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# SOCLE CONDITIONS FOR $QF$-1 RINGS

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**Let $R$ be an associative ring with 1. If $_RM$ is a left $R$-module, then $M$ can be considered as a right $\mathscr{C}$-module, where $\mathscr{C} = \mathrm{Hom}\,(_RM, {}_RM)$ is the centralizer of $_RM$. There is a canonic ring homomorphism $\rho$ from $R$ into the double centralizer $\mathscr{D} = \mathrm{Hom}\,(M_{\mathscr{C}}, M_{\mathscr{C}})$ of $_RM$. For a faithful module $_RM$, the homomorphism $\rho$ is injective, and $_RM$ is called balanced (or to satisfy the double centralizer condition) if $\rho$ is surjective. An artinian ring $R$ is called a $QF$-1 ring if every finitely generated faithful $R$-module is balanced. This definition was introduced by R. M. Thrall as a generalization of quasi-Frobenius rings, and he asked for an internal characterization of $QF$-1 rings.**

**The paper establishes three properties of $QF$-1 rings which involve the left socle and the right socle of the ring; in particular, it is shown that $QF$-1 rings are very similar to $QF$-3 rings. The socle conditions are necessary and sufficient for a (finite dimensional) algebra with radical square zero to be $QF$-1, and thus give an internal characterization of such $QF$-1 algebras. Also, as a consequence of the socle conditions, D. R. Floyd's conjecture concerning the number of indecomposable finitely generated faithful modules over a $QF$-1 algebra is verified. In fact, a $QF$-1 algebra has at most one indecomposable finitely generated faithful module, and, in this case, is a quasi-Frobenius algebra.**

An artinian ring $R$ is called a $QF$-1 ring if every finitely generated faithful $R$-module is balanced. This definition goes back to R. M. Thrall [15] who asked for an internal description of $QF$-1 algebras. The aim of this paper is to prove the following theorem.

THEOREM. *Let $R$ be a $QF$-1 ring with left socle $L$ and right socle $J$. If $e$ and $f$ are primitive idempotents with $f(L \cap J)e \neq 0$, then*

(1) *either $\partial_l Je = 1$ or $\partial_r fL = 1$,*
(2) *we have $\partial_l Le \times \partial_r fJ \leqq 2$, and*
(3) *$\partial_l Le = 2$ implies $Je \subseteqq Le$.*

Here, $\partial_l I$ denotes the length of (a composition series of) the left ideal $I$, whereas $\partial_r K$ denotes the length of the right ideal $K$.

The second socle condition shows that $QF$-1 rings are very similar to $QF$-3 rings, because an artinian ring is a $QF$-3 ring if and only if for every pair $e, f$ of primitive idempotents with $f(L \cap J)e \neq 0$, we have $\partial_l Le = 1 = \partial_r fJ$ (c.f. [15], [8]); however it is known [11], that there are $QF$-1 rings which are not $QF$-3 rings.

For a finite dimensional algebra $R$ with radical square zero, the socle conditions above are necessary and sufficient in order that $R$ is $QF$-1. The proof uses the fact that an algebra with radical square zero which satisfies the second socle condition is of cyclic-cocyclic representation type, as H. Tachikawa [13] has shown. As a consequence, an algebra $R$ with radical square zero is $QF$-1 if and only if $R$ is of cyclic-cocyclic representation type and coincides both with its complete ring of left quotient and its complete ring of right quotients.

Besides an internal description of at least the $QF$-1 algebras with radical square zero, we can derive from the socle conditions the verification of a conjecture concerning the number of indecomposable finitely generated faithful modules over a $QF$-1 algebra. D. R. Floyd has conjectured that for a given $QF$-1 algebra the length of such modules is bounded. J. P. Jans proved this under a rather technical condition on the indecomposable finitely generated modules [10]. Here we show that a $QF$-1 algebra has at most one indecomposable finitely generated faithful module, and, if such a module exists, is even a quasi-Frobenius algebra.

The methods used here are similar to those developed in the joint work with V. Dlab on balanced rings ([3], [4], [5]) and the author would like to thank him for various discussions during the preparation of this paper. Most of it was written while the author was a member of the summer research institute of the Canadian Mathematical Congress in Kingston, Ontario.

**1. Preliminaries.** Throughout the paper, $R$ denotes a ring with unity, $R^*$ its opposite. Algebras are always assumed to be defined over a field and to be finite dimensional. By an $R$-module we understand a unital $R$-module and the symbols ${}_RM$ of $M_R$ will be used to underline the fact that $M$ is a left or a right $R$-module, respectively. Usually left $R$-modules will be considered, but it should be noted that homomorphisms always act from the opposite side as the operators; in particular, every left $R$-module $M$ defines a right $\mathscr{C}$-module, where $\mathscr{C}$ is the endomorphism ring of ${}_RM$. The ring $\mathscr{C}$ is called the centralizer of $M$. The double centralizer $\mathscr{D}$ is the endomorphism ring of $M_{\mathscr{C}}$. Again, $\mathscr{D}$ operates from the opposite side as $\mathscr{C}$, that is from the left. There is a canonic ring homomorphism from $R$ into $\mathscr{C}$; if this homomorphism is surjective, then $M$ is called balanced, or to have the double centralizer property. If every finitely generated faithful (left or right) $R$-module is balanced, then $R$ is called a $QF$-1 ring [15].

Given an $R$-module $M$, denote by $\operatorname{Rad} M$ the intersection of all maximal submodules; $\operatorname{Rad} M$ is the set of all nongenerators. The radical of the ring $R$ is by definition $\operatorname{Rad}_R R$, it will consistently be denoted by $W$ (the radical of $\mathscr{C}$ will be denoted by $\mathscr{W}$). If $R/W$ is artinian, then $WM = \operatorname{Rad} M$ for every left $R$-module $M$. If $\operatorname{Rad} M$ is the only (proper) maximal submodule of $M$, then $M$ is called local; and, if ${}_R R$ (and equivalently $R_R$) is local, then $R$ is called a local ring. Corresponding to the notion of a local module is that of a colocal module. $M$ is called colocal, if $M$ has exactly one minimal submodule. Generally, the union of all minimal submodules of $M$ is the socle $\operatorname{Soc} M$. If $R/W$ is artinian, then $\operatorname{Soc} M = \{m \in M \mid Wm = 0\}$ for every left $R$-module $M$. Considering ${}_R R$, we get the left socle $L = \operatorname{Soc}{}_R R$ of $R$; considering $R_R$, we get the right socle $J = \operatorname{Soc} R_R$ of $R$. Also, we denote by $S$ the intersection of left socle and right socle of the ring $R$. The length of a composition series of a left ideal $I$ will be denoted by $\partial_l I$; similarly, $\partial_r K$ denotes the length of the right ideal $K$.

If $e$ is an idempotent of $R$, then $Re$ will be considered as a left $R$-module. It is well-known that for two idempotents $e$ and $e'$ the morphisms $Re \to Re'$ (i.e. the $R$-homomorphisms) can be identified with the elements in $eRe'$. In particular, the endomorphism ring of $Re$ is given by $eRe$. If the idempotent $e$ is primitive, then $eRe$ is a local ring and $eWe$ its radical. If $R$ is a (left and right) artinian ring, then $1 = \sum_{i=1}^{n} \sum_{j=1}^{f(n)} e_{ij}$, where the $e_{ij}$'s are primitive and pairwise orthogonal idempotents and $Re_{ij} \cong Re_{kl}$ if and only if $i = k$. The ring $ERE$ with $E = \sum_{i=1}^{n} e_{i1}$ is called a basis subring of $R$. The rings $R$ and $ERE$ are Morita equivalent. An artinian ring $R$ is called a basis ring if it coincides with a basis subring of itself. This is equivalent to the assertion that for orthogonal idempotents $e$ and $e'$, $Re$ and $Re'$ are never isomorphic. Basis rings have several pleasant properties: for any idempotent $e$ we have $eR(1-e) \subseteq W$, and, if $X$ is a simple left $R$-module with $eX \neq 0$, then $(1-e)X = 0$; in particular, $eL$ is a two-sided ideal. Also, the radical of a basis ring $R$ is the set of all nilpotent elements in $R$. In the proof of the socle conditions we will always assume that the ring $R$ is a basis ring. This is possible, because, on the one hand, the property to be a $QF$-1 ring is Morita equivalent [12], whereas, on the other hand, the socle conditions are true for $R$ if and only if they are true for a basis subring of $R$.

The left $R$-module $M$ is called indecomposable, if $M$ cannot be written as the direct sum of two proper submodules. If $M$ is indecomposable and of finite length, then the centralizer $\mathscr{C}$ of $M$ is a local ring. Moreover, if $\mathscr{W}$ denotes the radical of $\mathscr{C}$, then there exists a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i\mathscr{W} \subseteqq M_{i-1}$ for all $i$ [1]. Thus,

$$M_1 \subseteqq \operatorname{Soc} M_{\mathscr{C}} \text{ and } M\mathscr{W} = \operatorname{Rad} M_{\mathscr{C}} \subseteqq M_{n-1} .$$

It should be stressed that, for a local ring $R$ with radical $W$, $R/W$ is a division ring. Thus, the semisimple modules over a local ring behave like vector spaces. In particular, applying this to the centralizer $\mathscr{C}$ of an indecomposable module $M$ of finite length, there exists to every element $m \in M\backslash M\mathscr{W}$ and $x \in \operatorname{Soc} M_{\mathscr{C}}$ a $\mathscr{C}$-homomorphism of the form

$$M_{\mathscr{C}} \xrightarrow{\varepsilon} M/M\mathscr{W} \longrightarrow \operatorname{Soc} M_{\mathscr{C}} \xrightarrow{\iota} M_{\mathscr{C}} ,$$

(where $\varepsilon$ denotes the canonic epimorphism and $\iota$ the inclusion) mapping $m$ onto $x$. This will be used frequently throughout the paper, and in similar cases, $\varepsilon$ and $\iota$ will always denote the canonic morphisms.

