THE JACOBIAN OF A GROWTH TRANSFORMATION

Donald Steven Passman
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D. S. PASSMAN

The transformation $T$, described in a paper of Baum and Eagon, is frequently a growth transformation which affords an iterative technique for maximizing certain functions. In this paper, the Jacobian matrix $J$ of $T$ is studied. It is shown, for example, that the eigenvalues of $J$ are real and nonnegative in a large number of cases. In addition, these eigenvalues are considered at critical points of $T$. One necessary assumption used throughout is that the function $P$ to be maximized is homogeneous in the variables involved.

The author would like to thank P. Stebe for a number of stimulating conversations and useful suggestions.

1. Notation. Let $P$ be a function of the variables $x_{ij}$ with domain of definition $D$ given by

$$x_{ij} > 0 \quad \text{and} \quad \sum_j x_{ij} = 1.$$ 

Assume that on this domain both $P$ and all its partial derivatives $\partial P/\partial x_{ij}$ are positive. Moreover we assume that the second partial derivatives of $P$ exist and are continuous. Then the particular transformation $T$ of $P$ which we study here is given by (see [1])

$$T(x_{ij}) = \frac{x_{ij} \partial P/\partial x_{ij}}{\sum_k x_{ik} \partial P/\partial x_{ik}}.$$ 

Clearly $T$ maps $D$ into $D$.

We say that $P$ is row homogeneous if for each $i$ $P$ is homogeneous of degree $w_i > 0$ in the variables $x_{i1}, x_{i2}, \ldots$. In this case (1.1) simplifies by means of Euler's formula and we obtain

$$T(x_{ij}) = \frac{x_{ij} \partial P/\partial x_{ij}}{w_i P}.$$ 

Let us assume now that $P$ is row homogeneous. While the double subscript on the symbol $x_{ij}$ makes the domain $D$ easier to visualize, it turns out that a single subscript makes our later computations neater. Therefore we make the following notational change. Observe that the variables $\{x_{ij}\}$ are not all independent because of the constraints $\sum_j x_{ij} = 1$. Thus in each row of the array $(x_{ij})$ there is one variable which is dependent upon the others. Let us suppose now
that there are a total of \( n' \) variables of which \( n \) are independent. We can then write the variables \( \{x_{ij}\} \) as

\[
x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n'}
\]

where \( x_1, x_2, \ldots, x_n \) are independent and the remaining ones are dependent. Now the set \( \{x_{n+1}, x_{n+2}, \ldots, x_{n'}\} \) clearly contains precisely one variable from each row so we can certainly use the subscripts \( n+1, n+2, \ldots, n' \) to designate these rows. Finally we introduce the function

\[
f: \{1, 2, \ldots, n\} \rightarrow \{n+1, n+2, \ldots, n'\}
\]

so that \( f(i) \) indicates the row containing \( x_i \).

In this single subscripted notation we see that for \( i, j \leq n, x_i \) and \( x_j \) are in the same row if and only if \( f(i) = f(j) \). Thus the fact that the sum of the variables in the row containing \( x_i \) is 1 becomes

\[
1 - x_{f(i)} = \sum_{j=1}^{n} \delta_{f(i)f(j)} x_j \quad \text{for } i \leq n.
\]

Also setting \( y_i = T(x_i) \) equation (1.2) now reads

\[
y_i = \frac{x_i \partial P/\partial x_i}{w_{f(i)} p} \quad \text{for } i \leq n.
\]

Now suppose that \( Q \) is a function of \( x_1, \ldots, x_n \). Then we use as above \( \partial Q/\partial x_i \) to denote the partial derivative of \( Q \) with respect to \( x_i \). On the other hand, \( Q \) can be viewed as a function of the independent variables \( x_1, \ldots, x_n \). If we do this, then we use \( dQ/dx_i \) to denote the partial derivative of \( Q \) with respect to \( x_i \) for \( i \leq n \). It follows from (1.3) and the chain rule that

\[
\frac{dQ}{dx_i} = \frac{\partial Q}{\partial x_i} - \frac{\partial Q}{\partial x_{f(i)}} \quad \text{for } i \leq n.
\]

The Jacobian of the growth transformation \( T \) is the \( n \times n \) matrix

\[
J = \left[ \begin{array}{cccc}
\frac{dy_1}{dx_1} & & & \\
& \ddots & & \\
& & \ddots & \\
& & & \frac{dy_n}{dx_n}
\end{array} \right] \quad i, j \leq n.
\]

It is the matrix which we plan to study.

