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Tsuyoshi Andô

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CLOSED RANGE THEOREMS FOR CONVEX SETS AND LINEAR LIFTINGS

T. Ando

Let M be a closed subspace of a Banach space E such that its annihilator $M^{\perp}$ is the range of a projection P. Given a closed convex subset S containing 0, the first problem of this paper is to find a condition for $\tau(S)$ to be closed where τ is the canonical map from E to E/M. Closure is guaranteed if S is splittable in the sense that the polar S^0 coincides with the norm-closed convex hull of $P(S^0) \cup Q(S^0)$, where $Q = 1 - P$. The second problem is to give a condition for existence of a linear map φ, called a linear lifting, from E/M to E such that $\tau\circ\varphi = 1$ and $\varphi\circ\tau(S) \subseteqq S$. A linear lifting exists if and only if M is the kernel of a projection making S invariant. Of special interest is the case where S is a ball or a cone. When the unit ball is splittable, existence of a linear lifting of norm one is guaranteed under suitable conditions on E/M, which are satisfied by separable L_p and $C(X)$ on compact metrizable X. If further E is an ordered Banach space, and if both P and Q are positive, M is shown to be the kernel of a positive projection of norm one.

Though the closed range theorem (Theorem 1) yields immediately an abstract version of the Rudin-Carleson-Bishop theorem on norm-preserving extensions of functions defined on a peak set, in §2 further modification (Theorem 2) is shown to include Gamelin's extension [5] of the Rudin-Carleson-Bishop theorem in abstract form. Recently a different approach to generalization of the Gamelin theorem was made by Alfsen and Hirsberg [1]. In §3 it is indicated how the closed range theorem is applied to give unified proofs for results of Davies [4] and Perdrizet [9] on closedness of a cone in a quotient space and on order-preserving extensions. In §4 an idea of Pełczynski-Michael [8] is further developed for the closed range theorem to produce existence of linear liftings under suitable conditions. The Pełczynski-Michael theorems are generalized in abstract form (Theorems 5 and 6).

1. **Preliminary.** Let E be a real or complex Banach space with unit ball U. E^* and E^{**} are its dual and second dual respectively, and E is always imbedded canonically into E^{**}. $x, y, z, \cdots$ are vectors in E or E^{**} while $f, g, h, \cdots$ are functionals in E^*. For $x \in E^{**}$ and $f \in E^*$ $f(x)$ is used instead of $x(f)$. The weak topology $\sigma(E^*, E)$ on E^* is called the *weak** topology while $\sigma(E^{**}, E^*)$ on E^{**} is the *weak*** topology. For a subset S of E its norm-closure and its weak**

closure (in E^{**}) are denoted by $\bar{S}$ and $S^{\sim}$ respectively.

The *polar* S^0 is defined as the set of all f such that $\mathrm{Re}\, f(x) \leqq 1$ or $f(x) \leqq 1$ on S according as the scalar field is complex or real. When S is a subspace its polar coincides with its *annihilator* $S^{\perp}$ consisting of all f vanishing on it. The following basic facts are used frequently in this paper. Proofs are found, for instance, in [10]. Let S_1 and S_2 be closed convex subsets of E containing 0. $(S_1 \cap S_2)^0$ coincides with the weak* closure of conv $(S_1^0 \cup S_2^0)$ where conv $(\cdot)$ denotes the convex hull. The weak** closure $S_1^{\sim}$ coincides with the polar of S_1^0 in duality $\langle E^{**}, E^*\rangle$ and $S_1 = E \cap S_1^{\sim}$. Thus coincidence $(S_1 \cap S_2)^{\sim} = S_1^{\sim} \cap S_2^{\sim}$ occurs if and only if the weak* and the norm closure of conv $(S_1^0 \cup S_2^0)$ coincide. In some case conv $(S_1^0 \cup S_2^0)$ becomes itself weak* closed. Here the Krein-Smulian theorem is quite useful: conv $(S_1^0 \cup S_2^0)$ is weak* closed if (and only if) $\gamma U^0 \cap \mathrm{conv}\,(S_1^0 \cup S_2^0)$ is weak* closed for every $0 \leqq \gamma < \infty$. If S_1 contains 0 in its interior then S_1^0 is weak* compact and the norm closure of conv $(S_1^0 \cup S_2^0)$ is weak* closed. If S_1 is a subspace or a cone, the weak* closure of conv $(S_1^0 \cup S_2^0)$ is just that of $S_1^{\perp} + S_2^0$. In case both S_1 and S_2 are subspaces, $S_1^{\perp} + S_2^{\perp}$ is weak* closed if and only if $S_1 + S_2$ is norm-closed.

Suppose now that E is a real Banach space provided with a closed proper cone E_+. E_+ gives rise to natural ordering in E under which it becomes the set of all *positive* vectors: $x \leqq y$ means $y - x \in E_+$. In this respect E_+ is called the *positive cone.* The dual positive cone E_+^* is defined as the set of f nonnegative on E_+, or equivalently $E_+^* = -E_+^0$. E is called an *ordered Banach space* if $E = E_+ - E_+$ and if there is $\gamma < \infty$ with $(U - E_+) \cap (U + E_+) \subseteqq \gamma U$. The latter condition is equivalent to that every subset of the form $\{x; y_1 \leqq x \leqq y_2\}$ is norm-bounded. For notational convenience the relation $x \leqq y + \varepsilon$ in an ordered Banach space means that there is $z \geqq 0$ such that $\|z\| < \varepsilon$ and $x \leqq y + z$. An ordered Banach space or its norm is called *regular* if $\|x\| = \inf\{\|y\|; -y \leqq x \leqq y\}$ for every x. A regular norm is monotone on the positive cone in the sense that $0 \leqq x \leqq y$ implies $\|x\| \leqq \|y\|$. An ordered Banach space admits an equivalent regular norm. In fact, the functional $\|x\|_0 = \inf\{\|y\|; -y \leqq x \leqq y\}$ gives a regular norm.

It is known (cf. [2] and [4]) that E is regular if and only if E^* is regular. An ordered Banach space is said to have the *Riesz interpolation property* if for $y_i \geqq x_j$ $(i, j = 1, 2, \cdots, n)$ there is z such that $x_i \leqq z \leqq y_i (i = 1, 2, \cdots, n)$. A regular ordered Banach space is called a *Banach lattice* if it is lattice under the ordering. A Banach lattice has the Riesz interpolation property. It is known (cf. [2] and [4]) that E has the Riesz interpolation property if and only if E^* is a lattice. A continuous linear operator between ordered Banach spaces is called *positive* if it transforms a positive cone into another.

2. Closed range theorems. E is a real or complex Banach space with unit ball U and M is a closed subspace. The canonical map from E to the quotient space E/M is denoted by τ.

Throughout this section it is assumed:

There is a continuous projection P from E^* to $M^\perp$, and Q stands for $1 - P$.

