A CLASS OF OPERATORS ON EXCESSIVE FUNCTIONS

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Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^*)$ be a special standard Markov process with state space $(E, \mathcal{G})$ and transition semigroup $(P_t)$. We emphasize here that the $\mathcal{F}_t$ are the usual completions of the natural $\sigma$-fields for the process. In this paper, we associate with certain multiplicative functionals of $X$ operators on the class of excessive functions which are related to the operators $P_M$ but which are a bit unusual in probabilistic potential theory in that they are not generally determined by kernels on $E \times \mathcal{G}$. An application is given to a problem treated by P.-A. Meyer concerning natural potentials dominated by an excessive function.

2. The operator associated with a natural multiplicative functional. By a multiplicative functional of $X$, we mean a progressively measurable process $M$ which satisfies, in addition to the standard conditions ([1], III, (1.1)) the following condition:

\begin{equation}
(2.1) \text{almost surely, } M_t = 0, t \to M_t \text{ is decreasing on } [0, \infty) \text{ and if } S = \inf \{t > 0: M_t = 0\}, \text{ then } t \to M_t \text{ is right continuous on } [0, S), \text{ and } M_t M_s \circ \theta_t = M_{t+S} \text{ a.s. for all } t \geq 0.
\end{equation}

A simple example which illustrates some possibilities is obtained by considering $X$ to be uniform motion to the right on the real line $L$ and $M_t = f(X_t)/f(X_0)$ on $\{f(X_0) > 0\}$, $M_t = 0$ for all $t$ on $\{f(X_0) = 0\}$, where $f$ is a decreasing positive function on the line, $f(0+) = 0$, $f$ is right continuous on $(-\infty, 0)$ and $f(0) \leq f(0-)$. If $M$ is a multiplicative functional, then $S$ is a terminal time and so $M_{t\downarrow [0,S]}(\omega)$ is a multiplicative functional which is right continuous. For a given $M$, the modified functional will be denoted $\tilde{M}$. Let us denote by $E_M$ the set $\{x \in E: P^*_x(S > 0) = 1\} = E_{\tilde{M}}$ and call $M$ exact if $\tilde{M}$ is exact. Note that $M$ and $\tilde{M}$ generate the same resolvent, but not necessarily the same semigroup.

It should be emphasized that one will not have the freedom to replace $M$ by an equivalent multiplicative functional, for the operator to be associated with $M$ will not respect equivalence.

Let $M$ be a given $MF$; for almost all $\omega$, let $(-dM_t(\omega))$ denote the measure on $(0, \zeta(\omega))$ generated by the increasing function $t \to 1 - M_{t\wedge \sigma}(\omega)$. Care should be taken when computing with $(-dM_t)$, since $(-dM_t)$ is generally not the restriction of $(-d\tilde{M})$ to $(0, S)$.

\footnote{The reader is referred to the books of Blumenthal and Getoor [1] and Meyer [2] for unexplained terminology.}

361
DEFINITION 2.2. A multiplicative functional $M$ is called natural if, almost surely, the trajectories $t \to M_t$ and $t \to X_t$ have no common discontinuity on $[0, S)$, and $X_s = X_{s-}$ on $\{M_s < M_{s-}, S < \zeta\}$.

We now associate with a natural MF $M$ an operator $\tilde{P}_M^\alpha$ on the class $\mathcal{G}^\alpha$ of $\alpha$-excessive functions for $X$.

DEFINITION 2.3. If $M$ is a natural MF and $f \in \mathcal{S}^\alpha$, let

$$
\tilde{P}_M^\alpha f(x) = E^x \left\{ \int_{(0, \zeta)} e^{-st} f(X_t)_+ (-dM_t) + e^{-as} f(X_s) M_s \right\}, \quad x \in E_M
$$

$$
= f(x), \quad x \notin E_M.
$$

By $f(X_t)_-$ is meant the left limit of the trajectory $s \to f(X_s)$ at $t$ if $t > 0$, and $f(X_s)$ if $t = 0$. Recall that if $M$ is a right continuous MF, $\alpha \geq 0$ and $\mathcal{S}^\alpha$, one defines $P_M^\alpha f$ by

$$
P_M^\alpha f(x) = E^x \left\{ \int_{(0, \zeta)} e^{-st} f(X_t)(-dM_t) \right\}, \quad x \in E_M
$$