2. Real eigenvalues. Let \( N \) denote the \( n \times n \) symmetric matrix
\[ N = \left[ \frac{x_i}{w_{f(i)}} (\delta_{ij} - \delta_{f(i)f(j)}x_j) \right]. \]

The interplay of this matrix with \( J \) will prove to be of fundamental importance.

**Lemma 1.** \( N \) is a positive definite matrix.

**Proof.** Since the ordering of the variables \( x_1, x_2, \ldots, x_n \) does not effect the nature of \( N \), we may assume that the variables are grouped together according to the row of the array \( (x_{ij}) \) they are contained in. Then \( N \) clearly becomes a block diagonal matrix with each block corresponding to a row of \( (x_{ij}) \). Since it clearly suffices to show that each of these blocks is a positive definite matrix, it therefore suffices to consider the case in which \( (x_{ij}) \) has only one row. Thus \( n' = n + 1 \) and

\[ w_{n+1}N = [\delta_{ij}x_i - x_ix_j]. \]

Let \( z \) be the real row vector \( z = [z_1, z_2, \ldots, z_n] \neq 0 \) and set

\[
\begin{align*}
  u &= [\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_n}] \\
  v &= [\sqrt{x_1z_1}, \sqrt{x_2z_2}, \ldots, \sqrt{x_nz_n}].
\end{align*}
\]

Then using \((,\)\) for the usual inner product of vectors we have

\[
z(w_{n+1}N)z^T = \sum_{i=1}^{n} x_i z_i^2 - \left( \sum_{i=1}^{n} x_i z_i \right)^2 \\
= (v, v) - (u, v)^2 > (u, u)(v, v) - (u, v)^2 \geq 0
\]

by Cauchy's inequality and the fact that \((u, u) = x_1 + x_2 + \cdots + x_n = 1 - x_{n+1} < 1\).

The lemma is proved.

**Theorem 2.** Let \( P \) be a row homogeneous function. Then

\[ JN = \left[ \frac{x_i}{w_{f(j)}} \frac{\partial y_j}{\partial x_j} \right] \]

is a symmetric matrix.

**Proof.** From (1.4) it is clear that \( y_i \) is row homogeneous of degree zero. Thus Euler's equation yields

\[ x_{f(j)} \frac{\partial y_j}{\partial x_{f(j)}} + \sum_{k=1}^{n} \delta_{f(j)f(k)} x_k \frac{\partial y_j}{\partial x_k} = 0. \]

Let \( JN = [h_{ij}] \). Then by (1.5) and the symmetry of \( N \) we have
Thus (2.2) and (1.3) yield
\[ w_{f(j)}h_{ij} = \sum_{k=1}^{n} \frac{dy_i}{dx_k} \cdot x_k (\delta_{kj} - \delta_{f(k)f(j)x_j}) \]
\[ = x_j \frac{dy_i}{dx_j} - x_j \sum_{k=0}^{n} \delta_{f(k)f(j)x_k} \frac{dy_i}{dx_k} \]
\[ = x_j \left( \frac{\partial y_i}{\partial x_j} - \frac{\partial y_i}{\partial x_{f(j)}} \right) - x_j \sum_{k=1}^{n} \delta_{f(k)f(j)x_k} \frac{\partial y_i}{\partial x_k} \]
\[ + x_j \frac{\partial y_i}{\partial x_{f(j)}} \sum_{k=1}^{n} \delta_{f(k)f(j)x_k} \cdot \]

Therefore we have
\[ h_{ij} = x_j/w_{f(j)} \frac{\partial y_i}{\partial x_j} . \]

It remains to show that \( h_{ij} = h_{ji} \) and to do this we may assume that \( i \neq j \). Then by (1.4)
\[ h_{ij} = \frac{x_i x_j}{w_{f(i)} w_{f(j)} P^2} (\delta^2 P/\partial x_i \partial x_j - (\partial P/\partial x_i)(\partial P/\partial x_j)) \]
so the result clearly follows.

**Corollary 3.** Let \( P \) be a row homogeneous function. Then \( J \) is diagonalizable and all eigenvalues of \( J \) are real.

**Proof.** Write \( JN = A \). Since \( N \) is positive definite we have \( N = QQ^T \) for some real nonsingular matrix \( Q \). Set \( R = Q^{-1} \). Then we have easily
\[ RJR^{-1} = RAR^T . \]

Since \( RAR^T \) is real and symmetric by Theorem 2, it is diagonalizable with all real eigenvalues. Thus (2.3) yields the result.