Remark that the adjoint Q^* projects E^{**} onto $M^\sim$ but M is not necessarily range of any projection. S, S_1 and S_2 will denote closed convex subsets of E containing 0. S is said to be *splittable*, or more precisely, P-splittable if its polar S^0 coincides with the norm-closure of $\mathrm{conv}\,(P(S^0) \cup Q(S^0))$.

LEMMA 1. *The following conditions are equivalent.*

(a) *S is splittable.*

(b) *$S^\sim = \{x \in E^{**}; P^*x \in S^\sim$ and $Q^*x \in S^\sim\}$.*

(c) $\theta(f) = \theta(Pf) + \theta(Qf)$ $(f \in E^*)$

where $\theta(f)$ is defined by $\theta(f) = \sup\{\mathrm{Re}\, f(x); x \in S\}$.

Proof. Since the polar of $P(S^0)$(resp. of $Q(S^0)$) in E^{**} coincides with the set $\{x \in E^{**}; P^*x$(resp. $Q^*x) \in S^\sim\}$ equivalence of (a) and (b) is clear (cf. §1).

(b) $\Rightarrow$ (c). Obviously $\theta(f)$ can be defined by

$$\theta(f) = \sup\{\mathrm{Re}\, f(x); x \in S^\sim\}\ .$$

Take x and y in S. Then by (b) $P^*x + Q^*y$ belongs to $S^\sim$ so that

$$\mathrm{Re}\, Pf(x) + \mathrm{Re}\, Qf(y) = \mathrm{Re}\, f(P^*x + Q^*y) \leqq \theta(f)\ ,$$

leading to $\theta(Pf) + \theta(Qf) \leqq \theta(f)$. The reverse inequality is obvious.

(c) $\Rightarrow$ (b). Since the functional θ is nonnegative because of $S \ni 0$, (c) implies $P(S^0) \cup Q(S^0) \subseteqq S^0$. Therefore $S^\sim$ is contained in the set $\{x \in E^{**}; P^*x \in S^\sim$ and $Q^*x \in S^\sim\}$. Take x with P^*x, $Q^*x \in S^\sim$. Then by (c)

$$\mathrm{Re}\, f(x) = \mathrm{Re}\, Pf(P^*x) + \mathrm{Re}\, Qf(Q^*x) \leqq \theta(Pf) + \theta(Qf) = \theta(f)\ .$$

Thus x belongs to the polar of S^0 in E^{**}.

COROLLARY 1. *The unit ball U is splittable if and only if*

$$\|f\| = \|Pf\| + \|Qf\|\ \ (f \in E^*)\ .$$

COROLLARY 2. *If both S_1 and S_2 are splittable and $(S_1 \cap S_2)^\sim = S_1^\sim \cap S_2^\sim$, then $S_1 \cap S_2$ is splittable.*

COROLLARY 3. *A closed subspace (resp. cone) is splittable if and only if its polar is invariant under P (resp. under P and Q).*

Proof. Let N be a closed cone. $P(N^0) \subseteqq N^0$ and $Q(N^0) \subseteqq N^0$ implies $N^0 = P(N^0) + Q(N^0) = \operatorname{conv}(P(N^0) \cup Q(N^0))$. If N is further a subspace, $Q(N^0) \subseteqq N^0$ follows already from $P(N^0) \subseteqq N^0$.

LEMMA 2. *If S_1 and S_2 are splittable, for any $\varepsilon > 0$ and $\rho > 0$ the following inclusion relation holds:*

$$\begin{aligned} &\{S_1 \cap (S_2 + \varepsilon U) + \rho U\} \cap \overline{(S_1 + M)} \cap \overline{(S_2 + M)} \\ &\subseteqq \overline{S_1 \cap (S_2 + \alpha\varepsilon U) + \alpha\rho U \cap M} \end{aligned}$$

where $\alpha = \|Q\|$.

Proof. Take any x in the set on the left hand side. Then it follows by splittability that

$$Q^*x \in S_1^\sim \cap (S_2^\sim + \alpha\varepsilon U^\sim \cap M^\sim) + \alpha\rho U^\sim \cap M^\sim .$$

There is $y \in \alpha\rho U^\sim \cap M^\sim$ such that

$$Q^*(x - y) = Q^*x - y \in S_1^\sim \cap (S_2^\sim + \alpha\varepsilon U^\sim \cap M^\sim)$$

and

$$P^*(x - y) = P^*x \in P^*\overline{(S_1 + M)} \cap P^*\overline{(S_2 + M)} \subseteqq S_1^\sim \cap S_2^\sim .$$

Then by Lemma 1 $x - y \in S_1^\sim$ and there is $z \in \alpha\varepsilon U^\sim \cap M^\sim$ such that $x - y - z \in S_2^\sim$. Finally in view of arguments of §1 x belongs to

$$\begin{aligned} &E \cap \{S_1^\sim \cap (S_2^\sim + \alpha\varepsilon U^\sim) + \alpha\rho U^\sim \cap M^\sim\} \\ &\subseteqq E \cap \{S_1 \cap (S_2 + \alpha\varepsilon U) + \alpha\rho U \cap M\}^\sim \\ &= \overline{S_1 \cap (S_2 + \alpha\varepsilon U) + \alpha\rho U \cap M} . \end{aligned}$$

By definition of the quotient topology $\tau(x)$ belongs to the closure $\overline{\tau(S)}$ if and only if x is contained in $\overline{S + M}$. In particular, $\tau(S)$ is closed if and only if $S + M$ is closed.

LEMMA 3. *Suppose that S_1 and S_2 are splittable. If $\tau(x)$ belongs to $\overline{\tau(S_1)} \cap \overline{\tau(S_2)}$ and $\|x - S_1 \cap S_2\| < \gamma$ there is $y \in S_1$ such that $\tau(x) = \tau(y)$ and $\|x - y\| < \gamma\|Q\|$. In case $\|Q\| = 1$ for any $\varepsilon > 0$ y can be chosen in $S_1 \cap (S_2 + \varepsilon U)$.*

