CLOSED RANGE THEOREMS FOR CONVEX SETS AND LINEAR LIFTINGS

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Let $M$ be a closed subspace of a Banach space $E$ such that its annihilator $M^\perp$ is the range of a projection $P$. Given a closed convex subset $S$ containing 0, the first problem of this paper is to find a condition for $\tau(S)$ to be closed where $\tau$ is the canonical map from $E$ to $E/M$. Closure is guaranteed if $S$ is splittable in the sense that the polar $S^o$ coincides with the norm-closed convex hull of $P(S^o) \cup Q(S^o)$, where $Q = 1 - P$. The second problem is to give a condition for existence of a linear map $\varphi$, called a linear lifting, from $E/M$ to $E$ such that $\tau \circ \varphi = 1$ and $\varphi \circ \tau(S) \subseteq S$. A linear lifting exists if and only if $M$ is the kernel of a projection making $S$ invariant. Of special interest is the case where $S$ is a ball or a cone. When the unit ball is splittable, existence of a linear lifting of norm one is guaranteed under suitable conditions on $E/M$, which are satisfied by separable $L_p$ and $C(X)$ on compact metrizable $X$. If further $E$ is an ordered Banach space, and if both $P$ and $Q$ are positive, $M$ is shown to be the kernel of a positive projection of norm one.

Though the closed range theorem (Theorem 1) yields immediately an abstract version of the Rudin-Carleson-Bishop theorem on norm-preserving extensions of functions defined on a peak set, in §2 further modification (Theorem 2) is shown to include Gamelin’s extension [5] of the Rudin-Carleson-Bishop theorem in abstract form. Recently a different approach to generalization of the Gamelin theorem was made by Alfsen and Hirsberg [1]. In §3 it is indicated how the closed range theorem is applied to give unified proofs for results of Davies [4] and Perdrizet [9] on closedness of a cone in a quotient space and on order-preserving extensions. In §4 an idea of Pełczyński-Michael [8] is further developed for the closed range theorem to produce existence of linear liftings under suitable conditions. The Pełczyński-Michael theorems are generalized in abstract form (Theorems 5 and 6).

1. Preliminary. Let $E$ be a real or complex Banach space with unit ball $U$. $E^*$ and $E^{**}$ are its dual and second dual respectively, and $E$ is always imbedded canonically into $E^{**}$. $x, y, z, \cdots$ are vectors in $E$ or $E^{**}$ while $f, g, h, \cdots$ are functionals in $E^*$. For $x \in E^{**}$ and $f \in E^* f(x)$ is used instead of $x(f)$. The weak topology $\sigma(E^*, E)$ on $E^*$ is called the weak* topology while $\sigma(E^{**}, E^*)$ on $E^{**}$ is the weak** topology. For a subset $S$ of $E$ its norm-closure and its weak**
closure (in $E^{**}$) are denoted by $\bar{S}$ and $S^-$ respectively.

The polar $S_0^\circ$ is defined as the set of all $f$ such that $\text{Re } f(x) \leq 1$ or $f(x) \leq 1$ on $S$ according as the scalar field is complex or real. When $S$ is a subspace its polar coincides with its annihilator $S_\perp$ consisting of all $f$ vanishing on it. The following basic facts are used frequently in this paper. Proofs are found, for instance, in [10]. Let $S_1$ and $S_2$ be closed convex subsets of $E$ containing 0. $(S_1 \cap S_2)^\circ$ coincides with the weak* closure of $\text{conv } (S_1^\circ \cup S_2^\circ)$ where $\text{conv } (\cdot)$ denotes the convex hull. The weak** closure $S_1^\sim$ coincides with the polar of $S_1^\circ$ in duality $\langle E^{**}, E^* \rangle$ and $S_1 = E \cap S_1^\sim$. Thus coincidence $(S_1 \cap S_2)^\sim = S_1^\sim \cap S_2^\sim$ occurs if and only if the weak* and the norm closure of $\text{conv } (S_1^\circ \cup S_2^\circ)$ coincide. In some case $\text{conv } (S_1^\circ \cup S_2^\circ)$ becomes itself weak* closed. Here the Krein-Smulian theorem is quite useful: $\text{conv } (S_1^\circ \cup S_2^\circ)$ is weak* closed if (and only if) $\gamma U^\circ \cap \text{conv } (S_1^\circ \cup S_2^\circ)$ is weak* closed for every $0 \leq \gamma < \infty$. If $S_1$ contains 0 in its interior then $S_1^\circ$ is weak* compact and the norm closure of $\text{conv } (S_1^\circ \cup S_2^\circ)$ is weak* closed. If $S_1$ is a subspace or a cone, the weak* closure of $\text{conv } (S_1^\circ \cup S_2^\circ)$ is just that of $S_1^\circ + S_2^\circ$. In case both $S_1$ and $S_2$ are subspaces, $S_1^\circ + S_2^\circ$ is weak* closed if and only if $S_1 + S_2$ is norm-closed.