Two other useful tools which are by now well-known shall be mentioned here. The first is Morita's criterion for faithful modules to be balanced. Let $M$ and $N$ be two left $R$-modules. Then $M$ is said to generate $N$, if the images of all morphisms $M \to N$ generate $N$; and $M$ is said to cogenerate $N$, if the intersection of the kernels of all morphisms $N \to M$ is zero. With there definitions we can formulate:

Morita's criterion [11]: *Let $M$ be faithful and balanced, and let $N$ be indecomposable. Then, $M \oplus N$ is balanced if and only if $M$ either generates or cogenerates $N$.*

The second method to be mentioned here is the trivial extension of morphisms. Assume, $M$ and $N$ are left $R$-modules, $\mathscr{C}$ is the centralizer of $M$, and $M'$ and $M''$ are $\mathscr{C}$-submodules of $M$. Assume also, that there is defined a $\mathscr{C}$-homomorphism $\psi$ of the form

$$M_{\mathscr{C}} \xrightarrow{\varepsilon} M/M' \longrightarrow M'' \xrightarrow{\iota} M_{\mathscr{C}} .$$

We want to extend $\psi$ to an element of the double centralizer of $M \oplus N$.

Trivial extension: *If the image of every $R$-homomorphism $N \to M$ is contained in $M'$, and if $M''$ is contained in the kernel of every $R$-homomorphism $M \to N$, then $\begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix}: M \oplus N \to M \oplus N$ defines an element of the double centralizer of $M \oplus N$.*

The proofs are omitted; they may be found in several papers dealing with double centralizers. Some other definitions and remarks which will be needed only in §7, will be given there.

**2. Construction of indecomposable modules.** An essential tool in the study of $QF$-1 rings are indecomposable modules. Here, we prove that certain amalgamations of two principal indecomposable modules are indecomposable.

LEMMA. *Let $R$ be left artinian with left socle $L$ and right socle $J$. Let $e_1$, $e_2$ and $f$ be primitive idempotents such that $e_1$ and $e_2$ are either equal or orthogonal. Let $a_1 \in f(L \cap J)e_1$ and $a_2 \in fJe_2$ be nonzero elements with $a_1R \cap a_2R = 0$. Then*

$$M = (Re_1 \oplus Re_2)/(R(a_1, a_2))$$

*is an indecomposable left $R$-module.*

*Proof*. We may suppose that $R$ is a basis ring. For, without loss of generality, we may assume that $e_1 = e_2$, if $Re_1$ is isomorphic to $Re_2$, and, similarly, that $f = e_1$ or $f = e_2$, if $Rf$ is isomorphic to $Re_1$ or to $Re_2$, respectively. But then there exists a basis subring $R_0$ of $R$, containing all the elements $e_1, e_2, f, a_1$ and $a_2$. We may apply the lemma to $R_0$ and the Morita equivalence of $R_0$ and $R$ gives the result for $R$.

First, let us assume that $e_1$ and $e_2$ are orthogonal. The endomorphisms of $M$ are induced by matrices $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ with entries $r_{ij} \in e_iRe_j$ for $1 \leqq i, j \leqq 2$. The fact that $R$ is a basis ring implies that both $r_{12}$ and $r_{21}$ belong to $W$, because $e_1$ and $e_2$ are orthogonal. If $(r_{ij})$ induces an endomorphism of $M$, then there exists $\lambda \in R$ with

$$(a_1, a_2)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = (a_1r_{11}, a_2r_{22}) = (\lambda a_1, \lambda a_2) .$$

Here we have used that $a_1$ and $a_2$ are elements of $J$. We want to show that $r_{11} \in W$ if and only if $r_{22} \in W$. If $r_{11} \in W$, then $\lambda a_1 = a_1r_{11} = 0$, because $a_1 \in J$. But since $a_1$ and $a_2$ both are in $fR$, we may assume that $\lambda \in Rf$. Then, $\lambda a_1 = 0$ implies $\lambda \in W$. The equation $\lambda a_2 = a_2r_{22}$ implies $\lambda^n a_2 = a_2r_{22}^n$ for all natural $n$. Thus, the nilpotency of $\lambda$ shows that $a_2r_{22}^n = 0$ for some $n$, and $r_{22}$ cannot be a unit in $eRe$. Conversely, assume that $r_{22} \in W$. Then we conclude from $\lambda a_2 = a_2r_{22} = 0$ that $\lambda f \in W$ which in turns implies $a_1r_{11} = \lambda a_1 = 0$, because $a_1 \in L$, and thus $r_{11} \in W$. As a consequence, an endomorphism of $M$ is either nilpotent (if the corresponding matrix has only entries in $W$) or is an isomorphism (if the elements $r_{11}$ and $r_{22}$ of the corresponding matrix are units in $e_1Re_1$ and $e_2Re_2$, respectively). This shows that the endomorphism ring of $M$ is a local ring, and therefore, $M$ is indecomposable.

Next, we assume that $e_1 = e_2$ and denote it simply by $e$. In the case where also $a_2$ belongs to $L$, a length-counting arguement gives the result. For, if there is a proper direct decomposition of $M$, then $\partial_l(M/\operatorname{Rad} M) = 2$ implies that there are elements $x$ and $y$ in $M$ with $M = Rx \oplus Ry$. We may assume $x = ex$ and $y = ey$, because if $x + \operatorname{Rad} M$ and $y + \operatorname{Rad} M$ generate $M/\operatorname{Rad} M$, also $ex + \operatorname{Rad} M$ and $ey + \operatorname{Rad} M$ generate $M/\operatorname{Rad} M$; thus $ex$ and $ey$ generate $M$, and, of course $Rex \cap Rey = 0$. So $Rx$ and $Ry$ are isomorphic to quotient modules of $Re$. In particular, we have either $Rx \approx Re$ or $Ry \approx Re$, because $\partial_l M = 2 \cdot \partial_l Re - 1$. Thus, we have an epimorphism $M \to Re$. But given elements $s_1$ and $s_2$ in $eRe$ such that

$$Re \oplus Re \xrightarrow{\binom{s_1}{s_2}} Re$$

maps $(a_1, a_2)$ into 0, the equality

$$a_1 s_1 + a_2 s_2 = 0$$

implies that $a_1 s_1 = 0 = a_2 s_2$ and both elements $s_1$ and $s_2$ belong to $W$. Therefore, no morphism $M \to Re$ is surjective. This contradiction shows that $M$ is indecomposable.

There only remains the case where $e = e_1 = e_2$ and $a_2$ does not belong to the left socle $L$. Let $c$ be a nonzero element in $Ra_2 \cap L$. Of course, $c$ belongs to $Wa_2$, so we find $w \in W$ with $c = wa_2$.

Again, we will represent the endomorphisms of $M$ by matrices $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$, now with entries in $eRe$. First, let us show $r_{21} \in W$ for any matrix arising in this way. The element $(0, c) = (wa_1, wa_2)$ of $Re \oplus Re$ is mapped under $(r_{ij})$ onto

$$(0, c)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = (cr_{21}, cr_{22}) ,$$

and this element has to belong to $R(a_1, a_2)$. That is, we find $\lambda \in Rf$ with

$$(cr_{21}, cr_{22}) = (\lambda a_1, \lambda a_2) .$$

If we assume that $r_{21}$ is a unit in $eRe$, then $\lambda a_1 = cr_{21} \neq 0$, so $f\lambda$ is a unit in $fRf$. But $f\lambda a_2 = f c r_{22}$ belongs to $L$, and $a_2 = fa_2 \notin L$, so $f\lambda$ cannot be a unit in $fRf$.

Next, let us show that $r_{22} \notin W$ implies $r_{11} \notin W$. For, assuming $r_{11} \in W$ and applying $(r_{ij})$ to $(a_1, a_2)$, we get $\lambda' \in Rf$ with

$$(a_1, a_2)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = (0, a_1 r_{12} + a_2 r_{22}) = (\lambda' a_1, \lambda' a_2) .$$

Now $\lambda' a_1 = 0$ implies that $\lambda' \in W$, so we have the relation

$$a_2 r_{22} = \lambda' a_2 - a_1 r_{12} \in W a_2 + L .$$

If $r_{22}$ is a unit in $eRe$, then we have

$$a_2 \in W a_2 R + L .$$

But then we also have $a_2 \in W^n a_2 R + L$ for all natural $n$, so $a_2 \in L$, because $W$ is nilpotent. This contradicts the assumption on $a_2$.

Now we want to derive that for any natural $n$ there is $r_n \in eRe$ with

$$(a_1, a_2)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}^n = (a_1 r_{11}^n, a_1 r_n + a_2 r_{22}^n) .$$

For $n = 1$, we take $r_n = r_{12}$ and note that $a_2 r_{21} = 0$, because $a_2 \in J$ and $r_{21} \in W$. If we assume the equality for $n$, we get

$$\begin{aligned} (a_1, a_2)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}^{n+1} &= (a_1 r_{11}^n, a_1 r_n + a_2 r_{22}^n)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \\ &= (a_1 r_{11}^{n+1}, a_1(r_{11}^n r_{12} + r_n r_{22}) + a_2 r_{22}^{n+1}) , \end{aligned}$$

using that both $a_1, a_2 \in J$ and $r_{21} \in W$.

Since $(a_1, a_2)$ is always mapped into $R(a_1, a_2)$, we find for any natural $n$ an element $\lambda_n \in Rf$ with

$$(a_1 r_{11}^n, a_1 r_n + a_2 r_{22}^n) = (\lambda_n a_1, \lambda_n a_2) .$$

As a consequence, if $r_{11} \notin W$, then also $r_{22} \notin W$. For, $\lambda_n a_1 = a_1 r_{11}^n \neq 0$ implies that $\lambda_n$ does not belong to $W$, so $f\lambda_n$ is a unit in $fRf$. But if $r_{22} \in W$, then $r_{22}$ is nilpotent, and thus, for some $n$, we have $a_1 r_n = \lambda_n a_2$ and also $a_1 r_n = f\lambda_n a_2$. Since $a_1$ belongs to the left socle $L$, and $a_2 \notin L$, we conclude that $f\lambda_n$ cannot be a unit in $fRf$.

