at $z_0$, it then follows that $g'(z_0) \neq 0$.

Suppose now that $g'(z_0) \neq 0$. We must show $g(z) \neq 0$. Since the zeros of $g$ and $g'$ are isolated, it is clear that we may choose (even if $g(z_0) = 0$) an arc $\gamma$ ending at $z_0$ such that (4.2) holds for $z \in \gamma$, $z \neq z_0$, and such that $g'(z) \neq 0$ for $z \in \gamma$. Therefore, for $z \in \gamma$,

$$\lim_{z \to z_0} \left| \frac{g(z)}{z} \right|^{a-1} = \left| \frac{p(z_0)}{g'(z_0)} \left( \frac{h(z_0)}{z_0} \right)^a \right|,$$

and hence (since $\alpha \neq 1$) $g(z_0) \neq 0$, which establishes our claim.

To prove the lemma, it is now sufficient to show that $g$ vanishes only at the origin. Suppose not; that is, suppose $g(z) = (z - z_0)^m q(z)$ where $m \geq 1$, $q(z_0) \neq 0$ and $z_0 \neq 0$. We choose an arc $\gamma$ ending at $z_0$ such that for $z \in \gamma$ ($z \neq z_0$) we have $g(z) \neq 0$, $g'(z) \neq 0$, and such that (4.2) holds. Then with $z \in \gamma$,

$$(z - z_0)^{m-1} \left( \frac{q(z)}{z} \right)^{a-1} \left[ (z - z_0)q'(z) + mq(z) \right] = p(z) \left( \frac{h(z)}{z} \right)^a.$$ 

We now allow $z \to z_0$, and we find that $m \alpha = 1$. We now define $G$ for $z \in U$ by $G(z)^m = g(z^m)$. From (4.1) it follows that $G$ is close-to-convex with respect to $H$, given by $H(z)^m = h(z^m)$ where $h$ is as in (4.1). But $G(z_0)^m = g(z_0) = 0$ and $z_0^m \neq 0$, which contradicts the fact that $G$ is schlicht. This proves the lemma.

We now define $K_0(\beta)$ to be that subclass of $K(\beta)$ such that in (1.3) we have $c = 1$ and $p(0) = 1$. Therefore, $f \in K_0(\beta)$ iff

(4.3) $$f'(z) = p(z)^{\beta} \frac{h(z)}{z}$$

where $p \in \mathcal{P}$ and $h \in \mathcal{S}^\alpha$. We also assume $\beta > 0$.

**Theorem 4.1.** If $f \in K_0(\beta)$, then $g \in B(1/\beta)$ where

$$g(z) = \left\{ \frac{1}{\beta} \int_{0}^{z} \left( \frac{\xi f'((\xi))}{\xi} \right)^{1-1/\beta} d\xi \right\}^{\beta}.$$ 

Conversely, if $g \in B(\alpha)$, then $f \in K_0(1/\alpha)$ where

$$f(z) = \int_{0}^{z} \left( \frac{g(\xi)}{\xi} \right)^{1-1/\alpha} (g'(\xi))^{1/\alpha} d\xi.$$ 

**Proof.** Suppose first that $f \in K_0(\beta)$ and is given by (4.3). Then

$$f'(z)^{1/\beta} = p(z)^{1/\beta} \left( \frac{h(z)}{z} \right)^{1/\beta},$$ 

and from the definition of $B(1/\beta)$ it follows that $g$ defined as in the
Now we suppose \( g \in B(\alpha) \), and we define \( f \) as in theorem. By Lemma 4.1 \( f \) is regular in \( U \), and since \( g \in B(\alpha) \) we have from the definition of \( f \) that

\[
f'(z)^\alpha = p(z)\left(\frac{h(z)}{z}\right)^\alpha
\]

where \( p \in \mathcal{P} \) and \( h \in \mathcal{S}^* \). Hence \( f \in K_\alpha(1/\alpha) \).

Note that although for \( \beta > 1 \) \( f \) may be of arbitrarily high valence, it is always true that the corresponding \( g \) is schlicht. Also note that since \( V_k \subset K(\kappa/2 - 1) \), we have a relation between \( V_k \) and \( B(2/(k - 2)) \).

We now investigate the geometry of \( B(\alpha) \). We shall assume that \( g \) is regular and locally schlicht in \( U \), is normalized as in (1.1), and vanishes only at the origin. Also, for \( 0 < r < 1 \), we define the curve \( C(r) = \{ g(re^{i\theta})^\alpha: 0 \leq \theta < 2\pi \} \).

**Theorem 4.2.** With the above notation and hypothesis on \( g \), we have that \( g \in B(\alpha) \) iff for all \( 0 < r < 1 \) the tangent to \( C(r) \) never turns back on itself as much as \( \pi \) radians.