Suppose now that $E$ is a real Banach space provided with a closed proper cone $E_+$. $E_+$ gives rise to natural ordering in $E$ under which it becomes the set of all positive vectors: $x \leq y$ means $y - x \in E_+$. In this respect $E_+$ is called the positive cone. The dual positive cone $E_+^*$ is defined as the set of $f$ nonnegative on $E_+$, or equivalently $E_+^* = -E_+^\circ$. $E$ is called an ordered Banach space if $E = E_+ - E_+$ and if there is $\gamma < \infty$ with $(U - E_+) \cap (U + E_+) \subseteq \gamma U$. The latter condition is equivalent to that every subset of the form $\{x; y, \leq x \leq y\}$ is norm-bounded. For notational convenience the relation $x \leq y + \varepsilon$ in an ordered Banach space means that there is $z \geq 0$ such that $||z|| < \varepsilon$ and $x \leq y + z$. An ordered Banach space or its norm is called regular if $||x|| = \inf \{||y||; y \leq x \leq y\}$ for every $x$. A regular norm is monotone on the positive cone in the sense that $0 \leq x \leq y$ implies $||x|| \leq ||y||$. An ordered Banach space admits an equivalent regular norm. In fact, the functional $||x||_0 = \inf \{||y||; y \leq x \leq y\}$ gives a regular norm.

It is known (cf. [2] and [4]) that $E$ is regular if and only if $E^*$ is regular. An ordered Banach space is said to have the Riesz interpolation property if for $y_i \geq x_j$ ($i, j = 1, 2, \ldots, n$) there is $z$ such that $x_i \leq z \leq y_i$ ($i = 1, 2, \ldots, n$). A regular ordered Banach space is called a Banach lattice if it is lattice under the ordering. A Banach lattice has the Riesz interpolation property. It is known (cf. [2] and [4]) that $E$ has the Riesz interpolation property if and only if $E^*$ is a lattice. A continuous linear operator between ordered Banach spaces is called positive if it transforms a positive cone into another.
2. Closed range theorems. $E$ is a real or complex Banach space with unit ball $U$ and $M$ is a closed subspace. The canonical map from $E$ to the quotient space $E/M$ is denoted by $\tau$.

Throughout this section it is assumed:

There is a continuous projection $P$ from $E^*$ to $M^\perp$, and $Q$ stands for $1-P$.

Remark that the adjoint $Q^*$ projects $E^{**}$ onto $M^\perp$, but $M$ is not necessarily range of any projection. $S$, $S_1$ and $S_2$ will denote closed convex subsets of $E$ containing 0. $S$ is said to be splittable, or more precisely, $P$-splittable if its polar $S^0$ coincides with the norm-closure of $\text{conv} \,(P(S^0) \cup Q(S^0))$.

**Lemma 1.** The following conditions are equivalent.

(a) $S$ is splittable.

(b) $S^\sim = \{x \in E^{**}; P^*x \in S^\sim \text{ and } Q^*x \in S^\sim \}$.

(c) $\theta(f) = \theta(Pf) + \theta(Qf)$ $(f \in E^*)$

where $\theta(f)$ is defined by $\theta(f) = \sup \{\Re f(x); x \in S\}$.

**Proof.** Since the polar of $P(S^0)$(resp. of $Q(S^0)$) in $E^{**}$ coincides with the set $\{x \in E^{**}; P^*x$(resp. $Q^*x) \in S^\sim \}$ equivalence of (a) and (b) is clear (cf. §1).