So we have shown that $r_{11} \in W$ if and only if $r_{22} \in W$. Consequently, an endomorphism of $M$ induces on $M/\mathrm{Rad}\, M = Re/We \oplus Re/We$ either a nilpotent endomorphism (if the elements $r_{11}$, $r_{21}$ and $r_{22}$ of the corresponding matrix are in $W$) or an isomorphism (if for the corresponding matrix, $r_{21} \in W$ but $r_{11}$ and $r_{22}$ both are units in $eRe$). Thus, the endomorphism of $M$ itself is either nilpotent or an isomorphism. This proves the lemma in the remaining case.

## 3. The third socle condition.

We assume throughout § 3, 4 and 5 that $R$ is a basis ring with left socle $L$, right socle $J$ and that $e$ and $f$ are primitive idempotents with $f(L \cap J)e \neq 0$ We denote by $S$ the intersection $S = L \cap J$ of left socle and right socle. Also we will assume that $R$ is a $QF$-1 ring.

Our first aim is to prove the third socle condition, or formally more general, we show that $\partial_l Le \geqq 2$ implies $Je \subseteqq Le$. This we will use in the proof of the second socle condition, from which it then follows that always $\partial_l Le \leqq 2$. Also it should be noted that in this section the assumption of the existence of $f$ with $f(L \cap J)e \neq 0$ is irrelevant; if $f(L \cap J)e = 0$ for all primitive idempotents $f$, then, trivially, $Je \subseteqq Le$.

Let us now assume $\partial_l Le \geqq 2$. We distinguish two cases (which are not mutually exclusive).

*Case* 1. *There is a minimal left ideal $C$ contained in $Se$ with $CeRe \not\subseteqq C$.* Thus, we find an element $r \in eRe$ with $Cr \not\subseteqq C$, and $r$ is a unit of $eRe$, because $C$ is contained in $S$. Also, $R/C$ is a faithful left $R$-module.

The elements of the centralizer of $R/C$ can be lifted to elements of the ring

$$T = \{t \in R;\, Ct \subseteqq C\} .$$

Because $C \subseteqq Se$, the radical $W$ is contained in $T$. Also, if $t_1$ and $t_2$ are elements of $eRe$ with $t_1 t_2 = e$, then $t_1$ belongs to $T$ if and only if $Ct_2 = C$.

Let us verify the following inclusion

$$(e + C)eTe \cap (r + C)eTe \subseteqq We .$$

Since $e$ and $C$ both belong to $T$, we have $(e + C)T \subseteqq T$. So let us assume that $\Sigma(r + c_i)t_i \in T$, with $c_i \in C$ and $t_i \in eTe$. Now, $\Sigma(r + c_i)t_i = r(\Sigma t_i) + \Sigma c_i t_i$ together with $C \subseteqq T$ implies that $r(\Sigma t_i) \in T$. We want to show that $t = \Sigma t_i$ belongs to $W$. If not, then $Ct = C$ and also $Ct' = C$ for $t'$ with $tt' = e$; but since $rt \in T$, we conclude from $Crt \subseteqq C$ that $Cr \subseteqq Ct' = C$. This contradiction shows that $t \in W$. Together with the fact $c_i \in W$ for all $i$, this implies that $\Sigma(r + c_i)t_i = rt + \Sigma c_i t_i \in W$.

Let us denote by $\mathscr{C}$ the centralizer of $Re/C$ and by $\mathscr{W}$ the radical of $\mathscr{C}$. The elements of $\mathscr{C}$ can be lifted to elements of $eTe$ and, in this way, the elements of $\mathscr{W}$ correspond to those in $eWe$. In particular, both $We/C$ and $Je/C$ are $\mathscr{C}$-submodules of $(Re/C)_{\mathscr{C}}$ and $We/C$ contains the radical $(Re/C)\mathscr{W}$ of the $\mathscr{C}$-module $(Re/C)_{\mathscr{C}}$, whereas $Je/C$ is contained in the socle of the $\mathscr{C}$-module $(Re/C)_{\mathscr{C}}$. We may, for arbitrary $x \in Je$, define *a* $\mathscr{C}$-homomorphism $\psi$ of the form

$$(Re/C)_{\mathscr{C}} \xrightarrow{\varepsilon} Re/We \longrightarrow Je/C \xrightarrow{\iota} (Re/C)_{\mathscr{C}}$$

which maps $e + C$ onto $0$ and $r + C$ onto $x + C$. This is possible, because $e + We$ and $r + We$ are $\mathscr{C}$-independent, according to the inclusion $(e + C)eTe \cap (r + C)eTe \subseteqq We$ proved above. Now, the image of every $R$-homomorphism $R(1 - e) \to Re/C$ is contained in $We/C$, whereas the kernel of every $R$-homomorphism $Re/C \to R(1 - e)$ contains $Je/C$. This follows from the fact that such morphisms are given by elements in $(1 - e)Re$ and $eR(1 - e)$, respectively, and both $(1 - e)Re$ and $eR(1 - e)$ are contained in $W$. As a consequence, the trivial extension of $\psi$ to $Re/C \oplus R(1 - e)$ belongs to the double centralizer of $Re/C \oplus R(1 - e)$. Because $R$ is a $QF$-1 ring, we find an element $\rho \in R$ which induces this element of the double centralizer. In particular,

$$\rho e \in C \text{ and } \rho r - x \in C .$$

Taking into account that $r = er$, we see that

$$x \in C + Cr \subseteqq Le .$$

This proves the inclusion $Je \subseteqq Le$ in the first case.

*Case* 2. *There is a minimal left ideal $A$ contained in $Re$ with $AeRe \subseteqq A$.* Let $B$ be a minimal left ideal contained in $Re$, different from $A$. It is easy to see that an element $r \in eRe$ with $Ar \subseteqq B$ or with $Br \subseteqq A$ belongs to $W$. For, in the first case, $Ar \subseteqq A \cap B = 0$, thus $r$ cannot be a unit of $eRe$; similarly, in the second case, $r$ cannot be a unit of $eRe$, because otherwise the element $r' \in eRe$ with $rr' = e$ would satisfy $Ar' = B$.

Now let $\mathscr{C}$ be the centralizer of $Re/A$, and $\mathscr{W}$ the radical of $\mathscr{C}$. Both $We/A$ and $Je + A/A$ are $\mathscr{C}$-submodules of $Re/A$, and $We/A$ contains the radical $(Re/A)\mathscr{W}$ of the $\mathscr{C}$-module $(Re/A)_{\mathscr{C}}$, whereas $Je + A/A$ is contained in the socle of $(Re/A)_{\mathscr{C}}$. This follows from the fact that the elements of $\mathscr{W}$ can be lifted to certain elements in $eWe$. As a consequence, we may for arbitrary $x \in Je$ define a $\mathscr{C}$-homomorphism

$$(Re/A)_{\mathscr{C}} \xrightarrow{\varepsilon} Re/We \longrightarrow Je + A/A \xrightarrow{\iota} (Re/A)_C$$

mapping $e + A$ onto $x + A$.

Let us consider the trivial extension of $\psi$ to $Re/A \oplus Re/B \oplus R(1-e)$. Since every $r \in eRe$ with $Br \subseteqq A$ belongs to $W$, we know that the image of every $R$-homomorphism $Re/B \to Re/A$ is contained in $We/A$. Also, the image of every $R$-homomorphism $R(1 - e) \to Re/A$ is contained in $We/A$, because we may lift the morphism to get an element in $(1 - e)Re \subseteqq W$. On the other hand, $Je + A/A$ is contained in the kernel of every $R$-homomorphism $Re/A \to Re/B$ and $Re/A \to R(1 - e)$; for, these morphisms correspond to elements in $eWe$ or $eR(1 - e)$, respectively. Thus, the trivial extension of $\psi$ to $Re/A \oplus Re/B \oplus R(1 - e)$ belongs to the double centralizer of $Re/A \oplus Re/B \oplus R(1 - e)$.

By assumption, $R$ is a $QF$-1 ring, therefore we find an element $\rho \in R$ inducing this element of the double centralizer. In particular,

$$\rho e - x \in A \text{ and } \rho e \in B ,$$

where the last relation follows from the fact that $\rho(Re/B) = 0$. This implies that

$$x \in A + B \subseteqq Le .$$

So we proved the inclusion $Je \subseteqq Le$ also in the second case.

**4. The first socle condition.** We assume, as we have mentioned before, that the basis ring $R$ is a $QF$-1 ring with left socle $L$, right socle $J$ and that $e$ and $f$ are primitive idempotents with $fSe \neq 0$, where $S = L \cap J$. We want to show that either $\partial_l Je = 1$ or $\partial_r fL = 1$; thus, for the contrary, let us assume both $\partial_l Je > 1$ and $\partial_r fL > 1$. First, we prove

$$(1) \qquad \partial_r fSe = 1 .$$

If we assume $\partial_r fSe > 1$, then we find elements $a$ and $b$ in $fSe$ such that $aR$ and $bR$ are independent right ideals. We consider the indecomposable left $R$-module

$$M = (Re \oplus Re)/R(a, b) .$$

Let $\mathscr{C}$ be its centralizer, and $\mathscr{W}$ the radical of $\mathscr{C}$.