Proof. Let $\alpha = \|Q\|$ and take ε' with $0 < \varepsilon' < \varepsilon$. By hypothesis x is contained in

$$\{S_1 \cap (S_2 + \varepsilon' U) + \gamma' U\} \cap \overline{(S_1 + M)} \cap \overline{(S_2 + M)}$$

for some $\gamma > \gamma' > 0$. Choose $\varepsilon_n > 0$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \gamma - \gamma'$. By Lemma 2 there is $x_0 \in M$ such that $\|x_0\| \leqq \alpha\gamma'$ and $\|x + x_0 - S_1 \cap (S_2 + \alpha\varepsilon' U)\| < \varepsilon_1$. Then

$$x + x_0 \in \overline{S_i + M} + M = \overline{S_i + M}$$

and

$$x + x_0 \in S_1 \cap (S_2 + \alpha\varepsilon' U) + \varepsilon_1 U .$$

Now inductive procedure based on Lemma 2 makes it possible to find a sequence $\{x_n\}$ in M such that $\|x_n\| \leqq \alpha\varepsilon_n$ and $\|x + \sum_{i=0}^{n-1} x_i - S_1 \cap (S_2 + \alpha^n\varepsilon' U)\| < \varepsilon_n$. Since $\sum_{n=0}^{\infty} \|x_n\| < \infty$, $y = x + \sum_{n=0}^{\infty} x_n$ is well defined. Obviously y belongs to S_1, and in case $\alpha = 1$, to $S_1 \cap (\overline{S_2 + \varepsilon' U}) \subseteqq S_1 \cap (S_2 + \varepsilon U)$. Finally $\tau(x) = \tau(y)$ and $\|x - y\| \leqq \alpha \sum_{n=1}^{\infty} \varepsilon_n + \alpha\gamma' < \alpha\gamma$.

Now the main result of this paper is near at hand.

THEOREM 1. *Suppose that the annihilator of a closed subspace M is the range of a projection P and that S, S_1 and S_2 are closed convex subsets containg* 0. *Then* (a) *the image of S under the canonical map τ from E to E/M is closed whenever S is splittable.* (b) *If both S_1 and S_2 are splittable and $(S_1 \cap S_2)^{\sim} = S_1^{\sim} \cap S_2^{\sim}$ then $\tau(S_1) \cap \tau(S_2) = \tau(S_1 \cap S_2)$.* (c) *If both S_1 and S_2 are splittable and $\|1 - P\| \leqq 1$, then the inclusion $\tau(S_1) \cap \tau(S_2) \subseteqq \tau(S_1 \cap (S_2 + \varepsilon U))$ holds for any $\varepsilon > 0$.*

Proof. (a) follows from Lemma 3 with $S_1 = S_2 = S$. Also (c) is a direct consequence of Lemma 3. (b) $S_1 \cap S_2$ is splittable by Corollary 2 and $S_1 \cap S_2 + M$ is closed by (a). Now since by hypothesis

$$\begin{aligned} &P^*\{(S_1 + M) \cap (S_2 + M)\} \subseteqq P^*(S_1) \cap P^*(S_2) \subseteqq S_1^{\sim} \cap S_2^{\sim} \\ &= (S_1 \cap S_2)^{\sim} , \end{aligned}$$

it follows that

$$(S_1 + M) \cap (S_2 + M) \subseteqq E \cap (S_1 \cap S_2 + M)^{\sim} = S_1 \cap S_2 + M ,$$

showing $\tau(S_1) \cap \tau(S_2) \subseteqq \tau(S_1 \cap S_2)$. The reverse inclusion is obvious. This completes the proof.

It follows immediately from Theorem 1 that if the unit ball U is splittable and if N is a closed splittable subspace then $\tau(N \cap U)$ is closed and coincides with $\tau(N) \cap \tau(U)$. Let us show that the same conclusion holds for a non-splittable subspace under suitable conditions.

Since by Corollary 3 splittability of N is characterized by $P(N^{\perp}) \subseteqq N^{\perp} \cap M^{\perp}$, it follows by Corollary 1 that under the splittability of U, N is splittable if and only if

$$\|Qf\| = \|f - M^\perp \cap N^\perp\| \quad (f \in N^\perp) .$$

On the other hand, Corollary 1 implies $\|Qf\| = \|f - M^\perp\|$. Thus if the unit ball is splittable, splittability of N is characterized by

$$\|f - M^\perp \cap N^\perp\| \leqq \|f - M^\perp\| \quad (f \in N^\perp) .$$

LEMMA 4. *Let N_1 and N_2 be closed subspace. Then for $\rho > 0$ the following assertions are equivalent.*

(a) $\|x - N_1 \cap N_2\| \leqq \rho\|x - N_1\|$ $(x \in N_2)$

(b) $\|f - N_1^\perp \cap N_2^\perp\| \leqq \rho\|f - N_2^\perp\|$ $(f \in N_1^\perp)$

Proof. (a) means that

$$N_2 \cap (\overline{N_1 + U}) \subseteqq \overline{N_1 \cap N_2 + \rho U}$$

which implies by polar formation

$$(N_1^\perp + N_2^\perp) \cap U^0 \subseteqq N_2^\perp + N_1^\perp \cap \rho U^0 .$$

The last relation can be converted to

$$N_1 \cap (N_2^\perp + U^0) \subseteqq N_1^\perp \cap N_2^\perp + \rho U^0$$

which is nothing but (b). The reverse process can be pursued because (b) implies that $(N_1^\top + N_2^\perp) \cap U^0$ is weak* compact, hence by the Krein-Smulian theorem that $N_1^\perp + N_2^\perp$ is weak* closed.

COROLLARY 4. *Suppose that the unit ball is splittable. Then the following assertions for a closed subspace N are equivalent.*

(a) *N is splittable.*

(b) $\|x - N \cap M\| \leqq \|x - N\|$ $(x \in M)$

(c) $\|f - N^\perp \cap M^\perp\| \leqq \|f - M^\perp\|$ $(f \in N^\perp)$.

THEOREM 2. *Let M and N be closed subspaces, and suppose that the annihilator $M^\perp$ is the range of a projection P such that*

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*) . \tag{1}$$

If for some $1 \leqq \rho < 2$

$$\|x - N \cap M\| \leqq \rho\|x - N\| \quad (x \in M) , \tag{2}$$

then the images of the unit ball U, N and $N \cap U$ under the canonical map τ from E to E/M are closed and

$$\tau(N \cap U) = \tau(N) \cap \tau(U) .$$

Proof. Closedness of $\tau(U)$ follows from (1) by Corollary 1 and

Theorem 1. Let $Q = 1 - P$. Then in view of (1) relation (2) is converted by Lemma 4 to

$$\|Pf - M^\perp \cap N^\perp\| \leqq \gamma \|Qf\| \quad (f \in N^\perp) \tag{3}$$

where $\gamma = \rho - 1 < 1$. Then for any $f \in N^\perp$ and $g \in M^\perp$.