(b) $\Rightarrow$ (c). Obviously $\theta(f)$ can be defined by

$$\theta(f) = \sup \{\Re f(x); x \in S^\sim \}.$$ 

Take $x$ and $y$ in $S$. Then by (b) $P^*x + Q^*y$ belongs to $S^\sim$ so that

$$\Re Pf(x) + \Re Qf(y) = \Re f(P^*x + Q^*y) \leq \theta(f),$$

leading to $\theta(Pf) + \theta(Qf) \leq \theta(f)$. The reverse inequality is obvious.

(c) $\Rightarrow$ (b). Since the functional $\theta$ is nonnegative because of $S \ni 0$, (c) implies $P(S^0) \cup Q(S^0) \subseteq S^0$. Therefore $S^\sim$ is contained in the set $\{x \in E^{**}; P^*x \in S^\sim \text{ and } Q^*x \in S^\sim \}$. Take $x$ with $P^*x$, $Q^*x \in S^\sim$. Then by (c)

$$\Re f(x) = \Re Pf(P^*x) + \Re Qf(Q^*x) \leq \theta(Pf) + \theta(Qf) = \theta(f).$$

Thus $x$ belongs to the polar of $S^0$ in $E^{**}$.

**Corollary 1.** The unit ball $U$ is splittable if and only if

$$\|f\| = \|Pf\| + \|Qf\| \quad (f \in E^*).$$

**Corollary 2.** If both $S_1$ and $S_2$ are splittable, and $(S_1 \cap S_2)^\sim = S_1^\perp \cap S_2^\perp$, then $S_1 \cap S_2$ is splittable.
COROLLARY 3. A closed subspace (resp. cone) is splittable if and only if its polar is invariant under $P$ (resp. under $P$ and $Q$).

Proof. Let $N$ be a closed cone. $P(N^o) \subseteq N^o$ and $Q(N^o) \subseteq N^o$ implies $N^o = P(N^o) + Q(N^o) = \text{conv}(P(N^o) \cup Q(N^o))$. If $N$ is further a subspace, $Q(N^o) \subseteq N^o$ follows already from $P(N^o) \subseteq N^o$.

LEMMA 2. If $S_1$ and $S_2$ are splittable, for any $\varepsilon > 0$ and $\rho > 0$ the following inclusion relation holds:

$$\{S_1 \cap (S_2 + \varepsilon U) + \rho U\} \cap (S_1 + M) \cap (S_2 + M) \subseteq S_1 \cap (S_2 + \alpha \varepsilon U) + \alpha \rho U \cap M$$

where $\alpha = \|Q\|$.

Proof. Take any $x$ in the set on the left hand side. Then it follows by splittability that

$$Q^*x \in S_1^\sim \cap (S_2^\sim + \alpha \varepsilon U^\sim \cap M^\sim) + \alpha \rho U^\sim \cap M^\sim.$$ 

There is $y \in \alpha \rho U^\sim \cap M^\sim$ such that

$$Q^*(x - y) = Q^*x - y \in S_1^\sim \cap (S_2^\sim + \alpha \varepsilon U^\sim \cap M^\sim)$$

and

$$P^*(x - y) = P^*x \in P^*(S_1 + M) \cap P^*(S_2 + M) \subseteq S_1^\sim \cap S_2^\sim.$$ 

Then by Lemma 1 $x - y \in S_1^\sim$ and there is $z \in \alpha \varepsilon U^\sim \cap M^\sim$ such that $x - y - z \in S_2^\sim$. Finally in view of arguments of §1 $x$ belongs to

$$E \cap \{S_1^\sim \cap (S_2^\sim + \alpha \varepsilon U^\sim) + \alpha \rho U^\sim \cap M^\sim\} \subseteq E \cap \{S_1 \cap (S_2 + \alpha \varepsilon U) + \alpha \rho U \cap M\}^\sim = S_1 \cap (S_2 + \alpha \varepsilon U) + \alpha \rho U \cap M.$$ 

By definition of the quotient topology $\tau(x)$ belongs to the closure $\overline{\tau(S)}$ if and only if $x$ is contained in $\overline{S + M}$. In particular, $\tau(S)$ is closed if and only if $S + M$ is closed.

LEMMA 3. Suppose that $S_1$ and $S_2$ are splittable. If $\tau(x)$ belongs to $\overline{\tau(S_1)} \cap \overline{\tau(S_2)}$ and $\|x - S_1 \cap S_2\| < \gamma$ there is $y \in S_1$ such that $\tau(x) = \tau(y)$ and $\|x - y\| < \gamma \|\|Q\||$. In case $\|\|Q\|| = 1$ for any $\varepsilon > 0$ $y$ can be chosen in $S_1 \cap (S_2 + \varepsilon U)$.