(2.4)

$$
= f(x), \quad x \notin E_M.
$$

One obtains $P_M^\alpha U^\alpha f + V^\alpha f = U^\alpha f$, where $(V^\alpha)$ is the resolvent for the subprocess $(X, M)$ and it follows that if $M$ is exact, $P_M^\alpha g \in \mathcal{G}^\alpha$ for all $g \in \mathcal{S}^\alpha$. If $f \in \mathcal{S}^\alpha$ is regular, in particular if $f = U^\alpha g$ for some $g \in \mathcal{S}^\alpha$, then for $M$ natural, $\tilde{P}_M^\alpha f = P_M^\alpha f$. In general though, the trajectory $t \to f(X_t)$ can jump at the same time as does the trajectory $t \to M_t$ and $\tilde{P}_M^\alpha f$ will differ from $P_M^\alpha f$. Because of the assumption that $X$ is special standard, it follows from [1], IV, (4.21) that $f(T_t)_- \geq f(X_T)$ for any accessible stopping time $T$, and therefore

$$
\tilde{P}_M^\alpha f(x) \geq P_M^\alpha f(x) \text{ for all } x \text{ if } f \in \mathcal{S}^\alpha.
$$

(2.5)

We shall show that $\tilde{P}_M^\alpha f \leq f$ and $\tilde{P}_M^\alpha f \in \mathcal{S}^\alpha$ if $f \in \mathcal{S}^\alpha$. The fact that the action of $\tilde{P}_M^\alpha$ on $\alpha$-potentials is the same as that of $P_M^\alpha$, but that $\tilde{P}_M^\alpha$ may differ from $P_M^\alpha f$ shows that generally, $\tilde{P}_M^\alpha$ is not determined by a kernel on $E \times \mathcal{G}^\alpha$.

The first lemma shows that although it may not be determined by a kernel, $\tilde{P}_M^\alpha$ does respect certain increasing limits. Obviously $\tilde{P}_M^\alpha f \leq \tilde{P}_M^\alpha g$ if $f, g \in \mathcal{S}^\alpha$ and $f \leq g$.

**Lemma 2.6.** If $f \in \mathcal{S}^\alpha$, $\tilde{P}_M^\alpha (f \wedge n)$ increases to $\tilde{P}_M^\alpha f$ as $n \to \infty$.

**Proof.** It suffices to prove that $(f \wedge n)(X_t)_-$ increases to $f(X_t)_-$ for all $t \in (0, \zeta)$, almost surely. If the trajectory $s \to f(X_s)$ is right continuous and has left limits on $(0, \zeta)$, then for each $t < \zeta$, if $f(X_t)_- > \beta$, then there exists $\epsilon > 0$ such that $f(X_t) > \beta$ on $[t - \epsilon, t)$. Therefore, if $n > \beta$, $(f \wedge n)(X_t)_- > \beta$ on $[t - \epsilon, t)$ and hence $(f \wedge n)(X_t)_- \geq \beta$. 

We remark at this point that $\alpha \to \bar{P}_\alpha f(x)$ is right continuous for every fixed choice of $\alpha$, $f$, and $x$.

**Theorem 2.7.** If $M$ is an exact natural MF, $0 \leq \alpha < \infty$ and $f \in \mathcal{A}^\alpha$, then $\bar{P}_\alpha f \leq f$ and $\bar{P}_\alpha f \in \mathcal{A}^\alpha$.

**Proof.** Because of (2.6) it may be assumed that $f$ is bounded. We may also assume $\alpha > 0$, since the case $\alpha = 0$ will follow by a trivial limit argument. Let

$$N_t = M_t, \ t < S$$
$$= M_s, \ t \geq S \text{ on } \{S < \zeta\}$$
$$= M_{\zeta-}, \ t \geq \zeta \text{ on } \{S = \zeta\}.$$ 

One then has $-dN_t = -dM_t$ almost surely, and for $x \in E_M$, $\bar{P}_\alpha f(x) = E^\mu\left\{\int_0^\zeta e^{-s} f(X_t) \ldots \right\}$.

Define a family $\{T_s; 0 < s < 1\}$ of ($\mathcal{F}_t$) stopping times by

$$T_s = \inf \{u > 0: |1 - N_u| > s\}.$$ 

It is clear that $s \to T_s$ is almost surely increasing and right continuous, $T_s = \infty$ a.s. on $\{T_s > S\}$, $\{T_s = 0 \text{ for some } s\} = \{M_{\zeta-} = 0\}$ and $\{T_s \leq S\} = \{T_s < \zeta\}$ almost surely. By the change of variable formula,

$$\int_{(0, \zeta)} e^{-st} f(X_t) \ldots = \int_0^1 e^{-aT_s} f(X_t) \ldots 1_{\{T_s < \zeta\}} ds.$$ 