$$\begin{aligned} \|g - M^\perp \cap N^\perp\| &\leqq \|Pf - M^\perp \cap N^\perp\| + \|g - Pf\| \\ &\leqq \gamma\|Qf\| + \|g - Pf\| \leqq \|g - f\|, \end{aligned}$$

showing

$$\|g - M^\perp \cap N^\perp\| \leqq \|g - N^\perp\| \quad (g \in M^\perp),$$

which is converted by Lemma 4 to

$$\|x - M \cap N\| \leqq \|x - M\| \quad (x \in N). \tag{4}$$

This last relation means that the canonical map from the Banach space $N/M \cap N$ onto $\tau(N)$ has bounded inverse. Therefore $\tau(N)$ is closed. Further (4) implies

$$N \cap (U + M) \subseteqq \overline{U + N \cap M}.$$

Let us prove that really

$$N \cap (U + M) \subseteqq U + N \cap M$$

holds, which is equivalent to the required relation:

$$\tau(N) \cap \tau(U) \subseteqq \tau(N \cap U).$$

Suppose for contradiction that there exists x in $N \cap (U + M)$ with $(U - x) \cap N \cap M = \varnothing$. Since $N \cap M$ is a subspace, it follows that

$$\operatorname{conv}((U - x) \cup \{0\}) \cap N \cap M = \{0\}.$$

Since $\operatorname{conv}((U - x) \cup \{0\})$ is closed, the last relation implies by polar formation that $(U - x)^0 + N^\perp + M^\perp$ is weak* dense in E^*. Weak* closedness of $(U - x)^0 + N^\perp + M^\perp$, if proved, leads to

$$\operatorname{conv}((U^\sim - x) \cup \{0\}) \cap N^\sim \cap M^\sim = \{0\}$$

hence to a contradiction:

$$x \notin E \cap (U^\sim + N^\sim \cap M^\sim) = \overline{U + N \cap M}.$$

Now let us prove the above weak* closedness. To this end, in view of the Krein-Smulian theorem, it suffices to prove that for any $n > 0$

$$\{(U - x)^0 + N^\perp + M^\perp\} \cap nU^0 \subseteqq \delta U^0 \cap (U - x)^0 + \delta U^0 \cap N^\perp + M^\perp$$

where δ is a constant depending on n. Remark that $(U - x)^0$ consists of all f with $\|f\| \leqq \operatorname{Re} f(x) + 1$. Since $x \in U + M$ implies $\|P^*x\| \leqq 1$, it follows from (1) that

$$\begin{aligned} &1 + \operatorname{Re} f(x) - \|f\| \\ &\leqq 1 + \operatorname{Re} Qf(x) - \|Qf\| \\ &\quad - \{\|Pf\|\ \|P^*x\| - |Pf(P^*x)|\} \\ &\leqq 1 + \operatorname{Re} Qf(x) - \|Qf\| . \end{aligned}$$

This indicates that Q makes $(U - x)^0$ invariant. Now take $f \in (U - x)^0$, $g \in N^\perp$ and $h \in M^\perp$ with $\|f + g + h\| \leqq n$. Then by (1) $\|Qf + Qg\| \leqq n$. Since $x \in N \cap (U + M)$ and $g \in N^\perp$,

$$\begin{aligned} \operatorname{Re} Qf(x) &\leqq n\|x\| - \operatorname{Re} Qg(x) = n\|x\| + \operatorname{Re} Pg(x) \\ &\leqq n\|x\| + \|Pg - N^\perp \cap M^\perp\|\ \|x - M\| \\ &\leqq n\|x\| + \|Pg - N^\perp \cap M^\perp\| . \end{aligned}$$

Since Qf belongs to $(U - x)^0$ as f, it follows that

$$\begin{aligned} \|Qg\| &\leqq n + \|Qf\| \leqq n + \operatorname{Re} Qf(x) + 1 \\ &\leqq n(\|x\| + 2) + \|Pg - N^\perp \cap M^\perp\| . \end{aligned}$$

Then (3) applied to g yields

$$\|Pg - N^\perp \cap M^\perp\| \leqq \frac{\gamma(n\|x\| + 2)}{1 - \gamma} \equiv \delta_1$$

and consequently

$$\|Qf + g - N^\perp \cap M^\perp\| \leqq n + \delta_1 \equiv \delta_2 .$$

Since $x \in N$, $g \in N^\perp$ and $Qf \in (U - x)^0$,

$$\|Qf\| \leqq \operatorname{Re} Qf(x) + 1 \leqq \delta_2\|x\| + 1 \equiv \delta_3$$

and

$$\|g - N^\perp \cap M^\perp\| \leqq \delta_2 + \delta_3 \equiv \delta .$$

This implies that

$$\begin{aligned} f + g + h &= Qf + g + (Pf + h) \\ &\subseteqq \delta U^0 \cap (U - x)^0 + \delta U^0 \cap N^\perp + M^\perp . \end{aligned}$$

This completes the proof.

Consider the sup-norm Banach space $C(X)$ of continuous functions on a compact Hausdorff space X. By the Riesz theorem its dual is realized by the space of regular Borel measures on X with total-variation norm. Given a closed subset Y of X, let M be the subspace of functions in $C(X)$ vanishing on Y. Then $M^\perp$ is the set of measures with support in Y and becomes the range of a projection P: $Pm = \chi m$ for each measure m where χ is the characteristic function of Y. Obviously (1) is satisfied. Now let N be a closed subspace of $C(X)$. As shown in [5] (3) is equivalent to the property that for any $x \in N$ with $|x(t)| < 1 (t \in Y)$ and any closed subset $Z \subset X$ with $Y \cap Z = \emptyset$ there is $y \in N$ such that $x(t) = y(t)$ $(t \in Y)$, $|y(s)| < \gamma$ $(s \in Z)$ and $\|y\| < \max(1, \gamma)$. Remark that $\|x - M\|$ coincides with the norm of the restriction $x \mid Y$ of x to Y and that $x(t) = y(t) (t \in Y)$ is equivalent to $x - y \in M$. Thus Theorem 2 shows that if (3) with $\gamma < 1$, or equivalently (2) with $\rho < 2$, is satisfied then for any $x \in N$ there is $y \in N$ such that $x \mid Y = y \mid Y$ and $\|y\| = \|x \mid Y\|$. The case $\gamma = 0$ is the generalized Carleson-Rudin theorem (cf. [6] Chap. II). As Gamelin [5] shows, Theorem 2 can further yield the following: suppose that (3) with $\gamma < 1$, or equivalently (2) with $\rho < 2$, is satisfied and that $p \in C(X)$ satisfies $p(t) = 1$ $(t \in Y)$ and $p(s) > \gamma$ $(s \varepsilon X)$. Then if $x \in N$ satisfies $|x(t)| \leqq p(t)$ $(t \in Y)$ there is $y \in N$ such that $x(t) = y(t)$ $(t \in Y)$ and $|y(s)| \leqq p(s)$ $(s \in X)$. The case $\gamma = 0$ is the Bishop theorem (cf. [6] Chap. II). Generalization of the Gamelin theorem to other direction is treated by Alfsen and Hirsberg [1].