Proof. Let $\alpha = \|Q\|$ and take $\varepsilon'$ with $0 < \varepsilon' < \varepsilon$. By hypothesis $x$ is contained in

$$\{S_1 \cap (S_2 + \varepsilon' U) + \gamma' U\} \cap (S_1 + M) \cap (S_2 + M)$$
for some $\gamma > \gamma' > 0$. Choose $\varepsilon_n > 0$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \gamma - \gamma'$. By Lemma 2 there is $x_0 \in M$ such that $\|x_0\| \leq \alpha \gamma'$ and $\|x + x_0 - S_1 \cap (S_2 + \alpha \varepsilon' U)\| < \varepsilon_1$. Then

$$x + x_0 \in \overline{S_1 + M} + M = \overline{S_1 + M}$$

and

$$x + x_0 \in S_1 \cap (S_2 + \alpha \varepsilon' U) + \varepsilon_1 U.$$ 

Now inductive procedure based on Lemma 2 makes it possible to find a sequence $\{x_n\}$ in $M$ such that $\|x_n\| \leq \alpha \varepsilon_n$ and $\|x + \sum_{n=0}^{\infty} x_n - S_1 \cap (S_2 + \alpha^n \varepsilon' U)\| < \varepsilon_n$. Since $\sum_{n=0}^{\infty} \|x_n\| < \infty$, $y = x + \sum_{n=0}^{\infty} x_n$ is well defined. Obviously $y$ belongs to $S_1$, and in case $\alpha = 1$, to $S_1 \cap (S_2 + \varepsilon' U) \subseteq S_1 \cap (S_2 + \varepsilon U)$. Finally $\tau(x) = \tau(y)$ and $\|x - y\| \leq \alpha \sum_{n=1}^{\infty} \varepsilon_n + \alpha \gamma' < \alpha \gamma$.

Now the main result of this paper is near at hand.

**Theorem 1.** Suppose that the annihilator of a closed subspace $M$ is the range of a projection $P$ and that $S_1$, $S_2$, and $S_3$ are closed convex subsets containing 0. Then (a) the image of $S$ under the canonical map $\tau$ from $E$ to $E/M$ is closed whenever $S$ is splittable. (b) If both $S_1$ and $S_2$ are splittable and $(S_1 \cap S_2)^\perp = S_1^\perp \cap S_2^\perp$ then $\tau(S_1) \cap \tau(S_2) = \tau(S_1 \cap S_2)$. (c) If both $S_1$ and $S_2$ are splittable and $\|1 - P\| \leq 1$, then the inclusion $\tau(S_1) \cap \tau(S_2) \subseteq \tau(S_1 \cap (S_2 + \varepsilon U))$ holds for any $\varepsilon > 0$.

**Proof.** (a) follows from Lemma 3 with $S_1 = S_3 = S$. Also (c) is a direct consequence of Lemma 3. (b) $S_1 \cap S_2$ is splittable by Corollary 2 and $S_1 \cap S_2 + M$ is closed by (a). Now since by hypothesis

$$P^*(S_1 + M) \cap (S_2 + M) \subseteq P^*(S_1) \cap P^*(S_3) \subseteq S_1^\perp \cap S_2^\perp$$

it follows that

$$(S_1 + M) \cap (S_2 + M) \subseteq E \cap (S_1 \cap S_2 + M)^\perp = S_1 \cap S_2 + M,$$

showing $\tau(S_1) \cap \tau(S_2) \subseteq \tau(S_1 \cap S_2)$. The reverse inclusion is obvious. This completes the proof.

It follows immediately from Theorem 1 that if the unit ball $U$ is splittable and if $N$ is a closed splittable subspace then $\tau(N \cap U)$ is closed and coincides with $\tau(N) \cap \tau(U)$. Let us show that the same conclusion holds for a non-splittable subspace under suitable conditions.