3. **Critical points.** In this section we study in more detail the nature of \( J \) at a critical point. It follows from (1.4) and (1.5) that at such a point we have \( x_i = y_i \) and \( \partial P/\partial x_i = w_{f(i)} P \). Recall that a critical point is a point at which
\[ \frac{dP}{dx_i} = 0 \quad \text{for } i \leq n . \]
Theorem 4. At a critical point we have

\[ J = I + \frac{N}{P} \left[ \frac{d^2P}{dx_i dx_j} \right] \]

where \( I \) is the \( n \times n \) identity matrix.

Proof. We start with Euler's equation for the row homogeneity of \( P \). For \( i \leq n \) we have

\[ w_{f(i)} P = x_{f(i)} \partial P / \partial x_{f(i)} + \sum_{k=1}^{n} \delta_{f(i),f(k)} x_k \partial P / \partial x_k \]

and differentiating this identity with respect to \( x_j \) yields

\[ w_{f(i)} \frac{dP}{dx_j} = x_{f(i)} \frac{d}{dx_j} \partial P / \partial x_{f(i)} + \sum_{k=1}^{n} \delta_{f(i),f(k)} x_k \frac{d}{dx_j} \partial P / \partial x_k \]

\[ + \delta_{f(i),f(j)} (\partial P / \partial x_j - \partial P / \partial x_{f(i)}) . \]

Observe that the last term is just \( \delta_{f(i),f(j)} dP/dx_j \) so the above becomes

\[ (w_{f(i)} - \delta_{f(i),f(j)}) \frac{dP}{dx_j} = x_{f(i)} \frac{d}{dx_j} \partial P / \partial x_{f(i)} \]

\[ + \sum_{k=1}^{n} \delta_{f(i),f(k)} x_k \frac{d}{dx_j} \partial P / \partial x_k . \]

Now at a critical point \( dP/dx_j = 0 \) so

\[ 0 = x_{f(i)} \frac{d}{dx_j} \partial P / \partial x_{f(i)} + \sum_{k=1}^{n} \delta_{f(i),f(k)} x_k \frac{d}{dx_j} \partial P / \partial x_k . \]

By (1.5) for \( i, j \leq n \)

\[ \frac{d^2P}{dx_i dx_j} = \frac{d}{dx_j} \frac{\partial P}{\partial x_i} - \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(i)}} \]

and substituting

\[ \frac{d}{dx_j} \frac{\partial P}{\partial x_k} = \frac{d^2P}{dx_k dx_j} + \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(k)}} \]

into (3.2) yields

\[ 0 = x_{f(i)} \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(i)}} + \sum_{k=1}^{n} \delta_{f(i),f(k)} x_k \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(k)}} \]

\[ + \sum_{k=1}^{n} \delta_{f(i),f(k)} x_k \frac{d^2P}{dx_k dx_j} . \]

Now clearly
\[ \delta_{f(i)f(k)} x_k \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(k)}} = \delta_{f(i)f(k)} x_k \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(i)}} \]

so (3.4) becomes

\[ 0 = \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(i)}} \left( x_{f(i)} + \sum_{k=1}^{n} \delta_{f(i)f(k)} x_k \right) + \sum_{k=1}^{n} \delta_{f(i)f(k)} x_k \frac{d^2 P}{dx_k dx_j}. \]

Hence by (1.3) we have

\[ (3.5) \]

\[ \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(i)}} = - \sum_{k=1}^{n} \delta_{f(i)f(k)} x_k \frac{d^2 P}{dx_k dx_j}. \]

We now compute \( J \) at the critical point. By (1.4) and (3.1)

\[ \frac{dy_i}{dx_j} = \delta_{ij} \frac{\partial P}{\partial x_i} + \frac{x_i}{w_{f(i)} P} \frac{d}{dx_j} \frac{\partial P}{\partial x_i} \]

\[ = \delta_{ij} + \frac{x_i}{w_{f(i)} P} \frac{d}{dx_j} \frac{\partial P}{\partial x_i} \]

since at a critical point \( \partial P/\partial x_i = w_{f(i)} P \). Thus

\[ (3.6) \]

\[ J = I + \frac{1}{P} \left[ \frac{x_i}{w_{f(i)}} \frac{d}{dx_j} \frac{\partial P}{\partial x_i} \right]. \]

Let \( E = [e_{ij}] \) denote the latter matrix. Then using (3.3) and (3.5) we have

\[ e_{ij} = \frac{x_i}{w_{f(i)}} \left( \frac{d^2 P}{dx_i dx_j} + \frac{d}{dx_j} \frac{\partial P}{\partial x_{f(i)}} \right) \]

\[ = \frac{x_i}{w_{f(i)}} \sum_{k=1}^{n} \left( \delta_{ik} - \delta_{f(i)f(k)} x_k \right) \frac{d^2 P}{dx_k dx_j} \]

and this is the \((i, j)\)th entry in the matrix product

\[ \left[ \frac{x_i}{w_{f(i)}} \left( \delta_{ij} - \delta_{f(i)f(j)} x_j \right) \right] \left[ \frac{d^2 P}{dx_i dx_j} \right]. \]

In view of (3.6), the result follows.