3. **Ordered Banach spaces.** Let E be an ordered Banach space with positive cone E_+. A closed subspace M is called an *ideal* if $(M - E_+) \cap (M + E_+) \subseteqq M$. An ideal M is *hypostrict* if its annihilator $M^\perp$ is the range of a projection P such that $f \geqq Pf \geqq 0$ for every $f \geqq 0$ The requirement means that both P and $Q = 1 - P$ are positive. Perdrizet [9] shows that a closed subspace M is a hypostrict ideal if and only if the following two conditions are satisfied: (1) Given x_1, $x_2 \in M$ and $y \in E$ with $x_1, x_2 \leqq y$, for any $\varepsilon > 0$ there is $z \in M$ such that $x_1, x_2 \leqq z \leqq y + \varepsilon$, and (2) given $x \in M$ and $y_1, y_2 \in E_+$ with $x \leqq y_1 + y_2$ there are $x_1, x_2 \in M$ such that $x = x_1 + x_2$ and $x_i \leqq y_i + \varepsilon$ $i = 1, 2$. Under the Riesz interpolation property an ideal M is hypostrict if and only if it is positively generated in the sense: $M = M \cap E_+$ $M \cap E_+$.

When M is an ideal, the Banach space E/M is preordered by the cone $\tau(E_+)$ where τ is the canonical map from E to E/M. The following theorem was first proved by Davies [4] under the Riesz interpolation property and then by Perdrizet [9] in general case. Let us give a proof based on Theorem 1.

THEOREM 3. *Let E be an ordered Banach space with positive cone E_+. If M is a hypostrict ideal then E/M is an ordered Banach space with $\tau(E_+)$ as its positive cone. If E is regular in addition, so is E/M.*

Proof. Since hypostrictness means that $E_+^* = -E_+^0$ is invariant under both P and Q, E_+ is splittable by Corollary 3. Then $\tau(E_+)$ is closed by Theorem 1. $M^\perp$ is isometric to the dual of E/M, and the dual positive cone is identified with $M^\perp \cap E_+^*$. Suppose that E is regular. Then E^* is regular. Since P is positive and is of norm one in this case, $M^\perp$ is a regular ordered Banach space with $M^\perp \cap E_+^*$ as its positive cone. Therefore E/M is regular as stated in §1. This completes the proof because every ordered Banach space admits an equivalent regular norm.

COROLLARY 5. *Suppose that the positive cone E_+ has nonempty interior and that M is a hypostrict ideal. If a closed subspace N is splittable and if it contains an interior point of E_+ then $\tau(N \cap E_+)$ is closed and $\tau(N \cap E_+) = \tau(N) \cap \tau(E_+)$.*

Proof. Since E_+ is splittable, in view of Corollary 2 and Theorem 1 it suffices to prove that $(N \cap E_+)^0 = N^\perp + E_+^0$. Remark that f belongs to $(N \cap E_+)^0$ if and only if the restriction of $-f$ to N is positive. However it is known (cf. [10] Chap. V §5) that when N contains an interior point of E_+ every continuous positive linear functional on N admits a continuous positive linear extension to E, in other words, $-(N \cap E_+)^0 = -(N^\perp + E_+^0)$.

Since E/M is ordered by the cone $\tau(E_+)$, for any y, z with $\tau(z) \leqq \tau(y)$ there is y' such that $z \leqq y'$ and $\tau(y) = \tau(y')$. The next task is to treat the case $\tau(z) \leqq \tau(y) \leqq \tau(x)$ and $z \leqq x$ and to find a condition of existence y'' such that $z \leqq y'' \leqq x$ and $\tau(y) = \tau(y'')$.

LEMMA 5. *Let S_1 and S_2 be closed convex subsets containing* 0. *If for any $0 < \lambda < 1$, $f \in S_1^0$ and $g \in S_2^0$ there are $f' \in S_1^0$ and $g' \in S_2^0$ such that*

$$\lambda f + (1-\lambda)g = \lambda f' + (1-\lambda)g'$$

and

$$\lambda\|f'\|, (1-\lambda)\|g'\| \leqq \|\lambda f + (1-\lambda)g\|$$

*then $(S_1 \cap S_2)^\sim$ coincides with $S_1^\sim \cap S_2^\sim$ where $(\cdot)^\sim$ denotes the weak** closure.*

Proof. In view of the Krein-Smulian theorem it suffices to prove that for any $\gamma > 0$ the weak* closure of $\operatorname{conv}(S_1^0 \cup S_2^0) \cap \gamma U^0$ is contained in the norm closure of $\operatorname{conv}(S_1^0 \cup S_2^0)$. Suppose that $0 < \lambda_\alpha < 1$, $f_\alpha \in S_1^0$, $g_\alpha \in S_2^0$ and $\|\lambda_\alpha f_\alpha + (1-\lambda_\alpha)g_\alpha\| \leqq \gamma$ and that the net $\{\lambda_\alpha f_\alpha +$

$(1-\lambda_\alpha)g_\alpha\}$ weak* converges to h and the net $\{\lambda_\alpha\}$ converges to λ. By hypothesis $\{\lambda_\alpha f_\alpha\}$ and $\{(1-\lambda_\alpha)g_\alpha\}$ can be assumed to be bounded, hence to weak* converge to f' and g' respectively. If $0<\lambda<1$, $\{f_\alpha\}$ and $\{g_\alpha\}$ can be assumed to weak* converge to $f''\in S_1^0$ and $g''\in S_2^0$ respectively. Then $h=\lambda f''+(1-\lambda)g''$ belongs to conv $(S_1^0\cup S_2^0)$. In case $\lambda=0$, $h=f'+g''$ and nf' belongs to S_1^0 for any $n>0$. Therefore h, as the norm limit of $1/n(nf')+(1-1/n)g''$, belongs to the norm closure of conv $(S_1^0\cup S_2^0)$. The case $\lambda=1$ is treated similarly.

COROLLARY 6. *$(\bigcap_{i=1}^n(x_i-E_+))^\sim=\bigcap_{i=1}^n(x_i-E_+)^\sim$ whenever $x_i\geqq 0$ $i=1,2,\cdots,n$.*

Proof. E, hence E^*, can be assumed to be regular. $(x_i-E_+)^0$ consists of all $0\leqq f$ with $f(x_i)\leqq 1$. Suppose that $\lambda_i\geqq 0$, $\sum_{i=1}^n\lambda_i=1$ and $f_i\in(x_i-E_+)^0$. Since the norm is monotone on the dual positive cone by regularity, it follows that $\lambda_j\|f_j\|\leqq\|\sum_{i=1}^n\lambda_i f_i\|$ $j=1,2,\cdots,n$. Now inductive application of Lemma 5 yields the assertion.

The following theorem was proved by Perdrizet [9]. Let us give a proof based on Theorem 1.