The radical $M\mathscr{W}$ of the $\mathscr{C}$-module $M_{\mathscr{C}}$ is contained in $(We \oplus We)/R(a, b)$. Otherwise, either $(e, 0) + R(a, b)$ or $(0, e) + R(a, b)$ would be mapped under some $\varphi \in \mathscr{W}$ onto an element $m \in M\backslash(We \oplus We)/R(a, b)$. Now $m = em$, thus the natural map $Re \xrightarrow{\eta} Rm$ is surjective. The element $m$ together with either $(e, 0) + R(a, b)$ or with $(0, e) + R(a, b)$ generate $M$. Using the fact that $M$ is indecomposable, we see by a length counting argument that $\eta$ has to be an isomorphism. Let us set

$$M' = M\mathscr{W} + (We \oplus We)/R(a, b) .$$

This is a $\mathscr{C}$-submodule of $M$ and it follows from $M \neq M\mathscr{W}$ that we also have $M \neq M'$. Similarly, we form

$$M'' = \operatorname{Soc} M_{\mathscr{C}} \cap (Se \oplus Se)/R(a, b) .$$

It is easy to see that $(Se \oplus Se)/R(a, b)$ is a nonzero $\mathscr{C}$-submodule of $M$, thus it has a nontrivial intersection with $\operatorname{Soc} M_{\mathscr{C}}$. Because both $M/M'$ and $M''$ are nonzero semisimple $\mathscr{C}$-modules, there exists a nonzero $\mathscr{C}$-homomorphism $\psi$ of the form

$$M_{\mathscr{C}} \xrightarrow{\varepsilon} M/M' \longrightarrow M'' \xrightarrow{\iota} M_{\mathscr{C}} .$$

Every $R$-homomorphism $R(1-e)\to M$ maps into $(We\oplus We)/R(a, b)\subseteqq M'$, and every $R$-homomorphism $M\to R(1-e)$ vanishes on $M''\subseteqq(Se\oplus Se)/R(a, b)$. Thus the trivial extension $\psi'$ of $\psi$ vanishes on $Rm\oplus R(1-e)$, because $m\in M\mathscr{W}\subseteqq M'$. The module $Rm\oplus R(1-e)$ is isomorphic to $Re\oplus R(1-e)={}_RR$; thus, if $\psi'$ is induced by multiplication, then $\psi'$ has to be zero. This contradiction proves that $M\mathscr{W}\subseteqq(We\oplus We)/R(a, b)$.

As a consequence, $(Je\oplus Je)/R(a, b)$ is contained in $\operatorname{Soc} M_{\mathscr{C}}$. This follows from the fact that, if we lift the elements of $\mathscr{C}$ to $2\times 2$-matrices with entries in $eRe$, we get for the elements of $\mathscr{W}$ just the matrices with entries in $eWe$.

Both $(We\oplus We)/R(a, b)$ and $(Je\oplus Je)/R(a, b)$ are $\mathscr{C}$-submodules of $M_{\mathscr{C}}$. Thus given an element $x\in Je$, we may define a $\mathscr{C}$-homomorphism $\psi$ of the form

$$M_{\mathscr{C}}\xrightarrow{\varepsilon}(Re\oplus Re)/(We\oplus We)\longrightarrow(Je\oplus Je)/R(a, b)\xrightarrow{\iota}M_{\mathscr{C}}\ ,$$

mapping $(0, e)+R(a, b)$ onto $(x, 0)+R(a, b)$. Using the fact that the image of every $R$-homomorphism $R(1-e)\to M$ is contained in $(We\oplus We)/R(a, b)$ and that $(Je\oplus Je)/R(a, b)$ is contained in the kernel of every $R$-homomorphism $M\to R(1-e)$, we see that the trivial extension of $\psi$ to $M\oplus R(1-e)$ belongs to the double centralizer of $M\oplus R(1-e)$. Therefore, we find an element $\rho\in R$ with

$$(0, \rho e)-(x, 0)\in R(a, b)\ ,$$

in particular, $x\in Ra$. As a consequence, $Je\subseteqq Ra$. But this contradicts the assumption $\partial_l Je>1$. Thus we have proved (1).

$$fSe=fS\ . \tag{2}$$

Assume that we find a primitive idempotent $e'$ orthogonal to $e$, such that $fSe'\neq 0$. Let $a$ be a nonzero element in $fSe$, and $a'$ a nonzero element in $fSe'$. We form $R/R(a+a')$. It is easy to see that $R(a+a')$ is a minimal left ideal which is not twosided. For, $a+a'=f(a+a')$ implies that $R(a+a')\approx Rf/Wf$, so $R(a+a')$ is a minimal left ideal, and if it is twosided, it would contain $(a+a')e=a$ as well as $(a+a')e'=a'$. Thus, $R/R(a+a')$ is a faithful left $R$-module. The elements of the centralizer of $R/R(a+a')$ can be lifted to elements in $R$, and in this way we just get the elements of the ring

$$T=\{t\in R;\ (a+a')t\in R(a+a')\}\ .$$

The right ideals $aR$ and $a'R$ are independent, thus $M=R(e+e')/R(a+a')$ is an indecomposable left $R$-module. The elements of the centralizer $\mathscr{C}$ of $M$ can be lifted to the form $t = (e + e')t(e + e') \in T$. In this way, the elements of the radical $\mathscr{W}$ of $\mathscr{C}$ correspond to elements in $T \cap W$. For, if $t = (e + e')t(e + e') \in T$ induces a nilpotent endomorphism of $R(e + e')/R(a + a')$, then

$$(e + e')t^n \in R(a + a')$$

for some $n$, but then $t^{2n} = (e + e')t^n(e + e')t^n = 0$ and $t \in W$. This implies that

$$M\mathscr{W} \subseteq W(e + e')/R(a + a')$$

and

$$J(e + e')/R(a + a') \subseteq \operatorname{Soc} M_{\mathscr{C}} .$$

Now let $x$ be an element in $Je$. We may define a $\mathscr{C}$-homomorphism $\psi$ of the form

$$M_{\mathscr{C}} \xrightarrow{\varepsilon} R(e + e')/W(e + e') \longrightarrow J(e + e')/R(a + a') \xrightarrow{\iota} M_{\mathscr{C}} ,$$

mapping $e' + R(a + a')$ onto $x + R(a + a')$. Every $R$-homomorphism $R(1 - e - e') \rightarrow M$ maps into $W(e + e')/R(a + a')$, and every $R$-homomorphism $M \rightarrow R(1 - e - e')$ vanishes on $J(e + e')/R(a + a')$; thus, the trivial extension of $\psi$ to $R/R(a + a') = R(e + e')/R(a + a') \oplus R(1 - e - e')$ belongs to the double centralizer of $R/R(a + a')$. This implies that there is an element $\rho \in R$ with

$$\rho e' - x \in R(a + a') .$$

Multiplication from the right by $e$ gives $x \in Ra$. This shows that $Je \subseteq Ra$. But $a \in Je$, so we have proved that $Je = Ra$ is a minimal left ideal. This contradicts the assumption $\partial_l Je > 1$. Thus we have proved (2).

(3) $$Je = fSe .$$

According to (1) and (2), we know that $\partial_r fS = 1$. The assumption $\partial_r fL > 1$ therefore implies the existence of an element $c \in fL\backslash fS$.

We may choose such an element $c \in fL\backslash fS$ which satisfies moreover $cW \subseteq fS$ and either $c = ce$ or $c = c(1 - e)$, also we find $w \in We$ with $0 \neq yw$. The first assertion is easy to see: if we have chosen $c' \in fL\backslash fS$ and $c'W \nsubseteq fS$, then there is a largest integer $n$ such that $c'W^n \nsubseteq fS$, because $W$ is nilpotent. But then every element $c \in c'W^n\backslash fS$ belongs to $fL\backslash fS$ and satisfies $cW \subseteq c'W^{n+1} \subseteq fS$. One of the elements $ce$ and $c(1 - e)$ has the same properties. For the second assertion, take $w' \in W$ with $0 \neq cw'$. Such an element exists, because $c$ does not belong to $J$. Now $cw' \in cW \subseteq fS = fSe$, thus $cw' = cw'e$, and we may take $w = w'e$.

Let $C_1$ be a complement of $fSe \cap RcRe$ in the left $R$-module $RcRe$ that, in the case $c = ce$, contains $c$, and let $C_2 = RcR(1 - e)$. Then $C = C_1 \oplus C_2$ contains $c$ and has the following two properties

$$Ce \subseteq C \text{ and } C \oplus (fSe \cap RcRe) = RcR .$$

Also, $C$ does not contain a nonzero twosided ideal, because the fact $C \subseteq fL$ implies

$$C \cap S = C \cap fS = C \cap fSe = 0 .$$

Consequently, $R/C$ is a faithful left $R$-module.

The elements of the centralizer of $R/C$ can be lifted to the elements of

$$T = \{t \in R; Ct \subseteq C\} .$$

We have the following inclusion

$$R(T \cap W) \subseteq T .$$

For, assume $r \in R$ and $t \in T \cap W$. If $d$ is an arbitrary element of $C$, then $dr \in RcR = C \oplus (fSe \cap RcRe)$ can be written in the form $dr = d' + s$ for some $d' \in C$ and $s \in S$. Now $d't \in C$, because $t \in T$; and $st = 0$, because $s \in S$ and $t \in W$. Therefore, $drt = (d' + s)t = d't \in C$. This shows that $rt \in T$.

Both $e$ and $w$ are elements of $R \backslash R(T \cap W)$ and

$$eT \cap wT \subseteq R(T \cap W) .$$

For, the element $e$ does not belong to $W$, and $R(T \cap W) \subseteq W$. On the other side, $w$ has the property $0 \neq cw \in fSe$ and $fSe \cap C = 0$; thus $w$ does not belong to $T$, but, as we have seen above, $R(T \cap W) \subseteq T$. Because $Ce \subseteq C$, according to the construction of $C$, we have that $e \in T$. This implies that $eT \cap wT \subseteq T \cap W \subseteq R(T \cap W)$.

As an $R$-module,

$$R/C = Re/C_1 \oplus R(1 - e)/C_2 ,$$

and $Re/C_1$ is indecomposable. Let $\mathscr{C}$ be the centralizer of $Re/C_1$ and $\mathscr{W}$ the radical of $\mathscr{C}$. Then $R(T \cap W)e/C_1$ contains $(Re/C_1)\mathscr{W}$, and $Je + C_1/C_1$ is contained in the socle of $(Re/C_1)_{\mathscr{C}}$. Thus we may define for any $x \in Je$ a $\mathscr{C}$-homomorphism $\psi$ of the form

$$(Re/C_1)_{\mathscr{C}} \xrightarrow{\varepsilon} Re/R(T \cap W)e \longrightarrow Je + C_1/C_1 \xrightarrow{\iota} (Re/C_1)_{\mathscr{C}} ,$$

mapping $e + C_1$ onto $0$ and $w + C_1$ onto $x + C_1$. Here we used that the elements $e + R(T \cap W)e$ and $w + R(T \cap W)e$ are right independent.