Since by Corollary 3 splittability of $N$ is characterized by $P(N^\perp) \subseteq N^\perp \cap M^\perp$, it follows by Corollary 1 that under the splittability of $U$, $N$ is splittable if and only if
On the other hand, Corollary 1 implies \(|Qf| = |f - M^\perp|\). Thus if the unit ball is splittable, splittability of \(N\) is characterized by

\[ |f - M^\perp \cap N^\perp | \leq |f - M^\perp | \quad (f \in N^\perp) \]

**Lemma 4.** Let \(N_1\) and \(N_2\) be closed subspace. Then for \(\rho > 0\) the following assertions are equivalent.

(a) \(|x - N_1 \cap N_2| \leq \rho \|x - N_1\|\) \((x \in N_2)\)

(b) \(|f - N_1^\perp \cap N_2^\perp| \leq \rho \|f - N_1^\perp\|\) \((f \in N_1^\perp)\)

**Proof.** (a) means that

\[ N_2 \cap (N_1 + U) \subseteq N_1 \cap N_2 + \rho U \]

which implies by polar formation

\[ (N_1^\perp + N_2^\perp) \cap U^0 \subseteq N_2^\perp + N_1^\perp + \rho U^0 \]

The last relation can be converted to

\[ N_1 \cap (N_2^\perp + U^0) \subseteq N_1^\perp \cap N_2^\perp + \rho U^0 \]

which is nothing but (b). The reverse process can be pursued because (b) implies that \((N_1^\perp + N_2^\perp) \cap U^0\) is weak* compact, hence by the Krein-Smulian theorem that \(N_1^\perp + N_2^\perp\) is weak* closed.

**Corollary 4.** Suppose that the unit ball is splittable. Then the following assertions for a closed subspace \(N\) are equivalent.

(a) \(N\) is splittable.

(b) \(|x - N \cap M| \leq \|x - N\|\) \((x \in M)\)

(c) \(|f - N^\perp \cap M^\perp| \leq \|f - M^\perp\|\) \((f \in N^\perp)\).

**Theorem 2.** Let \(M\) and \(N\) be closed subspaces, and suppose that the annihilator \(M^\perp\) is the range of a projection \(P\) such that

\[ \|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*) \]

If for some \(1 \leq \rho < 2\)

\[ \|x - N \cap M\| \leq \rho \|x - N\| \quad (x \in M) \]

then the images of the unit ball \(U, N\) and \(N \cap U\) under the canonical map \(\tau\) from \(E\) to \(E/M\) are closed and

\[ \tau(N \cap U) = \tau(N) \cap \tau(U) \]

**Proof.** Closedness of \(\tau(U)\) follows from (1) by Corollary 1 and
Theorem 1. Let $Q = 1 - P$. Then in view of (1) relation (2) is converted by Lemma 4 to

$$
\|Pf - M^1 \cap N^1\| \leq \gamma \|Qf\| \quad (f \in N^1)
$$

where $\gamma = \rho - 1 < 1$. Then for any $f \in N^1$ and $g \in M^1$.

$$
\|g - M^1 \cap N^1\| \leq \|Pf - M^1 \cap N^1\| + \|g - P\| \\
\leq \gamma \|Qf\| + \|g - Pf\| \leq \|g - f\|
$$

showing

$$
\|g - M^1 \cap N^1\| \leq \|g - N^1\| \quad (g \in M^1)
$$

which is converted by Lemma 4 to

$$
\|x - M \cap N\| \leq \|x - M\| \quad (x \in N).
$$

This last relation means that the canonical map from the Banach space $N/M \cap N$ onto $\tau(N)$ has bounded inverse. Therefore $\tau(N)$ is closed. Further (4) implies

$$
N \cap (U + M) \subseteq \overline{U + N \cap M}.
$$

Let us prove that really

$$
N \cap (U + M) \subseteq U + N \cap M
$$

holds, which is equivalent to the required relation:

$$
\tau(N) \cap \tau(U) \subseteq \tau(N \cap U).
$$

Suppose for contradiction that there exists $x$ in $N \cap (U + M)$ with $(U - x) \cap N \cap M = \emptyset$. Since $N \cap M$ is a subspace, it follows that

$$
\text{conv } (((U - x) \cup \{0\}) \cap N \cap M = \{0\}.
$$

Since $\text{conv } (((U - x) \cup \{0\})$ is closed, the last relation implies by polar formation that $(U - x)^0 + N^1 + M^1$ is weak* dense in $E^*$. Weak* closedness of $(U - x)^0 + N^1 + M^1$, if proved, leads to