THEOREM 4. *Let E be an ordered Banach space with positive cone E_+. Suppose that M is a hypostrict ideal and E/M is ordered by the cone $\tau(E_+)$ where τ is the canonical map from E to E/M. If $z_i\leqq 0\leqq x_i$ and $\tau(z_i)\leqq\tau(y)\leqq\tau(x_i)$ $i=1,2,\cdots,n$, then for any $\varepsilon>0$ there is y' such that $z_i\leqq y'\leqq x_i+\varepsilon$ $i=1,2,\cdots,n$ and $\tau(y)=\tau(y')$. Further ε can be made 0 if every x_i is an interior point of E_+ or if E has the Riesz interpolation property.*

Proof. E is assumed to be regular, hence Q is of norm one. z_i+E_+ is a closed convex set containing 0. It is splittable because both P^* and Q^* are positive and z_i is negative. Similary x_i-E_+ is splittable. Let $S_1=\bigcap_{i=1}^n(z_i+E_+)$ and $S_2=\bigcap_{i=1}^n(x_i-E_+)$. Then by Corollaries 2, 6, and Theorem 1 both S_1 and S_2 are splittable and

$$\begin{aligned}&\textstyle\bigcap_{i=1}^n\tau(z_i+E_+)\cap\bigcap_{i=1}^n\tau(x_i-E_+)\\&=\tau(S_1)\cap\tau(S_2)\subseteqq\tau(S_1\cap(S_2+\varepsilon U))\end{aligned}$$

which is just the first assertion.

If every x_i is an interior point of E_+, S_2 contains 0 in its interior and by Corollary 2 and Theorem 1

$$\tau(S_1)\cap\tau(S_2)=\tau(S_1\cap S_2)\,.$$

Suppose finally that E has the Riesz interpolation property. Since E^{**} becomes a lattice as stated in §1, $S_1^{\sim}$ consists of all $w \in E^{**}$ with $\bigvee_{i=1}^n z_i \leqq w$, where $\bigvee_{i=1}^n z_i$ denotes the supremum of $z_1, \cdots, z_n$ in E^{**}. Then S_1^0 consists of all $0 \geqq f$ with $f(\bigvee_{i=1}^n z_i) \leqq 1$. Similarly S_2^0 consists of all $0 \leqq g$ with $g(\bigwedge_{i=1}^n x_i) \leqq 1$ where $\bigwedge_{i=1}^n x_i$ denotes the infimum of $x_1, \cdots x_n$, in E^{**}. Take $0 < \lambda < 1$, $f \in S_1^0$ and $g \in S_2^0$ and let $h = \lambda f + (1 - \lambda)g$. Since E^* is a Banach lattice as stated in §1, and since both $-f$ and g are positive, it follows that $0 \geqq h \wedge 0 \geqq \lambda f$ and $0 \leqq h \vee 0 \leqq (1 - \lambda)g$. Let $f' = (1/\lambda)(h \wedge 0)$ and $g' = (1)/(1 - \lambda)h \vee 0$. Then it follows from the above characterization of S_i^0 that $f' \in S_1^0$, $g' \in S_2^0$ and $h = \lambda f' + (1 - \lambda)g'$. Now since $||h \wedge 0||, ||h \vee 0|| \leqq ||h||$, Lemma 5 yields $(S_1 \cap S_2)^{\sim} = S_1^{\sim} \cap S_2^{\sim}$ and the assertion follows from Theorem 1.

4. Linear lifting. Let E be a Banach space with unit ball U and M a closed subspace. The canonical map from E to E/M is denoted by τ. A continuous linear map φ from E/M to E is called a *linear lifting* if $\tau \circ \varphi = 1$. If φ is a linear lifting, $\varphi \circ \tau$ is a projection with M as its kernel. Conversely, a linear lifting exists if M is the kernel of a continuous projection.

In this section it is assumed:

There is a projection P from E^* to $M^{\perp}$ such that

$$||f|| = ||Pf|| + ||f - Pf|| \quad (f \in E^*)$$

and Q stands for $1 - P$.

Let F be a finite dimensional Banach space with unit ball V. Consider the dual system $\langle F^* \otimes E, F \otimes E^* \rangle$ of tensor products. When $F^* \otimes E$ is provided with the Minkowski functional of $(V \otimes U^0)^0$ as norm, it is called the *inductive* tensor product of F^* and E and is denoted by $F^* \check{\otimes} E$. When $F \otimes E^*$ is provided with the Minkowski functional of conv$(V \otimes U^0)$ as norm, it is called the *projective* tensor product of F and E^* and is denoted by $F \hat{\otimes} E^*$. Let $B = B(F, E)$ denote the Banach space of all continuous linear maps from F to E, provided with operator-norm. Since F is finite dimensional, B is isometric to the inductive tensor product $F^* \check{\otimes} E$ under the canonical correspondence. The following lemma, whose proof is found in [10] Chap. IV §9, is basic in the subsequent development.

LEMMA 6. *The dual of $B(F, E)$ is isometric to the projective tensor product $F \hat{\otimes} E^*$ while the second dual is isometric to the inductive tensor product $F^* \otimes E^{**}$, hence to $B(F, E^{**})$.*

In view of Lemma 6 B^{**} is always identified with $B(F, E^{**})$.

In this case the imbedding of B to B^{**} is just the natural imbedding of $B(F, E)$ to $B(F, E^{**})$. In accordance with the terminology in §1 the weak** closure of a subset G of B is formed in $B(F, E^{**})$ and is denoted by $G^{\sim}$. When K and S are a subset of F and a closed convex subset of E containing 0 respectively, $G(K, S)$ and $\mathscr{G}(K, S^{\sim})$ denote the set of all $\varphi \in B$ with $\varphi(K) \subseteqq S$ and the set of all $\psi \in B^{**}$ with $\psi(K) \subseteqq S^{\sim}$. Obviously $G(K, S)$ is a closed convex subset of B containing 0 and its weak** closure is contained in $\mathscr{G}(K, S^{\sim})$.

COROLLARY 6. (a) $G(V, U)^{\sim} = \mathscr{G}(V, U^{\sim})$. (b) $\{G(H, 0) \cap G(F, N)\}^{\sim} = \mathscr{G}(H, 0) \cap \mathscr{G}(F, N^{\sim})$ *if* H *and* N *are closed subspaces of* F *and* E *respectively.* (c) $G(K, S)^{\sim} = \mathscr{G}(K, S^{\sim})$ *if* K *is a cone generated by a linearly independent basis* $\{x_1, \cdots, x_n\}$ *of* F *and* S *is a cone.*

Proof. (a) is an immediate consequence of Lemma 6. (b) $G(F, N)^{\sim} = \mathscr{G}(F, N^{\sim})$ follows from Lemma 6 applied to N instead of E. Since F is finite dimensional, H is the kernel of a projection σ. Then

$$\begin{aligned}\mathscr{G}(H, 0) \cap \mathscr{G}(F, N^{\sim}) &= \mathscr{G}(F, N^{\sim})\circ\sigma = G(F, N)^{\sim}\circ\sigma \\ &\subseteqq \{G(F, N)\circ\sigma\}^{\sim} = \{G(H, 0) \cap G(F, N)\}^{\sim} ,\end{aligned}$$

while the reverse inclusion is obvious. (c) Take any φ in $\mathscr{G}(K, S^{\sim})$ and let $y_i = \varphi(x_i)$ $i = 1, 2, \cdots, n$. Since each y_i belongs to $S^{\sim}$, there are nets $\{y_{i,\alpha}\}$ in S, weak** converging to y_i $i = 1, 2, \cdots, n$. Consider a net $\{\varphi_\alpha\}$ in B defined by $\varphi_\alpha(x_i) = y_{i,\alpha}$ $i = 1, 2, \cdots, n$. By hypothesis it is contained in $G(K, S)$ and weak** converges to φ. Thus $\mathscr{G}(K, S^{\sim})$ is contained in $G(K, S)^{\sim}$ with the reverse inclusion is obvious.