Let $Z_t = e^{-a(T_s S)} f(X_{T_s S})$. Since $\alpha > 0$,

$$\int_0^1 Z_{T_s} ds = \int_0^1 Z_{T_s} 1_{\{T_s \leq S\}} ds + \int_0^1 Z_{T_s} 1_{\{T_s = \infty\}} ds$$
$$= \int_0^1 e^{-aT_s} f(X_{T_s}) \ldots 1_{\{T_s \leq S\}} ds + \int_0^1 e^{-aT_s} f(X_{T_s}) \ldots 1_{\{T_s = \infty\}} ds$$
$$= \int_{(0, \zeta)} e^{-at} f(X_t) \ldots (-dM_t) + e^{-aT_s} f(X_{T_s}) M_{T_s}.$$ 

Upon checking separately the case $x \in E_M$, one finds

(2.8) $\bar{P}_\alpha f(x) = E^\mu\int_0^1 Z_{T_s} ds, \ x \in E$.

We now need a fact which will be of use at a subsequent point in the proof.

(2.9) For any initial measure $\mu$, the set of $s \in (0, 1)$ for which $T_s$ is a.s. $P^\mu$ equal to an accessible stopping time has full Lebesgue measure.

To demonstrate (2.9), we let
\(I(\omega) = \{\infty\} \cup \{0, \zeta(\omega)\} - \{t \in (0, \zeta(\omega)) : N_{t+\varepsilon}(\omega) < N_t(\omega)\}
\)
for all \(\varepsilon > 0\) and \(N_{t-\varepsilon}(\omega) = N_t(\omega)\) for some \(\varepsilon > 0\).

Obviously \([0, \zeta) - I\) is countable and 
\[
\int_{[0,\zeta) -I} (-dM_t) = 0 \text{ a.s., and consequently } \int_0^1 1_{\{T_s \in I\}} ds = 0 \text{ a.s., by the change of variable formula. If we prove that } T_s \text{ is accessible on } \{T_s \in I\}, \text{ we shall have proven (2.9), for by Fubini,}
\]
\[
0 = E^\mu \int_0^1 1_{\{T_s \in I\}} ds = \int_0^1 P^\mu\{T_s \in I\} ds.
\]
On \(\{T_s = 0\} \cup \{T_s = \infty\}\), \(T_s\) is trivially accessible. It is easy to check that \(\{T_s \in I, 0 < T_s < \zeta\} = \{0 < T_s = T_{s-} < \zeta\}\), and on \(\{T_s \in I, 0 < T_s < \zeta\} \cap \{X_s = X_{s-}\}\), \(T_s\) is accessible by the famous theorem of Meyer, whilst on \(\{T_s \in I, 0 < T_s < \zeta\} \cap \{X_s \neq X_{s-}\}\), \(N_{T_s} = N_{X_{s-}}\) since \(M\) is natural, and it follows that a.s., \(T_{s-} < T_s\) for all \(\varepsilon \in (0, s)\). The accessibility of \(T_s\) on \(\{T_s \in I\}\) is now evident.

To obtain \(\tilde{P}_H \alpha f \leq f\), we invoke (2.8) to see that 
\[
\tilde{P}_H \alpha f = \int_0^1 E^s Z_{T_{s-}} ds,
\]
and conclude by observing that \((Z_t, \mathcal{F}_t, P^s)\) is a bounded non-negative right-continuous supermartingale and that for almost all \(s \in (0, 1)\), \(T_s\) is a.s. \(P^s\) accessible to find 
\[
E^s Z_{T_{s-}} \leq E^s Z_0 = f(x)
\]
for almost all \(s\).

We prove next that \(\tilde{P}_H \alpha f\) is \(\alpha\)-super-mean-valued. It is enough to give a proof in case \(\alpha > 0\). From (2.8) we see that 
\[
\tilde{P}_H \alpha f = E^\mu \int_0^1 E^s e^{-at} E^t X_s Z_{T_{s-}} ds = \int_0^1 E^s e^{-at} Z_{T_{s-} \circ \theta_t} ds.
\]
Our first step is to show

\begin{equation}
\label{eqn:2.10}
P_t \alpha \tilde{P}_H \alpha f(x) \leq \int_0^1 E^s (Z_{t+T_{s-\theta_t}})_- ds, \quad x \in E.
\end{equation}