**Proof.** If \( g \in B(\alpha) \), then we see from Theorem 4.1 that \( f \in K_\alpha(1/\alpha) \) where

\[
f'(z) = \left(\frac{g(z)}{z}\right)^{1-1/\alpha} (g'(z))^{1/\alpha}.
\]

Denote by \( T(f, re^{i\theta}) \) the tangent to the curve \( f(|z| = r) \) at \( f(re^{i\theta}) \). Then with \( z = re^{i\theta} \),

\[
\arg T(f, re^{i\theta}) = (1 - 1/\alpha) \arg g(z) + (1/\alpha) \arg zg'(z) + \pi/2,
\]

from which it follows by a standard argument that

\[
\frac{\partial}{\partial \theta} \arg T(f, re^{i\theta}) = (1 - 1/\alpha) \text{Re} \frac{zg'(z)}{g(z)} + \frac{1}{\alpha} \text{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\}.
\]

Since \( f \in K_\alpha(1/\alpha) \),

\[
\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta} \arg T(f, re^{i\theta}) d\theta > -\pi/\alpha
\]

for any \( \theta_1 < \theta_2 < \theta_1 + 2\pi \), and so

\[
(\alpha - 1) \int_{\theta_1}^{\theta_2} \text{Re} \frac{zg'(z)}{g(z)} d\theta + \int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{zg''(z)}{g'(z)} \right) d\theta > -\pi.
\]

Noting that locally we have \((g^\alpha(z))' = \alpha g(z)^{\alpha-1} g'(z)\), we see by a standard
argument that (4.4) is equivalent to the fact that the tangent to \( C(r) \) never turns back on itself by as much as \( \pi \) radians.

To prove the converse, we have from Lemma 4.1 that for \( z \neq 0 \), \((g(z))^\alpha\) is locally regular, so we may assume that (4.4) holds. If \( f \) is defined by

\[
f(z) = \int_0^z \left( \frac{g(\xi)}{\xi} \right)^{\frac{1-1/\alpha}{\xi}} (g'(\xi))^\frac{1/\alpha}{\xi} d\xi,
\]

then \( f \) is regular in \( U \) and from (4.4) we have

\[
\left( 4.5 \right) \int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta} \arg T(f, re^{i\theta}) d\theta > -\pi / \alpha
\]

for any \( \theta_1 < \theta_2 < \theta_1 + 2\pi \). Since \( f' \) never vanishes, an argument due to Kaplan [9] shows that (4.5) implies \( f \in K_\alpha(1/\alpha) \), and thus

\[
f'(z) = p(z)^{1/\alpha} \frac{h(z)}{z}
\]

where \( p \in \mathcal{P} \) and \( h \in \mathcal{C}^* \). We now see from the definition of \( f \) that

\[
g(z) = \left\{ \alpha \int_0^z \xi^{\pi-1} p(\xi) \left( \frac{h(\xi)}{\xi} \right)^\frac{1/\alpha}{\xi} d\xi \right\}^{1/\alpha},
\]

and so \( g \in B(\alpha) \). This proves Theorem 4.2.

In conclusion, we prove geometrically that \( B(\alpha) \) contains only schlicht functions.

**Corollary 4.3.** \( B(\alpha) \) contains only schlicht functions.

**Proof.** Suppose \( g \in B(\alpha) \) and \( g \) is not schlicht. For each \( 0 < r < 1 \), let \( C(r) = \{g(re^{i\theta}) : 0 \leq \theta \leq 2\pi\} \), and let \( R = \inf \{r : C(r) \) is not a simple curve\}. Since \( g'(0) = 1 \), it is clear that \( R > 0 \). Also, \( R < 1 \), since it follows from the argument principle that there exists \( r < 1 \) such that \( g \) is not schlicht on \( |z| = r \).

Consider now the curve \( C(R) \). Clearly \( C(R) \) is nonsimple, and \( g \) is schlicht in \( \{z : |z| < R\} \). Hence we may choose \( w, z_1 = Re^{i\theta_1}, \) and \( z_2 = Re^{i\theta_2} \) (with \( \theta_1 < \theta_2 \)) such that \( g(z_1) = g(z_2) = w \), and such that the curve \( C(R) \) is simple for \( \theta \in (\theta_1, \theta_2) \).
By Lemma 4.1 $g$ is locally schlicht and vanishes only at the origin, so from Theorem 4.2, with $z = Re^{i\theta}$,

$$(\alpha - 1)\int_{\theta_1}^{\theta_2} d\arg g + \int_{\theta_1}^{\theta_2} d\arg zg'(z) > -\pi .$$

However, by the choice of $\theta_1$ and $\theta_2$ we have $\int_{\theta_1}^{\theta_2} d\arg g = 0$, and so

$$(4.6) \int_{\theta_1}^{\theta_2} d\arg zg' > -\pi .$$

But it is clear geometrically that between $\theta_1$ and $\theta_2$ the argument of the tangent vector to $C(R)$ turns back on itself by $\pi$ radians, which contradicts (4.6). Therefore $g$ must be schlicht.

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