Because every $R$-homomorphism $R(1-e)/C_2 \to Re/C_1$ maps into $R(T \cap W)e/C_1$, and every $R$-homomorphism $Re/C_1 \to R(1-e)C_2$ vanishes on $Je + C_1/C_1$, we may conclude that the trivial extension of $\psi$ to $ReC_1 \oplus R(1-e)/C_2$ belongs to the double centralizer of $Re/C_1 \oplus R(1-e)/C_2$. Thus, we find an element $\rho \in R$ with

$$\rho e \in C_1,\ \rho w - x \in C_1 \text{ and } \rho(1-e) \in C_2 .$$

The first and the last condition together imply that $\rho$ belongs to $C_1 \oplus C_2 \subseteqq fL$; consequently, the second condition shows that

$$x \in C_1 + fLw \subseteqq fL .$$

Thus, $Je \subseteqq fL$, and therefore $Je = fSe$.

This proves (3).

But applying (1) to the opposite ring $R^*$ of $R$, we get the equality

$$(1)^* \qquad \partial_l fSe = 1 ,$$

and therefore, (3) implies $\partial_l Je = 1$. This contradiction proves the first socle condition.

5. **The second socle condition.** As in the previous sections, we assume that the basis ring $R$ is a $QF$-1 ring with left socle $L$, right socle $J$, and that $e$ and $f$ are primitive idempotents with $fSe \neq 0$, where $S = L \cap J$. The aim of this section is to establish the inequality

$$\partial_l Le \times \partial_r fJ \leqq 2 .$$

First, we are going to show

$$(1) \qquad \partial_l Le \leqq 2 .$$

Assume for the contrary that there are three independent minimal left ideals $A$, $B$ and $C$ in $Re$. Because $Je \neq 0$, we may assume that $A \subseteqq Je$.

There is an element $r \in eRe \backslash W$ with $Br = C$. If not, then all elements $r \in eRe$ with $Br \subseteqq C$ or with $Cr \subseteqq B$ belong to $W$. As a consequence, the image of every $R$-homomorphism $Re/C \oplus R(1-e) \to Re/B$ is contained in $We/B$, and the kernel of every $R$-homomorphism $Re/B \to Re/C \oplus R(1-e)$ contains $Je + B/B$. Let $\mathscr{C}$ denote the centralizer of the $R$-module $Re/B$. The radical elements of $\mathscr{C}$ are induced by elements of $W$, thus, $We/B$ contains the radical of $(Re/B)_{\mathscr{C}}$ and $Je + B/B$ is contained in the socle of $(Re/B)_{\mathscr{C}}$. This shows that exists a $\mathscr{C}$-homomorphism $\psi$ of the form

$$(Re/B)_{\mathscr{C}} \xrightarrow{\varepsilon} Re/We \longrightarrow Je + B/B \xrightarrow{\iota} (Re/B)_{\mathscr{C}} ,$$

mapping $e+B$ onto an element $a+B$ with $0 \neq a \in A$. The trivial extension of $\psi$ onto $Re/B \oplus Re/C \oplus R(1-e)$ is an element of its double centralizer. Because $Re/B \oplus Re/C \oplus R(1-e)$ is a faithful left $R$-module, we get an element $\rho \in R$ with

$$\rho e - a \in B \text{ and } \rho e \in C .$$

This implies that $a \in B + C$, a contradiction to the independence of $A$, $B$ and $C$. Thus, we have shown that there is $r \in eRe\backslash W$ with $Br = C$, and, in particular, the left $R$-module $Re/B$ is faithful.

Let us consider $T = \{t \in R \mid Bt \subseteqq B\}$. Of course, $e$ belongs to $T$, whereas $r$ does not belong to $T$. The radical of $T$ is just $T \cap W$, because if $t \in T\backslash W$, then also $t^{-1}$ belongs to $T$. The elements of $T$ induce by right multiplication just the endomorphisms of ${}_R(Re/B)$; thus we may define a $\mathscr{C}$-homomorphism $\psi$ of the form

$$(Re/B)_{\mathscr{C}} \xrightarrow{\varepsilon} Re/We \longrightarrow Je + B/B \xrightarrow{\iota} (Re/B)_{\mathscr{C}} ,$$

mapping $r+B$ onto an element $a+B$ with $0 \neq a \in A$ and $e+B$ onto $B$. Here we use, that the elements $e + We$ and $r + We$ are independent in $(Re/We)_{\mathscr{C}}$. The trivial extension of $\psi$ onto $Re/B \oplus R(1-e)$ belongs to its double centralizer, because all morphisms between $Re/B$ and $R(1-e)$ are induced by elements of $W$. Thus, we get an element $\rho' \in R$ with

$$\rho' r - a \in B \text{ and } \rho' e \in B .$$

But $r = er$, so $a$ belongs to $Br + B = C + B$, a contradiction. This concludes the proof of (1).

Applying (1) to the opposite ring $R^*$ of $R$, we get

$$(1^*) \qquad \partial_r fJ \leqq 2 .$$

Thus, it remains to show that $\partial_l Le = 2$ yields $\partial_r fJ = 1$. So let us assume that $\partial_l Le = 2$. Our first aim is to prove

$$(2) \qquad fJ \subseteqq Le .$$

Accoding to §3, we know that $Je \subseteqq Le$, and we have to verify that for every primitive idempotent $e'$ which is orthogonal to $e$, we have $fJe' = 0$. So let us assume $fJe' \neq 0$. We distinguish two cases.

*Case* 1. *The socle $Le$ of $Re$ contains a minimal left ideal $Rc$ that is not twosided.* Thus, $R/Rc$ is a faithful left $R$-module. Also $fSe \not\subseteqq Rc$, because $fSe$ is nonzero and a twosided ideal. Let $a$ be an element of $fSe\backslash Rc$, let $b$ be a nonzero element in $fJe'$. Let us consider

$$M = (Re \oplus Re')/R(a, b) .$$

This is an indecomposable left $R$-module and, according to Morita's criterion, $M$ is either generated or cogenerated by $R/Rc$.

First, let us show that the image of every morphism $R/Rc \to M$ is contained in $(We \oplus Re')/R(a, b)$. Such a morphism maps $R(1-e)$ into $(We \oplus Re')/R(a, b)$, so it is enough to consider morphisms $Re/Rc \to M$. Thus, let us assume there are given two elements $r_1 \in eRe$ and $r_2 \in eRe'$ such that

$$Re \xrightarrow{(r_1, r_2)} Re \oplus Re'$$

maps $c$ into $R(a, b)$, that is

$$(cr_1, cr_2) = (\lambda a, \lambda b)$$

for some $\lambda \in R$. Because $a$ and $b$ belong to $fJ$, we may assume that $\lambda \in Rf$. If $u \notin J$, then $r_1$ is not a unit of $eRe$, because $cr_1 = \lambda a \in J$. Thus, in this case, $r_1 \in W$. If $c \in J$, then $\lambda b = cr_2 = 0$, because $r_2 \in eRe' \subseteqq W$. Therefore also $f\lambda b = 0$ and $f\lambda$ is not a unit in $fRf$; so $\lambda = f\lambda + (1-f)\lambda$ has to belong to $W$. But this implies that $cr_1 = \lambda a = 0$, because $a \in S$. Again, $r_1$ is not a unit of $eRe$ and therefore belongs to $W$. This shows that $R/Rc$ does not generate $M$.

Secondly, we prove that every morphism $M \to R/Rc$ maps $(a, 0) + R(a, b)$ into 0. We may restrict to morphisms $M \to Re/Rc$, because every morphism $M \to R(1-e)$ maps $(a, 0) + R(a, b)$ into 0 for trivial reasons. Thus we have given two elements $s_1 \in eRe$ and $s_2 \in e'Re$ such that

$$Re \oplus Re' \xrightarrow{\binom{s_1}{s_2}} Re$$

maps $(a, b)$ into $Rc$, that is

$$as_1 + bs_2 \in Rc\ .$$

But $b \in J$ and $s_2 \in eRe' \subseteqq W$, therefore $bs_2 = 0$ and $as_1 \in Rc$. Since $as_1$ is also the image of $(a, 0)$ under $\binom{s_1}{s_2}$, we conclude that $(a, 0)$ is mapped into $Rc$. This shows that $M$ is not cogenerated by $R/Rc$.

*Case* 2. *The minimal left ideals contained in Re are twosided ideals*. In particular, this implies that $Le = Se$. Let $a$ be a nonzero element in $fSe$, $b$ a nonzero element in $fJe'$, and $c$ an element in $Le \backslash Ra$. Again, we consider the left $R$-module

$$M = (Re \oplus Re')/R(a, b)\ ,$$

but this time we form $M \oplus R(1-e)$. This is a faithful left $R$-module, and, according to Morita's criterion, has to generate or to cogenerate the indecomposable $R$-module $Re/Rc$.

First, let us show that $M \oplus R(1-e)$ does not generate $Re/Rc$. Trivially, any morphism $R(1-e) \to Re/Rc$ maps into $We/Rc$; thus assume there are elements $r_1 \in eRe$ and $r_2 \in e'Re$ such that

$$Re \oplus Re' \xrightarrow{\binom{r_1}{r_2}} Re$$

maps $(a, b)$ into $Rc$. Thus, there is $\lambda \in R$ with

$$ar_1 + br_2 = \lambda c .$$

But $b \in J$ and $r_2 \in e'Re \subseteqq W$, thus $ar_1 = \lambda c$. According to our assumption, $Ra$ is a twosided ideal, and $Ra \cap Rc = 0$, thus $ar_1 = 0$ and $r_1$ is not a unit in $eRe$. This shows that $r_1 \in W$. As a consequence, the image of every morphism $M \oplus R(1-e) \to Re/Rc$ is contained in $We/Rc$.