On \(\{S \geq t + T_{s-\theta_t}\}\), either \(S > t\) or \(S = t\) and \(T_s \circ \theta_t = 0\). It is a matter of checking cases to see that 
\[
e^{-at} Z_{T_{s-} \circ \theta_t} = (Z_{t+T_{s-\theta_t}})_- \text{ on } \{S > t\},
\]
and a.s. on \(\{S = t, T_s \circ \theta_t = 0\}, \)
\[
e^{-at} Z_{T_{s-} \circ \theta_t} = e^{-at} f(X_t) = e^{-at} f(X_{t-}) \leq e^{-at} f(X_t) - = (Z_{t+T_{s-\theta_t}})_-.
\]
Hence 
\[
e^{-at} Z_{T_{s-} \circ \theta_t} \leq (Z_{t+T_{s-\theta_t}})_- \text{ a.s. on } \{S \geq t + T_{s-\theta_t}\}. \quad \text{On } \{S < t + T_{s-\theta_t}\}, \quad (Z_{t+T_{s-\theta_t}})_- = e^{-as} f(X_s), \text{ while}
\]
\[
e^{-at} Z_{T_{s-} \circ \theta_t} \leq e^{-at(T_{s-\theta_t})} f(X_{t+T_{s-\theta_t}})_- \text{ on } \{S < t + T_s \circ \theta_t, T_s \circ \theta_t \leq S \circ \theta_t\},
\]
\[
e^{-at(T_{s-\theta_t})} f(X_{t+T_{s-\theta_t}})_- \text{ on } \{S < t + T_{s-\theta_t}, T_{s-\theta_t} \circ \theta_t > S \circ \theta_t\}.
\]
One sees readily from (2.9) that for fixed \( x, t + T_s \circ \theta_t \) is a.s. \( P^x \) equal to an accessible stopping time for almost all \( s \) and so for almost all choices of \( s \), there exists an increasing sequence \( \{R_n\} \) of stopping times with limit \( t + T_s \circ \theta_t \) such that \( P^x(R_n < t + T_s \circ \theta_t) = 1 \) for every \( n \). Then \( L_n = R_n \wedge (t + S \circ \theta_t) \) increases to \( t + T_s \circ \theta_t \) strictly from below (a.s. \( P^x \)) on \( \{S < t + T_s \circ \theta_t, T_s \circ \theta_t \leq S \circ \theta_t\} \) and \( R_n \) is eventually equal to \( t + S \circ \theta_t \) on \( \{S < t + T_s \circ \theta_t, T_s \circ \theta_t > S \circ \theta_t\} \). One then has

\[
E^x e^{-at} Z_{T_s \circ \theta_t} = E^x \mathbb{1}_{\{S < t + T_s \circ \theta_t\}} + \lim_{n \to \infty} e^{-at} f(X_{L_n}) \mathbb{1}_{\{S < t + T_s \circ \theta_t\}}.
\]

But \( t + S \circ \theta_t \geq S \) a.s. and so \( L_n \geq S \) eventually, a.s., on \( \{S < t + T_s \circ \theta_t\} \) and it follows from the fact that \( \{e^{-at} f(X_t), \mathcal{F}_t, P^x\} \) is a bounded nonnegative right-continuous supermartingale that \( E^x e^{-at} Z_{T_s \circ \theta_t} \leq E^x(Z_{t + T_s \circ \theta_t}) \) for almost all \( s \in (0, 1) \). This proves (2.10).

Now observe that a.s., \( T_s \leq t + T_s \circ \theta_t \) on \( \{T_s \leq S\} \) and \( t + T_s \circ \theta_t > S \) on \( \{T_s > S\} \). For, on \( \{T_s \leq S\} \cap \{M_t > 0\} \),

\[
t + T_s \circ \theta_t = \inf \{u + t: u > 0, N_u \circ \theta_t < 1 - s\}
\]

\[
\geq \inf \{u + t: u > 0, M_u \circ \theta_t < 1 - s\}
\]

\[
= \inf \{v > t: M_v < (1 - s)M_t\}
\]

\[
\geq \inf \{v > 0: M_v < 1 - s\} = T_s,
\]

and on \( \{T_s > S\} \cap \{M_t = 0\} \), \( t \geq S \) so \( T_s \leq S \leq t \leq t + T_s \circ \theta_t \). On \( \{T_s > S\} \cap \{M_t > 0\} \), the same calculation as above gives \( t + T_s \circ \theta_t \geq \inf \{v > 0: M_v < 1 - s\} \) a.s., and so \( t + T_s \circ \theta_t \leq S \) would imply \( T_s \leq S \). On \( \{T_s > S\} \cap \{M_t = 0\}, M_s > 0 \) so \( t > S \) and \( t + T_s \circ \theta_t > S \) almost surely.