Since B^* is identified with the projective tensor product $F \hat{\otimes} E^*$ by Lemma 6, the operators $1 \otimes P$ and $1 \otimes Q$ are considered to define projections in B^*. The adjoints of $1 \otimes P$ and $1 \otimes Q$ are realized in $B(F, E^{**})$ according to the following formula:

$$(5) \qquad (1 \otimes P)^*\varphi = P^*\circ\varphi \text{ and } (1 \otimes Q)^*\varphi = Q^*\circ\varphi \ (\varphi \in B(F, E^{**})) .$$

LEMMA 7. *The annihilator* $G(F, M)^{\perp}$ *is the range of* $1 \otimes P$.

Proof. Since Q^* is a projection onto $M^{\sim}$, by (5) $(1 \otimes Q)^*$ projects B^{**} onto $\mathscr{G}(F, M^{\sim})$, which coincides with $G(F, M)^{\sim}$ by Corollary 6. Then $1 \otimes P$ is obviously a projection from B^* to $G(F, M)^{\perp}$.

On the basis of Lemma 7, a sentence "$G(K, S)$ is *splittable*" will always mean that $G(K, S)$ is $1 \otimes P$-splittable.

COROLLARY 7. *If* S *is splittable and* $G(K, S)^{\sim} = \mathscr{G}(K, S^{\sim})$ *then*

$G(K, S)$ is splittable.

Proof. This follows from (5) by Lemma 1.

The following lemma can be considered a development of a basic device in Michael and Pełczynski [8], treating linear lifting in a special case. The crucial requirement for P plays a decisive role in the proof.

LEMMA 8. *Suppose that S is splittable and $G(K, S)^{\sim} = \mathscr{G}(K, S^{\sim})$ for a subset K of F. If ψ belongs to*

$$G(\pi(K), S) \cap G(\pi(V), U) \cap G(K, S + M) \cap G(V, U + M)$$

where π is a projection of F to a subspace H, then for any $\varepsilon > 0$ there is φ in $G(K, S) \cap G(V, U)$ such that

$$\tau \circ \varphi = \tau \circ \psi \quad \textit{and} \quad \|(\varphi - \psi) \mid H\| < \varepsilon .$$

Proof. Remark first of all that the requirement for P means by Corollary 1 that the unit ball U is splittable.

Let $\psi_1 = \psi - Q^* \circ \psi \circ (1 - \pi)$. Since

$$Q^* \circ \psi_1(K) \subseteqq Q^* \circ \psi \circ \pi(K) \subseteqq Q^*(S) \subseteqq S^{\sim}$$

and

$$P^* \circ \psi_1(K) \subseteqq P^* \circ \psi(K) \subseteqq P^*(S + M) \subseteqq S^{\sim}$$

by splittability of S, ψ_1 belongs to $\mathscr{G}(K, S^{\sim})$ by Lemma 1, hence to $G(K, S)^{\sim}$ by hypothesis. Since U is splittable and $G(V, U)^{\sim} = \mathscr{G}(V, U^{\sim})$ by Corollary 6, the same argument shows that ψ_1 belongs also to $G(V, U)^{\sim}$. Moreover it belongs to $\{G(K, S) \cap G(V, U)\}^{\sim}$ because $G(V, U)$ is the unit ball of B. On the other hand, $Q^* \circ \psi \circ (1 - \pi)$ belongs to $\mathscr{G}(H, 0) \cap \mathscr{G}(F, M^{\sim})$, hence to $\{G(H, 0) \cap G(F, M)\}^{\sim}$ by Corollary 6. Thus ψ belongs to

$$\{G(K, S) \cap G(V, U) + G(H, 0) \cap G(F, M)\}^{\sim} .$$

It follows that ψ must be contained in the norm closure of

$$G(K, S) \cap G(V, U) + G(H, 0) \cap G(F, M) .$$

Therefore there is $\psi_2 \in B$ such that $\psi - \psi_2 \in G(H, 0) \cap G(F, M)$ and

$$\|\psi_2 - G(K, S) \cap G(V, U)\| < \varepsilon .$$

Since $G(K, S) \cap G(V, U)$ is splittable by hypothesis and Corollary 7, Lemma 3 guarantees that there is $\varphi \in G(K, S) \cap G(V, U)$ such that $\varphi - \psi_2 \in G(F, M)$ and $\|\varphi - \psi_2\| < \varepsilon$. Now $\psi_2 - \psi \in G(H, 0) \cap G(F, M)$

implies that $\tau\circ\varphi = \tau\circ\psi$ and

$$||(\varphi - \psi)\,|\,H|| = ||(\varphi - \psi_2)\,|\,H|| \leqq ||\varphi - \psi_2|| < \varepsilon\,.$$

Let S be a closed splittable subset of E and L a subset of $\tau(S)$. Suppose that there is a sequence of projections $\{\pi_n\}$ in E/M such that (1) the range F_n of π_n is of finite dimension, (2) $||\pi_n|| \leqq 1$, (3) $\pi_n\cdot\pi_m = \pi_n$ for $n \leqq m$, (4) $\pi_n(L) \subseteqq L$ and (5) π_n converges strongly to the identity as $n \to \infty$.

Let $\mathscr{G}_n$ denote the set of all $\varphi \in B(F_n, E)$ with $\varphi\circ\pi_n(L) \subseteqq S$ while G_n is the set of all $\psi \in B(F_n, E^{**})$ with $\psi\circ\pi_n(L) \subseteqq S^{\sim}$. As before, the second dual of $B(F_n, E)$ is identified with $B(F_n, E^{**})$.