Secondly, $M \oplus R(1-e)$ does not cogenerate $Re/Rc$. It is enough to show that every morphism $Re/Rc \to M$ and every morphism $Re/Rc \to R(1-e)$ maps $Le/Rc$ into $0$. This is obvious in the second case, because $Le = Se$. Thus, assume there are given elements $s_1 \in eRe$ and $s_2 \in eRe'$ such that

$$Re \xrightarrow{(s_1, s_2)} Re \oplus Re'$$

maps $c$ into $R(a, b)$. That means, we find an element $\mu \in R$ with

$$(cs_1, cs_2) = (\mu a, \mu b) .$$

Because $Rc$ is a twosided ideal and $Rc \cap Ra = 0$, we conclude that $cs_1 = 0$. But this implies that $s_1 \in W$. Trivially, also $s_2 \in W$; thus $Le = Se$ is mapped under $(s_1, s_2)$ into $0$.

In Case 1 as well as in Case 2, the assumption $fJe' \neq 0$ for a primitive idempotent $e'$ orthogonal to $e$, leads to a contradiction. This proves statement (2). Using this assertion, we are able to prove

(3) $$\partial_r fJ = 1 .$$

We know from (2) that $fJ = fSe$. If $Se = Le$, then

$$\partial_l Je \geqq \partial_l Se = 2 ,$$

and the first socle condition implies $\partial_r fL = 1$. But $fJ = fSe \subseteqq fL$, thus also $\partial_r fJ = 1$. Therefore, we may assume that $Se$ is a proper submodule of $Le$. Let $c$ be an element in $Le \backslash Se$. Then $Rc \cap Se = 0$, and $Rc$ is not a twosided ideal, because otherwise it must intersect $Se$ nontrivial. As a consequence, $R/Rc$ is a faithful left $R$-module.

Let us assume $\partial_r fJ > 1$. Then we find elements $a$ and $b$ in $fJ =$

$fSe$, such that $aR$ and $bR$ are independent right ideals of $R$. Let us form the indecomposable left $R$-module

$$M = (Re \oplus Re)/R(a, b) .$$

Morita's criterion implies again, that $R/Rc$ either generates or cogenerates $M$.

First, the image of every morphism $R/Rc \to M$ is contained in $(We \oplus Re')/R(a, b)$. Of course, it is enough to consider morphisms

$$Re \xrightarrow{(r_1, r_2)} Re \oplus Re$$

with

$$(cr_1, cr_2) = (\lambda a, \lambda b)$$

for some $\lambda \in R$, where $r_1$ and $r_2$ both belong to $eRe$. But $c \notin Se$, whereas $cr_1 = \lambda a \in Se$. This shows that $r_1$ is not a unit in $eRe$ and thus belongs to $W$. As a consequence, $R/Rc$ does not generate $M$.

Secondly, every morphism $M \to R(1-e)$ and every morphism $M \to Re/Rc$ maps $(a, 0) + R(a, b)$ into 0. We only have to consider the latter; thus there are two elements $s_1$ and $s_2$ in $eRe$ such that

$$Re \oplus Re \xrightarrow{(s_1, s_2)} Re$$

maps $(a, b)$ into $Rc$. That is,

$$as_1 + bs_2 \in Rc \cap Se = 0 .$$

Now the fact, that $aR \cap bR = 0$ implies that both $r_1$ and $r_2$ belong to $W$. Therefore, $Se \oplus Se$ belongs to the kernel of $(s_1, s_2)$; in particular, $(a, 0)$ is mapped into 0. This proves that $Re/Rc$ does not cogenerate $M$.

The assumption $\partial_r fJ > 1$ has led to a contradiction. This establishes statement (3) and completes the proof of the second socle condition.

**6. Indecomposable faithful modules.** The first application of the socle conditions gives the solution to a problem raised by D. R. Floyd ([6], [10]): whether a $QF$-1 algebra can have many types of indecomposable finitely generated faithful modules. He conjectured that, for a given $QF$-1 algebra, the length of all such modules is bounded. J. P. Jans [10] proved the conjecture under the assumption that the algebra has "large kernels", this is however, a rather technical condition concerning all indecomposable finitely generated modules. Here we are going to prove a stronger version of Floyd's conjecture: not only is the length of all indecomposable finitely generated faithful modules of a $QF$-1 algebra bounded, but there is at most one isomorphism class of such modules. And, proper $QF$-1 algebras ($QF$-1 algebras which are not quasi-Frobenius algebras) don't have any such modules.

THEOREM. *Let $R$ be a $QF$-1 algebra with an indecomposable finitely generated faithful module. Then $R$ is Morita equivalent to a local quasi-Frobenius algebra.*

*Proof.* We may assume that $R$ is a basis ring. Also we may assume that there is a primitive idempotent $e$ with $Je \neq 0$ and $\partial_l Le = 1$, where $L$ is the left socle and $J$ the right socle of $R$. This is a consequence of the second socle condition; for, $L \cap J \neq 0$, so we find primitive idempotents $e$ and $f$ with $f(L \cap J)e \neq 0$, and the second socle condition now implies that either $\partial_l Le = 1$ or $\partial_r fJ = 1$. In the second case, the opposite ring $R^*$ of $R$ satisfies the assumption. But an algebra $R$ has an indecomposable finitely generated faithful modules if and only if $R^*$ has one; also the opposite ring of a local quasi-Frobenius algebra is again a local quasi-Frobenius algebra.

Let $M$ be an indecomposable finitely generated faithful left $R$-module. Let $\mathscr{C}$ be its centralizer and $\mathscr{W}$ the radical of $\mathscr{C}$. First, let us show that there is an element $m = em$ in $M \backslash M\mathscr{W}$ such that $Sem \neq 0$, where $S = L \cap J$. The elements of the form $ex$ and $(1 - e)x$ generate the module $M$ additively, so we may take a minimal generating set $\{x_i; i \in I\}$ of the $\mathscr{C}$-module $M_{\mathscr{C}}$, consisting of elements of the form $x_i = ex_i$ or $x_i = (1 - e)x_i$. The minimality implies that no element $x_i$ belongs to the radical $M\mathscr{W}$ of $M_{\mathscr{C}}$. Every element of $M$ has the form $\Sigma x_i \varphi_i$ with $\varphi_i \in \mathscr{C}$. Therefore, if we assume $Sex_i = 0$ for all $i \in I$, we get

$$Se(\Sigma x_i \varphi_i) = \Sigma Sex_i \varphi_i = 0 .$$

But because $M$ is faithful, we have $SeM \neq 0$. This contradiction implies the existence of some $x_i$ with $Sex_i \neq 0$. According to our construction, we have either $x_i = ex_i$ or $x_i = (1 - e)x_i$. Since the latter is impossible, the element $m = x_i$ satisfies all requirements.

The submodule $Rm$ of $M$ is isomorphic to $Re$. For, the obvious homomorphism $Re \to Rem = Rm$ has trivial kernel; otherwise, the kernel would contain $Le$, since we have $\partial_l Le = 1$. But $m$ satisfies $Sem \neq 0$, therefore $Se$ is not contained in the kernel.

Now, let $f$ be a primitive idempotent with $fS \neq 0$. Because $M$ is faithful, we also have $fSM \neq 0$. The submodule $SM$ of $M$ is contained in the socle $\mathrm{Soc}_R M$ of ${}_R M$, so we have $f \cdot \mathrm{Soc}_R M \neq 0$. It is easy to see that $f \cdot \mathrm{Soc}_R M$ is a $\mathscr{C}$-submodule of $M_{\mathscr{C}}$, thus we have

$$f \cdot \operatorname{Soc}_R M \cap \operatorname{Soc} M_{\mathscr{C}} \neq 0 .$$

Let $s = fs$ be a nonzero element of $f \cdot \operatorname{Soc}_R M \cap \operatorname{Soc} M_{\mathscr{C}}$. We may define a $\mathscr{C}$-homomorphism $\psi$ of the form

$$M_{\mathscr{C}} \xrightarrow{\varepsilon} M/M\mathscr{W} \longrightarrow \operatorname{Soc} M_{\mathscr{C}} \xrightarrow{\iota} M_{\mathscr{C}} ,$$

mapping $m$ onto $s$. The $\mathscr{C}$-homomorphism $\psi$ belongs to the double centralizer of $M$, and is induced by multiplication by some $\rho \in R$, since $R$ is $QF$-1. Thus, we have $s = \rho m$, that is $s \in Rm$. But $s$ also belongs to $\operatorname{Soc}_R M$, so

$$s \in Rm \cap \operatorname{Soc}_R M = \operatorname{Soc} Rm .$$

Using the fact that $s = fs$ and using the isomorphism of $Rm$ and $Re$, we see that $fSe \neq 0$. We therefore have proved that a primitive idempotent $f$ with $fS \neq 0$ also satisfies $fSe \neq 0$. If $1 = \Sigma f_i$, where the $f_i$'s are primitive and orthogonal, then there is only one of the $f_i$'s with $f_iSe \neq 0$, since $\partial_l Se = 1$ and $R$ is a basis ring. For this idempotent, we have $f_iS \neq 0$, whereas $f_jS = 0$ for $j \neq i$. As a consequence, $S = f_iS$. In the following, we will denote this $f_i$ simply by $f$.

Using again the second socle condition, we conclude that $\partial_r fS \leqq 2$. We distinguish two cases.

*Case* 1. *There is a primitive idempotent $e$ with $fS = fSe$, so also $fS = Se$.* According to the first socle condition, either $\partial_r fL = 1$ or $\partial_l Je = 1$, a fortori we have either $fL = fS$ or $Je = Se$.

If we assume $fL = fS$, then we have

$$L = fL = fS = Se \subseteqq Re ,$$

where we use that $S = fS$ implies $L = fL$.

Similar, if we assume $Je = Se$, we have

$$J = Je = Se = fS \subseteqq fR ,$$

where we use that $S = Se$ implies $J = Je$.

In the first case, the whole left socle is contained in $Re$, thus $R = Re$; in the second case, the whole right socle is contained in $fR$, thus $R = fR$. Always we conclude that $R$ is a local ring.