For almost all \( s \in (0, 1), T_s \) and \( t + T_s \circ \theta_t \) are (a.s. \( P^x \)) accessible stopping times and it follows simply from the order relation observed above and the fact that \( \{Z_t, \mathcal{F}_t, P^x\} \) is bounded nonnegative supermartingale that \( E^x Z_{t + T_s \circ \theta_t} \leq E^x Z_{T_s \circ \theta_t} \) for almost all \( s \in (0, 1) \), whence \( P^t \tilde{P}_M f(x) \leq \tilde{P}_M f(x) \).

It remains to show \( P^t \tilde{P}_M f(x) \to \tilde{P}_M f(x) \) as \( t \to 0 \). If \( x \in E_M \), then \( X_t \in E_M \) a.s. on \( \{t < S\} \), and so

\[
P^t \tilde{P}_M f(x) = E^x e^{-at} \tilde{P}_M f(X_t)
\]

\[
\geq E^x e^{-at} \mathbb{1}_{\{t < S\}} \mathbb{E}_t \left\{ f(X_{\mathbb{1}_{\{t < S\}}}(-dM_s) + f(X_S)M_s e^{-as} \right\}
\]

\[
= E^x \mathbb{1}_{\{t < S\}} \left\{ f(X_{t+})_{\mathbb{1}_{\{t < S\}}}(-dM_s \circ \theta_t) + f(X_{t+} \circ \theta_t)M_s \circ \theta_t e^{-as \circ \theta_t} \right\}
\]

\[
= E^x \mathbb{1}_{\{t < S\}} M_t \left\{ f(X_{\mathbb{1}_{\{t < S\}}}(-dM_u) + f(X_S)M_s e^{-as} \right\}.
\]
By Fatou's lemma, if \( x \in E_M \)

\[
\lim_{(t_0, t] \to (t_0, 0]} P_t^a \bar{P}_M f(x) \\
\geq E^a \lim_{(t_0, t] \to (t_0, 0]} M_t^{-1}\left\{ \int_{(t_0, t]} e^{-au} f(X_u) (-dM_u) + f(X_0) M_0 e^{-as} \right\} \\
= E^a \left\{ \int_{(0, t]} e^{-au} f(X_u) (-dM_u) + e^{-as} M_0 f(X_0) \right\} = \bar{P}_M^a f(x).
\]

Consequently \( P_t^a \bar{P}_M f(x) \to \bar{P}_M f(x) \) if \( x \in E_M \). On the other hand, if \( x \in E - E_M, P_t^a \bar{P}_M f(x) \geq P_t^a \bar{P}_M f(x) \) which converges as \( t \to 0 \) to \( P_M f(x) = f(x) = \bar{P}_M f(x) \), using exactness of \( \bar{M} \). Our proof is now complete.

3. Application to a problem treated by Meyer. Meyer [3] proved that if \( u \) is a natural potential of \( X, f \in \mathcal{S} \) and \( u \leq f \), and if in addition \( u(X_t)_- \leq f(X_t) \) for all \( t \) such that \( X_t = x \), then \( u = P_M f \) for some exact terminal time \( R \) on a possibly larger sample space. We give here a similar representation using an operator of the type discussed in the preceding section, one advantage being that one may remain on the original sample space, using only the fields \( (\mathcal{F}_t) \), and another being that the last, somewhat unnatural, condition may be dropped.

**Theorem 3.1.** Let \( f \in \mathcal{S} \) be finite off a polar set and let \( u \) be a natural potential such that \( u \leq f \). Then there exists a natural exact MF \( M \) of \( X \) such that \( u = \bar{P}_M f \).

**Proof.** Let \( u = u_B, B \) a natural additive functional. Since \( u \) is finite, \( B \) is a.s. finite on \([0, \zeta] \), and by [1], IV, (4.29), if \( T \) is a stopping time which is accessible on \( A \), then \( B_T - B_{T-} = u(X_T)_- - u(X_T) \) a.s. on \( A \cap \{ T < \zeta \} \). For every \( \varepsilon > 0 \), let

\[
A^\varepsilon_t = \int_{0}^{t} (f(X_s)_- + \varepsilon - u(X_s))^{-1} dB_s,
\]