$$
\text{conv } (((U - x) \cup \{0\}) \cap N^* \cap M^* = \{0\}
$$

hence to a contradiction:

$$
x \notin E \cap (U^* + N^* \cap M^*) = \overline{U + N \cap M}.
$$

Now let us prove the above weak* closedness. To this end, in view of the Krein-Smulian theorem, it suffices to prove that for any $n > 0$

$$
\{(U - x)^0 + N^1 + M^1\} \cap nU^0 \subseteq \delta U^0 \cap (U - x)^0 + \delta U^0 \cap N^1 + M^1
$$
where $\delta$ is a constant depending on $n$. Remark that $(U - x)^o$ consists of all $f$ with $||f|| \leq \text{Re } f(x) + 1$. Since $x \in U + M$ implies $||P^*x|| \leq 1$, it follows from (1) that

$$
1 + \text{Re } f(x) - ||f|| \\
\leq 1 + \text{Re } Qf(x) - ||Qf|| \\
- (||Pf|| ||P^*x|| - |Pf(P^*x)|) \\
\leq 1 + \text{Re } Qf(x) - ||Qf|| .
$$

This indicates that $Q$ makes $(U - x)^o$ invariant. Now take $f \in (U - x)^o$, $g \in N^\perp$ and $h \in M^\perp$ with $||f + g + h|| \leq n$. Then by (1) $||Qf + Qg|| \leq n$. Since $x \in N \cap (U + M)$ and $g \in N^\perp$,

$$
\text{Re } Qf(x) \leq n ||x|| - \text{Re } Qg(x) = n ||x|| + \text{Re } Pg(x) \\
\leq n ||x|| + ||Pg - N^\perp \cap M^\perp|| ||x - M|| \\
\leq n ||x|| + ||Pg - N^\perp \cap M^\perp|| .
$$

Since $Qf$ belongs to $(U - x)^o$ as $f$, it follows that

$$
||Qg|| \leq n + ||Qf|| \leq n + \text{Re } Qf(x) + 1 \\
\leq n(||x|| + 2) + ||Pg - N^\perp \cap M^\perp|| .
$$

Then (3) applied to $g$ yields

$$
||Pg - N^\perp \cap M^\perp|| \leq \frac{\gamma(n ||x|| + 2)}{1 - \gamma} = \delta_1
$$

and consequently

$$
||Qf + g - N^\perp \cap M^\perp|| \leq n + \delta_1 = \delta_2 .
$$

Since $x \in N, g \in N^\perp$ and $Qf \in (U - x)^o$,

$$
||Qf|| \leq \text{Re } Qf(x) + 1 \leq \delta_2 ||x|| + 1 = \delta_3
$$

and

$$
||g - N^\perp \cap M^\perp|| \leq \delta_2 + \delta_3 = \delta .
$$

This implies that

$$
f + g + h = Qf + g + (Pf + h) \\
\leq \delta U^0 \cap (U - x)^o + \delta U^0 \cap N^\perp + M^\perp .
$$

This completes the proof.

Consider the sup-norm Banach space $C(X)$ of continuous functions on a compact Hausdorff space $X$. By the Riesz theorem its dual is realized by the space of regular Borel measures on $X$ with total-variation norm. Given a closed subset $Y$ of $X$, let $M$ be the subspace
of functions in $C(X)$ vanishing on $Y$. Then $M_1$ is the set of measures
with support in $Y$ and becomes the range of a projection $P$: $Pm = \chi m$ for each measure $m$ where $\chi$ is the characteristic function of $Y$.
Obviously (1) is satisfied. Now let $N$ be a closed subspace of $C(X)$.
As shown in [5] (3) is equivalent to the property that for any $x \in N$ with $|x(t)| < 1 (t \in Y)$ and any closed subset $Z \subset X$ with $Y \cap Z = \emptyset$ there is $y \in N$ such that $x(t) = y(t) (t \in Y)$, $|y(s)| < \gamma (s \in Z)$ and $\|y\| < \max(1, \gamma)$. Remark that $\|x - M\|$ coincides with the norm of the restriction $x|Y$ of $x$ to $Y$ and that $x(t) = y(t) (t \in Y)$ is equivalent to $x - y \in M$. Thus Theorem 2 shows that if (3) with $\gamma < 1$, or equivalently (2) with $\rho < 2$, is satisfied then for any $x \in N$ there is $y \in N$ such that $x|Y = y|Y$ and $\|y\| = \|x|Y\|$. The case $\gamma = 0$ is the generalized Carleson-Rudin theorem (cf. [6] Chap. II). As Gamelin [5] shows, Theorem 2 can further yield the following: suppose that (3) with $\gamma < 1$, or equivalently (2) with $\rho < 2$, is satisfied and that $\rho \in C(X)$ satisfies $p(t) = 1 (t \in Y)$ and $p(s) > \gamma (s \in X)$. Then if $x \in N$ satisfies $|x(t)| \leq p(t) (t \in Y)$ there is $y \in N$ such that $x(t) = y(t) (t \in Y)$ and $|y(s)| \leq p(s) (s \in X)$. The case $\gamma = 0$ is the Bishop theorem (cf. [6] Chap. II). Generalization of the Gamelin theorem to other direction is treated by Alfsen and Hirsberg [1].