*Case* 2. *There are two primitive idempotents $e_1$ and $e_2$ with $fS = fSe_1 \oplus fSe_2$, and so also $fS = Se_1 \oplus Se_2$.* Since in this case $\partial_r fL > 1$, the first socle condition implies $\partial_l Je_i = 1$, for $i = 1, 2$. In particular, we have $Je_i = Se_i$. It follows from $S = Se_1 \oplus Se_2$ that $J = Je_1 \oplus Je_2$, thus

$$J = Je_1 \oplus Je_2 = Se_1 \oplus Se_2 = fS \subseteqq fR .$$

Again, we conclude from the fact that the whole right socle is contained in $fR$ that $R = fR$. Consequently, $R$ is a local ring.

It is known that a local $QF$-1 algebra is a quasi-Frobenius algebra ([2], [3]), but this is also a consequence of the second socle condition. For, in this case, it shows that either the left socle is simple or the right socle is simple.

7. **$QF$-1 algebras with radical square zero.** As a second application of the socle conditions we give an internal characterization of $QF$-1 algebras with radical square zero. This answers partly the question of R. M. Thrall [15] to determine the class of $QF$-1 algebras "in the language of ideal theory". Until now, only for two other classes of algebras a characterization of those algebras which are $QF$-1 seems to be known: for serial (or "generalized uniserial") algebras [7] and for algebras which are direct sums of full matrix rings over local rings ([2], [3]). In what follows, let us assume that $R$ is a finite dimensional algebra with the radical $W$ and that $W^2 = 0$.

The algebra $R$ is said to be of local-colocal representation type (or of "cyclic-cocyclic" representation type) if every finitely generated module is either local or colocal (a module is colocal if its socle is a minimal submodule). H. Tachikawa [13] has characterized these algebras. Under the assumption $W^2 = 0$ we get that $R$ is of local-colocal representation type if and only if for every pair $e, f$ of primitive idempotents with $fWe \neq 0$, we have

$$\partial_l We \times \partial_r fW \leqq 2 ,$$

and that, in this case, every indecomposable module is of length $\leqq 3$ and either simple, or projective, or injective. Now let us again denote by $L$ the left socle and by $J$ the right socle of $R$. The assumption $W^2 = 0$ implies

$$W \subseteqq L \cap J .$$

Thus, if $R$ satisfies the second socle condition

(2) for primitive idempotents $e$ and $f$ with $f(L \cap J) \neq 0$ we have

$$\partial_l Le \times \partial_r fJ \leqq 2 ,$$

then $R$ is of local-colocal representation type.

In the theory of rings, certain double centralizers are of particular interest. If $M$ is an $R$-module, let us denote by $EM$ the injective envelop of $M$. The double centralizer of the left module $E_R R$ is called the complete ring of left quotients, and $R$ is said to coincide with its complete ring of left quotients if $E_R R$ is balanced. Similarly, $R$ is its complete ring of right quotients if the right module $ER_R$ is balanced.

With these definitions we can formulate the theorem that characterizes the $QF$-1 algebras with $W^2 = 0$. Besides the second socle condition we will also need the other two conditions

(1) for primitive idempotents $e$ and $f$ with $f(L \cap J)e \neq 0$, we have either

$$\partial_l Je = 1 \text{ or } \partial_r fL = 1 ,$$

(3) for every primitive idempotent $e$ with $\partial_l Le = 2$, we have

$$Je \subseteqq Le ,$$

and the condition dual to (3), namely

(3*) for every primitive idempotent $f$ with $\partial_r fJ = 2$, we have

$$fL \subseteqq fJ .$$

It should be noted that the conditions (1) and (2) are self-dual.

THEOREM. *Let $R$ be a finite dimensional algebra with the radical $W$. Assume $W^2 = 0$. Then the following conditions are equivalent:*

(i) *$R$ is a $QF$-1 algebra;*

(ii) *$R$ satisfies* (1), (2), (3) *and* (3*);

(iii) *$R$ is of local-colocal representation type and coincides both with its complete ring of left quotients and its complete ring of right quotients.*

*Proof*. The main theorem of this paper shows that (i) implies (ii). So let us assume (ii). As we have seen above, $R$ is of local-colocal representation type. If we prove that $R$ coincides with its complete ring of left quotients, then the same result holds for the opposite ring of $R$ and $R$ also coincide with its complete ring of right quotients. It is well-known that $R$ coincides with its complete ring of left quotients if and only if $E_R R/_R R$ (that means, of course, $(E_R R)/_R R$) is cogenerated by $E_R R$. The assumption $W^2 = 0$ implies that $E_R R$ is semisimple. Thus, we have to show that $E_R R/_R R$ is cogenerated by $_R R$. Equivalently, we have to show that for every primitive idempotent $e$ with $e(E_R R/_R R) \neq 0$, we have $eL \neq 0$.

So let us assume that $e$ is a primitive idempotent with $e(E_R R/_R R) \neq 0$. If $1 = \Sigma f_i$ where the $f_i$'s are primitive and orthogonal idempotents, then $E_R R/_R R = \bigoplus ERf_i/Rf_i$.

Therefore we find a primitive idempotent $f$ with

$$e(ERf/Rf) \neq 0 .$$

We want to show $e \notin J$. If $Rf$ is a minimal left ideal, then $Rf$ must be isomorphic to a proper submodule of $Re$, thus $fWe \neq 0$. Now $fL$ contains both $f$ and $fWe$, thus $\partial_r fL > 1$. So the socle condition (1) yields $\partial_l Je = 1$. Since $We \neq 0$, we conclude $e \notin J$. Now let us consider the case where $Wf \neq 0$. Because of (2) we have $\partial_l Wf \leqq 2$, and if $\partial_l Wf = 2$ then $ERf = Rf/A \oplus Rf/B$ where $A$ and $B$ are minimal left ideals. So, in the case $\partial_l Wf = 2$, we may assume $e = f$. But $\partial_l Wf = 2$ has, according to (3), the consequence $Jf \subseteqq Lf$, thus $e = f \notin J$. Finally, in the case $\partial_l Wf = 1$ take a primitive idempotent $f'$ with $f'Wf \neq 0$. The injective envelop $ERf$ can be considered as an amalgamation $Rf \oplus Re/R(a, b)$ with elements $a \in f'Wf$, $b \in f'We$. In particular, we have $\partial_r f'L > 1$. This together with $f'We \neq 0$ yields according to (1) that $\partial_l Je = 1$, thus $e \notin J$. But $e \notin J$, obviously, implies $eW \neq 0$ and thus $eL \neq 0$. This concludes the proof (ii) $\rightarrow$ (iii).

It remains to show that (iii) implies (i). First, it is obvious that we may assume that $R$ is a basis ring since both assertions (iii) and (i) are Morita-invariant. Also, we may restrict to rings which are twosided-indecomposable, i.e. rings which cannot be written as the direct sum of two proper twosided ideals. To avoid trivial cases we further assume that $R$ is not a division ring. Thus, in particular, if $f$ is a primitive idempotent with $Wf = 0$ we have $fW \neq 0$.

We assume that $R$ is of local-colocal representation type. So let us mention some consequences of Tachikawa's characterization which we will need in the sequel. If $e$ is a primitive idempotent, then $\partial_l Re \leqq 3$. If $\partial_l Re = 3$, and $A$ is a minimal left ideal contained in $Re$, then $Re/A$ is indecomposable and not projective, thus injective; as a consequence, $ERe = Re/A \oplus Re/B$ where $A$ and $B$ are different minimal left ideals in $Re$. If $\partial_l Re = 2$, and we assume that $Re$ is not injective, let $f$ be a primitive idempotent with $fWe \neq 0$. Then $\partial_r fW = 2$ and we find right independent elements $a \in fWe$ and $b \in fWe'$, where $e'$ is a primitive idempotent not necessarily distinct from $e$. Since $(Re \oplus Re')/R(a, b)$ is indecomposable and not projective, it has to be the injective envelop of $ERe$.

Let us show that, given two primitive idempotents $e$ and $f$ with $Wf = 0$ and $fWe \neq 0$, we may conclude that $Re$ is injective. Because $Wf = 0$, the simple right module $fR/fW$ does not occur as a submodule of $W_R$, and, since $fW \neq 0$, $fR/fW$ is also not projective. Thus, $fR/fW$ does not occur as a submodule of the right socle of $R$. Again using the result that a ring $R$ coincides with its complete ring of right quotients if and only if $ER_R/R_R$ is cogenerated by $ER_R$, we see that $fR/fW$ does not occur as a submodule of $ER_R/R_R$. Now $\partial_l Re = 2$. For, otherwise $\partial_l Re = 3$ and we find left independent elements $a \in fWe$ and $b \in f'We$, with $f'$ a primitive idempotent not necessarily distinct from $f$. Then $(fR \oplus f'R)/(a, b)R$ is injective and, in fact, the injective envelop of $f'R$. Thus $Ef'R/f'R \approx fR/fW$ would occur as a submodule of $ER_R/R_R$. Also, $\partial_r fW = 1$. For, otherwise $\partial_r fR = 3$ and $EfR = fR/A \oplus fR/B$ for some minimal right ideals $A$ and $B$. But this implies that $EfR/fR \approx fR/fW$ occurs as a submodule of $ER_R/R_R$. This shows that $Re$ is injective.