Clearly \( A^\varepsilon \) is a finite natural AF of \( X \), and if \( T \) is an accessible stopping time, \( A^\varepsilon_T - A^\varepsilon_{T-} = (f(X_T)_- + \varepsilon - u(X_T))^{-1}(u(X_T)_- - u(X_T)) \) a.s. on \( \{ T < \zeta \} \) and so \( A^\varepsilon_T - A^\varepsilon_{T-} < 1 \) for any accessible \( T \). There exists therefore a right continuous natural MF, \( M^\varepsilon \), such that \( S = \zeta \) and

\[
(M^\varepsilon)_-^{-1}(-dM^\varepsilon_t) = dA^\varepsilon_t, \quad t < \zeta.
\]

Let \( C_t = B_t^\varepsilon \), the continuous part of \( B \). Then for \( t < \zeta \)

\[
M^\varepsilon_t = \exp \left\{ \int_{0}^{t} (f(X_s)_- + \varepsilon - u(X_s))^{-1} dC_s \right\} \times \prod_{s \leq t} [1 - (f(X_s)_- + \varepsilon - u(X_s))^{-1} dB_s]
\]

and it is clear that a.s., \( M^\varepsilon_t \) decreases as \( \varepsilon \) decreases for all \( t \geq 0 \).
Let $M_t = \lim_{(t-\varepsilon)\to0} M^\varepsilon_t$, $S = \inf \{ t > 0 : M_t = 0 \}$. We propose to show that $M$ is a MF of the type considered in the second section. Obviously $M$ is adapted, multiplicative, a.s. decreasing, $M_t = 0$, $M_t \circ \theta_t = M_{t+\varepsilon-t} \circ \theta_t$, but it may well happen that $M_s > 0$. Upon taking the monotonic limit as $\varepsilon \to 0$ in the above representation, one sees that

$$
M_t = \exp \left\{ -\int_0^t [f(X_s)_- - u(X_s)]^{-1} dB_s \right\} 
\times \prod \left[ 1 - (f(X_s)_- - u(X_s))^{-1} d\Lambda_f \right]
$$

for all $t < \zeta$, and from (3.2) one finds

$$
S = \inf \left\{ t > 0 : \int_0^t [f(X_s)_- - u(X_s)]^{-1} dB_s = \infty \right\}.
$$

**Remark.** In the product term of (3.2), we take

$$
[f(X_s)_- - u(X_s)]^{-1} d\Lambda_f = 0 \text{ if } \Delta B_s = 0.
$$

It is almost surely true that if $M_t > 0$, $M_t / M_s \leq M^\varepsilon_t / M^\varepsilon_s$ for all $s \leq t$ whence $M^\varepsilon_t \to M_t$ uniformly on $[0, t]$ if $M_t > 0$. The right continuity of $M$ on $[0, S)$ follows immediately.

To see that $M$ is natural, use (3.2) to observe that on $[0, S)$, the only jumps of $M$ must occur at jump times of $B$, and that on $\{M_s < M_s^-, S < \zeta\}$, $\Delta B_s > 0$, implying that $S$ is accessible on $\{M_s^+ > M_s\}$.

The exactness of $M$ is a consequence of [1], III, (5.9) once it is established that if $P^v[S = 0] = 1$, then $E^vM_{v-t} \circ \theta_t \to 0$ as $t \to 0$, for all $v > 0$. However, $\tilde{M} \leq M$ and it is easy to see that $t \to M_{v-t} \circ \theta_t$ is an increasing function. Because of the monotonic convergence of $M^\varepsilon_t$ to $M_t$, it is legal to interchange limits to obtain

$$
\lim_{(t-\varepsilon)\to0} M^\varepsilon_{t-\varepsilon} \circ \theta_t = \lim_{(t-\varepsilon)\to0} \lim_{(t-\varepsilon-\varepsilon)\to0} M^\varepsilon_{t-\varepsilon} \circ \theta_t = \lim_{(t-\varepsilon\to0)} \lim_{(t-\varepsilon-\varepsilon)\to0} M^\varepsilon_{t-\varepsilon} \circ \theta_t = \lim_{(t-\varepsilon\to0)} \lim_{(t-\varepsilon\to0)} M^\varepsilon_t = 0 \text{ a.s. } P^s,
$$

using the exactness of $M^\varepsilon$.