LEMMA 9. *If the weak** closure of G_n coincides with $\mathscr{G}_n$, $n = 1, 2, \cdots$, then there is a linear lifting φ from E/M to E such that $\varphi(L) \subseteqq S$ and $||\varphi|| \leqq 1$.*

Proof. Let $\pi_0 = 0$ and $\varphi_0 = 0$. Assume that linear maps $\varphi_j \in B(F_j, E)$ $j = 0, 1, \cdots, n$ have been found in such a way that $\tau\circ\varphi_j = 1$ on F_j, $||\varphi_j|| \leqq 1$, $\varphi_j\circ\pi_j(L) \subseteqq S$ and $||(\varphi_{j-1} - \varphi_j)\,|\,F_{j-1}|| < 1/2^{j-1}$ $j = 0, 1, \cdots, n$. Since F_{n+1} is finite dimensional by hypothesis, there is $\psi \in B(F_{n+1}, E)$ such that $\tau\circ\psi = 1$ on F_{n+1}. Consider the map $\psi' = \varphi_n\circ\pi_n + \psi\circ(1 - \pi_n)$ from F_{n+1} to E. Then by assumption

$$\psi'\circ\pi_n(\pi_{n+1}(L)) = \varphi_n\circ\pi_n(L) \subseteqq S$$

and in view of $||\pi_n|| \leqq 1$

$$\psi'\circ\pi_n(V_{n+1}) = \varphi_n(V_n) \subseteqq U$$

where V_i denotes the unit ball of F_i. Since $V_{n+1} \subseteqq \tau(U)$ by Theorem 1 and $\pi_{n+1}(L) \subseteqq \tau(S)$,

$$\psi'(V_{n+1}) \subseteqq U + M \text{ and } \psi'(\pi_{n+1}(L)) \subseteqq S + M\,.$$

Since the weak** closure of G_n coincides with $\mathscr{G}_n$ by hypothesis, Lemma 8, applied to $F_{n+1}, \pi_{n+1}(L)$ and π_n instead of F, K and π, yields that there is $\varphi_{n+1} \in G_{n+1}$ such that $||\varphi_{n+1}|| \leqq 1, \tau\circ\varphi_{n+1} = 1$ on F_{n+1} and $||(\varphi_{n+1} - \varphi_n)\,|\,F_n|| < 1/2^n$, completing induction. Now the sequence $\{\varphi_n\circ\pi_n\}$ is uniformly bounded and

$$\sum_{k=n}^{\infty}\left|\left|(\varphi_{k+1} - \varphi_k)\right|F_n\right|\right| \leqq \sum_{j=n}^{\infty} 1/2^k < \infty$$

guaranteeing convergence of $\varphi_k(x)$ for every $x \in F_n$ as $k \to \infty$. Then $\{\varphi_n\circ\pi_n\}$ converges strongly to some map φ from E/M to E. Obviously φ is a required linear lifting.

It is better to introduce some terminology before stating the main result on linear lifting. A Banach space E is called a *π-space* if there is a sequence $\{F_n\}$ of finite dimensional subspaces such that $F_1 \subseteqq F_2 \subseteqq \cdots$ with $\overline{\bigcup_{n=1}^{\infty} F_n} = E$ and each F_n is the range of a projection of norm one. Here projections π_n can be assumed to have the property that $\pi_n \pi_m = \pi_n$ for $n \leqq m$ and that π_n converges strongly to the identity as $n \to \infty$. An ordered Banach space is called a *Π-space* if, in addition, projections can be chosen positive and if each F_n has the positive cone generated by a linearly independent basis.

THEOREM 5. *Suppose that the annihilator of a closed subspace M is the range of a projection P such that*

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*) .$$

If the quotient space E/M becomes a π-space then there is a linear lifting of norm one, or equivalently, M is the kernel of a projection of norm one.

Proof. Since the unit ball U is splittable by Corollary 1, all requirements in Lemma 9 are fulfilled with $S = U$ and $L = \tau(U)$ by Corollary 7.

COROLLARY 8. *Let N and M be closed subspaces and suppose that $M^\perp$ is the range of a projection P such that $P(N^\perp) \subseteqq N^\perp$ and*

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*) .$$

If the quotient space $N/N \cap M$ is a π-space, there is a linear lifting of norm one from $N/N \cap M$ to N.

Proof. In view of Theorem 5 it suffices to prove that the annihilator of $N \cap M$ in N^* is the range of a projection $\mathscr{P}$ such that

$$\|g\| = \|\mathscr{P}g\| + \|g - \mathscr{P}g\| \quad (g \in N^*) .$$

When N^* is identified with $E^*/N^\perp$, the annihilator of $N \cap M$ becomes the image of $(N \cap M)^\perp$ under the canonical map from E^* to $E^*/N^\perp$. Since hypothesis implies splittability of N by Corollary 3, $N + M$ is closed by Theorem 1 so that $N^\perp + M^\perp$ is weak* closed and coincides with $(N \cap M)^\perp$. Therefore the annihilator of $N \cap M$ in N^* becomes the image of $M^\perp$ in $E^*/N^\perp$. Since $N^\perp$ is invariant under P, there arises a natural projection $\mathscr{P}$ from N^* to the annihilator $N \cap M$. Since by hypothesis

$$\|f - N^\perp\| \geqq \|Pf - N^\perp\| + \|f - Pf - N^\perp\| ,$$

$\mathscr{P}$ is easily seen to have the required property.

When E is the space of continuous functions on a compact set and M consists of functions vanishing on a fixed closed subset, Corollary 8 was proved by Michael and Pełczynski [8].

THEOREM 6. *Let M be a closed subspace of an ordered Banach space E. Suppose that M is the range of a projection P such that $f \geqq Pf \geqq 0 \quad (f \geqq 0)$ and*

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*) .$$

If E/M is a Π-space under the canonical ordering, there is a positive linear lifting of norm one, or equivalently, M is the kernel of a positive projection of norm one.

Proof. Since the positive cone E_+ is splittable by Corollary 3, all requirements in Lemma 9 are fulfilled with $S = E_+$ and $L = \tau(E_+)$ by definition of a Π-space and Corollary 6.

To be a π-space or a Π-space is not so severe restriction. Let us prove:

Separable complex (resp. real) $L_p(1 \leqq p < \infty)$ and complex (resp. real) $C(X)$ on compact metrizable X are π-spaces (resp. Π-spaces).

In fact, it suffices for the first part to treat a L_p space on a finite measure space $(\mathscr{B}, \mu)$. Since the Borel field is separable with respect to μ, there is an increasing sequence $\{\mathscr{B}_n\}$ of finite Borel subfields such that $\bigcup_{n=1}^{\infty} L_p(\mathscr{B}_n)$ is dense in L_p where $L_p(\mathscr{B}_n)$ is the subspace of $\mathscr{B}_n$-measurable functions. Each $L_p(\mathscr{B}_n)$ is finite dimensional and the conditional expection relative to $\mathscr{B}_n$ becomes a (positive) projection of norm one from L_p to $L_p(\mathscr{B}_n)$(cf. [3]). The assertion for $C(X)$ is proved in [7] by using peaked partition.

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