Since $W^2 = 0$, we know that $E_R R/_R R$ is semisimple. Now a simisimple module is called square-free if two isomorphic submodules always coincide, or equivalently, if the homogenious components are of length 1. We claim that $E_R R/_R R$ is square-free. In order to prove this, we will embed $E_R R/_R R$ into $_R R/_R W$, since for a basis ring $R$ the module $_R R/_R W$ is obviously square-free. So let $1 = \Sigma e_i$, where the $e_i$'s are primitive and orthogonal idempotents. If $\partial_l Re_i = 1$, then $We_i = 0$ implies that there is some $j \neq i$ with $e_i We_j \neq 0$. Then, as we have seen above, $Re_j$ is injective. Thus $Re_j$ is the injective hull of $Re_i$ and

$$(*) \qquad ERe_i/Re_i \oplus ERe_j/Re_j \approx Re_j/We_j \oplus 0 = Re_j/We_j \;.$$

If $\sigma_l Re_i = 2$ and $Re_i$ is not injective, then for a primitive idempotent $f$ with $fW_i \neq 0$ we have $\partial_r fW = 2$. Now in the case where $fW = fWe_i$, we have

$$(**) \qquad ERe_i/Re_i \approx Re_i/We_i \;,$$

whereas otherwise we find $j \neq i$ with $fW = fW_i \oplus fWe_j$ and since $ERe_i/Re_i \approx Re_j/We_j$ and $ERe_j/Re_j \approx Re_i/We_i$, we have

$$(***) \qquad ERe_i/Re_i \oplus ERe_j/Re_j \approx Re_i/We_i \oplus Re_j/We_j \;.$$

Finally, for $\partial_l Re_i = 3$ we have $ERe_i = Re_i/A_i \oplus Re_i/B_i$ for minimal left ideals $A_i$ and $B_i$, thus we get again (**). But the three cases (*), (**) and (***) together define the embedding of $E_R R/_R R \approx \bigoplus ERe_i/Re_i$ into $\bigoplus Re_i/We_i \approx {_R R}/_R W$.

In [14], Corollary 3.4, Tachikawa has shown that for a ring $R$ for which $E_R R/_R R$ is semisimple and square-free with $E_R R$ also every module $M$ satisfying $_R R \subseteq M \subseteq E_R R$ is balanced. Since we assume that $R$ coincides with its complete ring of left quotients and since we have proved that $E_R R/_R R$ is semisimple and square-free, we may use this result.

Let us call two left $R$-modules $M$ and $N$ equivalent if there are decompositions $M = \bigoplus M_i$ and $N = \bigoplus N_j$ such that for every $i$ there is some $j$, and conversely, for every $j$ there is some $i$, with $M_i \approx N_j$. If $M$ and $N$ are equivalent, then $M$ is balanced if and only if $N$ is balanced [9]. An $R$-module $X$ is called minimal faithful, if $X$ is faithful but no proper direct summand of $X$ is faithful. We want to show that every finitely generated, minimal faithful left $R$-module is equivalent to a module $M$ with ${}_RR \subseteqq M \subseteqq E_RR$.

Let $X$ be a finitely generated, minimal faithful left $R$-module. Let $X = \bigoplus X_j$ be a decomposition of $X$ into indecomposable modules. Let $1 = \Sigma e_i$, where the $e_i$'s are primitive and orthogonal idempotents. For every $i$, we will construct a module $M_i$ with $Re_i \subseteqq M_i \subseteqq ERe_i$ such that $M_i$ is either isomorphic to one of the modules $X_j$ or to the direct sum $X_j \bigoplus X_{j'}$ of two such modules. Since $M = \bigoplus M_i$ satisfies ${}_RR \subseteqq M \subset E_RR$, we see that $M$ is faithful. Thus we may conclude that every $X_j$ was used in the formation of some $M_i$, and thus is a direct summand of $M$ in the given decomposition; for otherwise we would get a contradiction to the minimality of $X$. As a consequence, $X$ and $M$ are equivalent.

So, let us construct for a given $i$ the module $M_i$. If $\partial_l Re_i = 2$, we find some $j$ with $We_iX_j \neq 0$ since $X$ is faithful. As a consequence, we may embed $Re_i$ into $X_j$. But $X_j$ is indecomposable, thus either isomorphic to $Re_i$ or to $ERe_i$. So either $M_i = Re_i$ or $M_i = ERe_i$ fulfills the requirements. If $\partial_l Re_i = 1$, then we find some $i'$ with $e_i We_{i'} \neq 0$. As we have seen above, $Re_{i'}$ is injective and one of the modules $X_j$ is isomorphic to $Re_{i'}$. So let us take $M_i = ERe_i$ ($\approx Re_{i'} \approx X_j$). Finally, for $\partial_l Re_i = 3$, we look again for $j$ with $We_iX_j \neq 0$. If $X_j$ is of length, 3, then $X_j$ has to be projective and isomorphic to $Re_i$. In this case, take $M_i = Re_i$. If every module $X_j$ with $We_iX_j \neq 0$ is of length 2, then all these modules are injective, and either $Re_i$ is embeddable in $X_i \bigoplus X_i$ for some $i$, or else we find two different $i$ and $i'$ with $Re_i$ embeddable in $X_i \bigoplus X_{i'}$. This means that we take $M_i = ERe_i$. So we have shown that any finitely generated, minimal faithful left $R$-module is equivalent to a module $M$ with ${}_RR \subseteqq M \subseteqq E_RR$ and thus is balanced.

To complete the proof, take an arbitrary finitely generated faithful left $R$-module $Y$. Let

$$Y = X \bigoplus Y_1 \bigoplus \cdots \bigoplus Y_n ,$$

where $X$ is minimal faithful and the modules $Y_i$ are indecomposable. We know that $X$ is balanced. In order to apply Morita's criterion, we have to show that every module $Y_i$ is either generated or cogenerated by $X$. But it can easily be verified that every projective module is cogenerated by $X$, whereas every injective module, and also every simple, but not projective, module is generated by $X$. This proves that $R$ is $QF$-1.

**8. Remarks and examples.** The following remarks try to shed some light on the possibility to improve the socle conditions. It will be shown that the three conditions—or rather the four conditions (1), (2), (3) and (3*)-are independent.

(a) It should be noted that condition (1) cannot be brought in a form similar to (2); this follows from the fact that there are $QF$-1 rings with $\partial_l Je > 2$. Indeed, a serial (or "generalized uniserial") algebra with the Kupisch series

$$1, 2, 3, 3, 3$$

has a primitive idempotent $e$ with $\partial_l Je = 3$ (namely $e = e_5$), and also a primitive idempotent $f$ with $\partial_r fL = 3$ (namely $f = e_1$). On the other hand, it follows from K. R. Fuller's characterization of serial $QF$-1 rings [7], that this algebra is $QF$-1. Similarly, for every natural $n$ we may consider a serial algebra with the Kupisch series

$$1, 2, \cdots n-1, n, n, \cdots n$$

with $n$ primitive and orthogonal idempotents $e_i$ such that $\partial_l Re_i = n$. Such an algebra is $QF$-1 and has idempotents $e$ and $f$ with $\partial_l Je = n$ and $\partial_r fL = n$.

(b) In order to show that the different socle conditions studied in this paper are independent, let us consider the following examples.

First, any $QF$-3 algebra satisfies the conditions (2), (3) and (3*). The ring of all upper-triangular $2 \times 2$-matrices over a field is a $QF$-3 algebra, but does not satisfy condition (1).

Then, let us start with a field and a subfield of index 2, say with the complex numbers $\boldsymbol{C}$ and the reells $\boldsymbol{R}$, and consider the ring $R_0$ of all triangular matrices with entries in $\boldsymbol{C}$ or in $\boldsymbol{R}$ according to

$$\begin{pmatrix} \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{C} \\ 0 & \boldsymbol{R} & \boldsymbol{R} \\ 0 & 0 & \boldsymbol{R} \end{pmatrix}.$$

Let $W_0$ be its radical, and define $R$ as $R = R_0/W_0^2$. It is easy to verify that $\boldsymbol{R}$ satisfies the conditions (1), (2) and (3), but not (3*). Let us remark that it is also an example of an algebra of local-colocal representation type which coincides with its complete ring of left quotients but not with the complete ring of right quotients.

Finally, let $R$ be the subalgebra of the ring of all $8 \times 8$—matrices over some field, generated by the elements

$$e_1 = a_{11} + a_{88},\ e_2 = a_{22} + a_{77},\ e_3 = a_{33} + a_{66},\ e_4 = a_{44} + a_{55}k$$
$$a_{21},\ a_{31},\ a_{41},\ a_{85},\ a_{86},\ a_{87}\ .$$

This algebra satisfies the condition (1), (3) and (3*), but not condition (2). Also, this is the example of an algebra $R$ which is not of local-colocal representation type but which coincides both with its complete ring of left quotients as well as its complete ring of right quotients. A simple example for the latter is of course any local algebra with radical $W$ and $W^2 = 0$ which has two different minimal left ideals which are twosided ideals.

Since all examples mentioned here are algebras with $W^2 = 0$, we see that the conditions in the theorem of §7 are independent.

(c) We have shown that the three socle conditions characterize the $QF$-1 algebras with radical square zero. However, if we drop the assumption on the radical, the assertion does not remain valid. In fact, a serial algebra with the Kupisch series

$$1, 2, 3, 3$$

satisfies all the properties (1), (2), (3) and (3*), but is, according to [7], not a $QF$-1 algebra.

On the other hand, for algebras which are direct sums of full matrix rings over local rings, even the socle condition (2) alone characterizes those which are $QF$-1.

(d) It is well-known that for a quasi-Frobenius ring the left length and the right length coincide. Also, the previously published examples of $QF$-1 algebras were either serial or had the property that every simple module was one dimensional over the ground field; thus, again, the left and the right length of the corresponding basis ring had to coincide. Using the characterization of $QF$-1 algebra with radical square zero, we show that in general the left length and the right length of a basis $QF$-1 algebra need not to be equal.

Let $R_0$ be the ring of all matrices with entries in $\boldsymbol{C}$ and $\boldsymbol{R}$ according to

$$\begin{pmatrix} \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{C} \\ 0 & \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{C} \\ 0 & 0 & \boldsymbol{R} & \boldsymbol{R} \\ 0 & 0 & 0 & \boldsymbol{R} \end{pmatrix}$$

and let $W_0$ be the radical of $R_0$. $R_0$ is an algebra over $\boldsymbol{R}$ and $R = R_0/W_0^2$ satisfies all the conditions (1), (2), (3) and (3*), thus $R$ is a $QF$-1 ring. But we have $\partial_l R = 7$, whereas $\partial_r R = 8$.

Of course, the exceptional rings studies in [4] and [5] are also $QF$-1 rings (but not algebras) for which left length and right length does not coincide.

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