We remark at this point that $f(X_s) = u(X_s)$ a.s. on $\{S < \zeta\}$, for by (3.3), on $\{S < \zeta\}$, either $\Delta B_s > 0$ and $f(X_s)_- = u(X_s)$ or $\Delta B_s = 0$. In the first case, $S$ is accessible on $\{\Delta B_s > 0\}$ and so $f(X_s)_- = u(X_s)$, whence $u(X_s) = f(X_s)$. In case $\Delta B_s = 0$, $t \to A_t = \int_{[0, t]} [f(X_s)_- - u(X_s)]^{-1} dB_s$ is left continuous at $S$. If $A_S = \infty$ and $T_n = \inf \{ t > 0 : A_t \geq n \}$ then $T_n$ increases to $S$ a.s. on $\{S < \zeta\}$ and $T_n < S$ a.s. on $\{0 < S < \zeta\}$. Thus $S$ is accessible on $\{A_S = \infty, 0 < S < \zeta\}$ and a.s. on $\{A_S = \infty, 0 < S < \zeta\}$, lim $\inf_{t \to S-}[f(X_s)_- - u(X_s)] = 0$ which implies $f(X_s)_- = u(X_s)$. But $u(X_s)_- = u(X_s)$ since $\Delta B_s = 0$ and $f(X_s) \leq f(X_s)_-$ since $S$ is accessible. This shows that $u(X_s) = f(X_s)$ a.s. on
\[ \{0 < S < \zeta, A_S = \infty\} \]. On \( \{S < \zeta, A_S < \infty\} \), one sees from (3.3) that 
\[ \lim \inf (t_s+,(f(X_t)_- - u(X_t)) = 0, \] whence \( f(X_S) = u(X_S) \), proving finally that \( u(X_S) = f(X_S) \) a.s. on \( \{S < \zeta\} \).

From (3.2), we find that a.s. on \([0, S)\)
\[ (-dM_t)(f(X_t)_- - u(X_t)) = M_t dB_t \]
and a.s. on \( \{S < \zeta\} \)
\[ (M_S - M_s)f(X_s)_- = (M_S - M_s)u(X_s) + M_s dB_s. \]
Thus
\[
\int_{(0, \zeta)} f(X_t)_-( -dM_t) = \int_{(0, S)} f(X_t)_-( -dM_t) \]
\[ + \int_{(0, S)} [f'(X_t)(M_S - M_s) + f(X_S)M_s] 1_{(S < \zeta)} \]
\[ = \int_{(0, S)} u(X_t)( -dM_t) + \int_{(0, S)} M_t dB_t \]
\[ + \int_{(0, S)} [(M_S - M_s)u(X_s) + M_s dB_s + u(X_s)M_s] 1_{(S < \zeta)} \]
\[ = \int_{s} u(X_t)( -d\tilde{M}_t) + \int_{0}^{\infty} \tilde{M}_t dB_t. \]
Since \( u(X_T)1_{[T < \infty]} = E^*[((B_t - B_T)1_{[T < \infty]} | \mathscr{F}_T] \) for all stopping times \( T \), Meyer’s integration lemma ([2], VII, T.15) applies to give 
\[ E^*[\int_{0}^{\infty} u(X_t)( -d\tilde{M}_t) = E^*[\int_{0}^{\infty} (B_t - B_T)( -d\tilde{M}_t). \]
Thus, for \( x \in E_M \),
\[ \bar{P}_f(x) = E^*[\int_{0}^{\infty} (B_t - B_T)( -d\tilde{M}_t) + \int_{0}^{\infty} \tilde{M}_t dB_t \]
\[ = E^*[B_\omega] + E^*[\int_{0}^{\infty} \tilde{M}_t dB_t - \int_{0}^{\infty} B_t(-d\tilde{M}_t) \]
\[ = u(x) \]
upon integrating by parts.

If \( x \in E_M \), \( \bar{P}_f(x) = f(x) = E^*[f(X_S) = E^*[u(X_S) = u(x) \), and the theorem is completely proven.

4. REMARKS. It is natural to ask for a specification of the class \( \{\bar{P}_f : M \) a natural exact MF\}, for a given \( f \in \mathscr{F} \). The following example shows that although it contains \( f \) and all natural potentials, it need not include all excessive functions dominated by \( f \). Let \( X \) be uniform motion to the right on the real line, let \( f \equiv 1 \) and \( u \equiv 1/2 \). Obviously \( \bar{P}_f(x) = P_{\bar{f}}1(x) \) for all \( x \), and because we can write down (up to equivalence) the form of \( \tilde{M} \), it is a simple matter to check that \( P_{\bar{f}}1 = 1/2 \) has no solution for \( \tilde{M} \).
A particular example of an operator $\bar{P}_M$ which may be of interest is obtained by taking, for a fixed Borel subset $B$ of $E$,