Let \( B \) denote the matrix

\[ (3.7) \]

\[ B = \left[ \frac{d^2 P}{dx_i dx_j} \right]. \]

**COROLLARY 5.** Suppose that at a critical point we have \( \det B \neq 0 \) and let \( \lambda \) be an eigenvalue of \( J \). Then
(i) at a minimum, $\lambda > 1$
(ii) at a maximum, $\lambda < 1$.

Proof. By Theorem 4, $\lambda = 1 + \mu/P$ where $\mu$ is an eigenvalue of $NB$ and by Corollary 3, $\lambda$ is real. Thus it suffices to show that $\mu$ is positive at a minimum and negative at a maximum.

Let $v$ be a real column eigenvector for $\mu$. Then $NBv = \mu v$ yields easily

\[ v^T Bv = \mu (u^T Nu) \]

where $v = Nu$. Since $N$ is positive definite by Lemma 1 we have $u^T Nu > 0$.

Now at a minimum, since $\det B \neq 0$, we see that $B$ is a positive definite matrix. Thus $v^T Bv > 0$ and $\mu > 0$ by (3.8). Similarly at a maximum, $B$ is negative definite so $\mu < 0$. This completes the proof.

4. Polynomials. We assume here that $P$ is a row homogeneous function and use the notation of the preceding sections. In addition, we assume that $P$ is a polynomial with positive coefficients so that (in single subscripted variables)

\[ P = \sum_a m_a \]

where

\[ m_a = e_a x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n'}, \quad e_a > 0 \]

Here, of course, $a$ designates the $n'$-tuple. $a = (a_1, a_2, \cdots, a_n')$. Let $\mathcal{A}$ denote the set of all such $a$'s which occur in $P$.

Fix some ordering of the $a$'s and let $\alpha$ denote a subset $\{a, b\}$ of $\mathcal{A}$ with $b > a$. For each such $\alpha$ set

\[ m_\alpha = m_a m_b, \quad \alpha_i = b_i - a_i. \]

Since $P$ is row homogeneous we have for $i \leq n$

\[ \alpha_{f(i)} + \sum_{j=1}^n \delta_{f(i),f(j)} a_j = w_{f(i)} \]

and hence (4.3) yields

\[ \alpha_{f(i)} + \sum_{j=1}^n \delta_{f(i),f(j)} \alpha_j = 0. \]

For the row homogeneous polynomial $P$ we define the vector subspace $V(P) \subseteq R^{n'}$ to be the subspace of $R^{n'}$ spanned by all $n'$-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_n')$ with $\alpha = (a, b), a, b \in \mathcal{A}$. In view of (4.4) we
have certainly

$$\dim V(P) \leq n.$$ 

**Theorem 6.** Let $P$ be a row homogeneous polynomial. Then $JN$ is a positive semi-definite symmetric matrix with

$$\text{rank } JN = \dim V(P).$$

**Proof.** Let $P$ be given by (4.1) and (4.2). Then by (1.4)

$$w_{f(i)}y_i = \frac{\sum a_i m_a}{\sum m_a}$$

and we have

$$w_{f(i)}P^z x_j \frac{\partial y_i}{\partial x_j} = (\sum a_i m_a) (\sum b_i b_j m_b) - (\sum a_i m_a) (\sum b_i m_b)$$

$$= \sum_{a,b} m_a m_b (b_i b_j - b_i a_j).$$

Observe that the inner summand vanishes at $a = b$. Thus if we sum over $a < b$ then we obtain

$$w_{f(i)}P^z x_j \frac{\partial y_i}{\partial x_j} = \sum_{a,b} m_a m_b (b_i b_j - b_i a_j + a_i a_j - a_i b_j)$$

$$= \sum_a m_a \alpha_i \alpha_j$$

in the notation of (4.3). Thus by Theorem 2

$$(4.5)\quad P^z JN = \left[ \sum_a m_a \alpha_i \alpha_j / w_{f(i)} w_{f(j)} \right].$$

Let $z = [z_1, z_2, \ldots, z_n]$ be a row vector of real entries. Then

$$(4.6)\quad z(P^z JN)z = \sum_{i,j} \sum_a m_a \alpha_i \alpha_j z_i z_j / w_{f(i)} w_{f(j)}$$

$$= \sum_a m_a (\sum_i \alpha_i z_i / w_{f(i)})^2.$$

Thus clearly $P^z JN$ and hence $JN$ is positive semi-definite.