$$M_t = 1_{\{0, T_B < \zeta\}}(t) + 1_{\{t = T_B < \zeta, X_t \neq X_{T_B}\}}.$$

Then $S = \inf \{t > 0: M_t = 0\} = T_B \wedge \zeta$, and using the fact that $S$ is totally inaccessible on $\{X_S \neq X_{S-}, S < \zeta\}$, $P^x\{t = T_B < \zeta, X_t \neq X_{T_B}\} = 0$ for all $t \geq 0$ and $x \in E$. It follows readily that $M$ is a MF satisfying (2.1). Define, for $f \in \mathscr{S}$,

$$\bar{P}_B f(x) = \bar{P}_M f(x) = E^x(f(X_{T_B})); T_B < \zeta, X_{T_B} = X_{T_B-}$$

$$+ E^x(f(X_{T_B})); T_B < \zeta, X_{T_B} \neq X_{T_B-}.$$

Because of Theorem (2.7), $\bar{P}_B f \in \mathscr{S}$ if $f \in \mathscr{S}$.

One simple use of the operator $\bar{P}_B$ is afforded by the following example. Let $B$ be a finely closed Borel subset of $E$ and let $f$ be a uniformly integrable excessive function. Assume that $X$ is a Hunt process. Let $f^B$ be the lower envelope of the family of excessive functions which dominate $f$ on a (variable) neighborhood of $B$. In [1], VI, (2.12)-(2.15), it is shown, under different hypotheses, that $f^B = P_B f$ off a certain exceptional set provided $f$ is “admissible”. However, under the hypotheses given above without assuming $f$ to be admissible, it is a simple matter, using [1], I, (11.3) together with certain facts from [1], VI, (2.12)-(2.15), to obtain $\bar{P}_B f \leq f^B$ everywhere, and $\bar{P}_B f(x) = f^B(x)$ except possibly on $B - B^\ast$. It does not seem to be easy to remove the restrictions imposed above to obtain a general representation of $f^B$.

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Jimmy T. Arnold, Power series rings over Prüfer domains ................. 1
Maynard G. Arsove, On the behavior of Pincherle basis functions .......... 13
Jan William Auer, Fiber integration in smooth bundles ................... 33
George Bachman, Edward Beckenstein and Lawrence Narici, Function algebras
over valued fields .................................................. 45
Gerald A. Beer, The index of convexity and the visibility function .......... 59
James Robert Boone, A note on mesocompact and sequentially mesocompact
spaces ............................................................. 69
Selwyn Ross Caradus, Semiclosed operators ................................ 75
John H. E. Cohn, Two primary factor inequalities .......................... 81
Mani Gagrat and Somashekhar Amrith Naimpally, Proximity approach to
semi-metric and developable spaces ................................ 93
John Grant, Automorphisms definable by formulas .......................... 107
Walter Kurt Hayman, Differential inequalities and local valency .......... 117
Wolfgang H. Heil, Testing 3-manifolds for projective planes .............. 139
Melvin Hochster and Louis Jackson Ratliff, Jr., Five theorems on Macaulay
rings .............................................................. 147
Thomas Benton Hoover, Operator algebras with reducing invariant subspaces . . . 173
James Edgar Keesling, Topological groups whose underlying spaces are separable
Fréchet manifolds .................................................. 181
Frank Leroy Knowles, Idempotents in the boundary of a Lie group ....... 191
George Edward Lang, The evaluation map and EHP sequences ........... 201
Everette Lee May, Jr, Localization of the spectrum ......................... 211
Frank Belsley Miles, Existence of special K-sets in certain locally compact abelian
groups ............................................................. 219
Susan Montgomery, A generalization of a theorem of Jacobson, II ....... 233
T. S. Motzkin and J. L. Walsh, Equilibrium of inverse-distance forces in
three-dimensions .................................................. 241
Arunava Mukherjea and Nicolas A. Tserpes, Invariant measures and the converse
of Haar’s theorem on semitopological semigroups ........................ 251
James Waring Noonan, On close-to-convex functions of order β ........... 263
Donald Steven Passman, The Jacobian of a growth transformation ....... 281
Dean Blackburn Priest, A mean Stieltjes type integral ...................... 291
Joe Bill Rhodes, Decomposition of semilattices with applications to topological
lattices ............................................................. 299
Claus M. Ringel, Socle conditions for QF − 1 rings ......................... 309
Richard Rochberg, Linear maps of the disk algebra ...................... 337
Roy W. Ryden, Groups of arithmetic functions under Dirichlet convolution .... 355
Michael J. Sharpe, A class of operators on excessive functions .......... 361
Erling Stormer, Automorphisms and equivalence in von Neumann algebras .... 371
Philip C. Tonne, Matrix representations for linear transformations on series
analytic in the unit disc ........................................... 385