It remains to compute the rank of $JN$. Let $W(P) \subseteq \mathbb{R}^n$ be the subspace of $\mathbb{R}^n$ spanned by all $n$-tuples $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. In view of (4.4) we have clearly

$$(4.7)\quad \dim W(P) = \dim V(P).$$

Let $[ , ]$ be the inner product on $\mathbb{R}^n$ defined by

$$[u_1, u_2, \ldots, u_n], (v_1, v_2, \ldots, v_n] = \sum_{i=1}^n u_i v_i / w_{f(i)}.$$
Then (4.6) becomes
\[ z(P^2JN)z^T = \sum_m a_m [\tilde{a}, z]^2. \]
Thus \( z(P^2JN)z^T = 0 \) if and only if \( z \in W(P)^\perp \), the orthogonal complement of \( W(P) \). Finally since \( P^2JN \) is positive semi-definite we have
\[ \text{rank } P^2JN = n - \text{dim } W(P)^\perp = \text{dim } W(P) \]
and the result follows from (4.7).

**COROLLARY 7.** Let \( P \) be a row homogeneous polynomial. Then all eigenvalues of \( J \) are non-negative real numbers and hence \( \det J \geq 0 \).

*Proof.* This follows immediately from (2.3) and Theorem 6.

Observe that Theorem 6 implies that \( \det J > 0 \) if \( \text{dim } V(P) = n \) and \( \det J = 0 \) otherwise. This fact is an unpublished result of L. Baum.

5. Examples. In this section, we consider a number of examples with \( P \) not homogeneous. Suppose \( y_1 \) is given by

\[ y_1 = \frac{x_i \partial P/\partial x_i}{x_i \partial P/\partial x_1 + x_2 \partial P/\partial x_2}. \]

Then

\[ \frac{1}{1 - y_1} = 1 + \frac{x_i \partial P/\partial x_i}{x_2 \partial P/\partial x_2}. \]

Differentiating with respect to some variable \( x \) then yields

\[ \frac{dy_1}{dx} = (1 - y_1)^2 \frac{d}{dx} \left( \frac{x_i \partial P/\partial x_i}{x_2 \partial P/\partial x_2} \right). \]

This formula enables the following computations to be done easily.

Let \( P(x_1, x_2) = x_1 + x_2^2. \) Then

\[ \det J = \frac{dy_1}{dx_1} = \frac{(1 - y_1)^2}{(x_1 x_2)^2} (2x_1 - 1) \]

and this changes sign at \( x_1 = 1/2. \) Thus Corollary 7 requires that \( P \) be homogeneous.

Now let

\[ P(x_1, x_2, x_3, x_4) = x_1 x_2^2 + x_3^2 x_4 + x_4^2 x_1 \]
subject to the constraints $x_1 + x_3 = 1$, $x_2 + x_4 = 1$. Then

$$J = \begin{bmatrix}
\frac{(1 - y_1)^2 x_2^2}{x_1 x_3^2} (2x_1 - 1) & \frac{(1 - y_1)^2 2x_2}{x_1 x_3} \\
\frac{(1 - y_2)^2 2}{x_4} & \frac{(1 - y_2)^2 2x_1}{x_4^2}
\end{bmatrix}.$$  

Thus

$$\frac{x_4^2}{(1 - y_1)^2 (1 - y_2)^2} \det J = \begin{vmatrix}
x_2^2 & 2x_3 \\
\frac{2x_1}{x_1 x_3} & (2x_1 - 1)
\end{vmatrix}.$$  

Finally let $x_2 \sim 1$ so $x_4 \sim 0$ and the right hand determinant is approximately equal to

$$\frac{2(2x_1 - 1)}{x_1 x_3^2},$$  

which changes sign at $x_1 = 1/2$. Thus we see that even though $P$ is homogeneous, Corollary 7 can still fail unless $P$ is row homogeneous.

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Institute for Defense Analyses
PRINCETON, NEW JERSEY
AND
UNIVERSITY OF WISCONSIN, MADISON
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