3. Ordered Banach spaces. Let $E$ be an ordered Banach space with positive cone $E_+$. A closed subspace $M$ is called an ideal if $(M - E_+) \cap (M + E_+) \subseteq M$. An ideal $M$ is hypostRICT if its annihilator $M^\perp$ is the range of a projection $P$ such that $f \geq Pf \geq 0$ for every $f \geq 0$. The requirement means that both $P$ and $Q = 1 - P$ are positive. Perdrizet [9] shows that a closed subspace $M$ is a hypostRICT ideal if and only if the following two conditions are satisfied: (1) Given $x_1, x_2 \in M$ and $y \in E$ with $x_1, x_2 \leq y$, for any $\varepsilon > 0$ there is $z \in M$ such that $x_1, x_2 \leq z \leq y + \varepsilon$, and (2) given $x \in M$ and $y_1, y_2 \in E_+$ with $x \leq y_1 + y_2$ there are $z_1, z_2 \in M$ such that $x = z_1 + z_2$ and $x_i \leq y_i + \varepsilon$ for $i = 1, 2$. Under the Riesz interpolation property an ideal $M$ is hypostRICT if and only if it is positively generated in the sense: $M = M \cap E_+ \cap E_+$. When $M$ is an ideal, the Banach space $E/M$ is preordered by the cone $\tau(E_+)$ where $\tau$ is the canonical map from $E$ to $E/M$. The following theorem was first proved by Davies [4] under the Riesz interpolation property and then by Perdrizet [9] in general case. Let us give a proof based on Theorem 1.

**Theorem 3.** Let $E$ be an ordered Banach space with positive cone $E_+$. If $M$ is a hypostRICT ideal then $E/M$ is an ordered Banach space with $\tau(E_+)$ as its positive cone. If $E$ is regular in addition, so is $E/M$. 
Proof. Since hypostrictness means that $E^*_+ = - E_+^*$ is invariant under both $P$ and $Q$, $E_+$ is splittable by Corollary 3. Then $\tau(E_+)$ is closed by Theorem 1. $M^+$ is isometric to the dual of $E/M$, and the dual positive cone is identified with $M^+ \cap E^*_+$. Suppose that $E$ is regular. Then $E^*$ is regular. Since $P$ is positive and is of norm one in this case, $M^1$ is a regular ordered Banach space with $M^1 \cap E^*_+$ as its positive cone. Therefore $E/M$ is regular as stated in §1. This completes the proof because every ordered Banach space admits an equivalent regular norm.

**COROLLARY 5.** Suppose that the positive cone $E_+$ has nonempty interior and that $M$ is a hypostrict ideal. If a closed subspace $N$ is splittable and if it contains an interior point of $E_+$ then $\tau(N \cap E_+)$ is closed and $\tau(N \cap E_+) = \tau(N) \cap \tau(E_+)$.

**Proof.** Since $E_+$ is splittable, in view of Corollary 2 and Theorem 1 it suffices to prove that $(N \cap E_+)^0 = N^+ + E_+^0$. Remark that $f$ belongs to $(N \cap E_+)^0$ if and only if the restriction of $-f$ to $N$ is positive. However it is known (cf. [10] Chap. V §5) that when $N$ contains an interior point of $E_+$ every continuous positive linear functional on $N$ admits a continuous positive linear extension to $E$, in other words, $- (N \cap E_+)^0 = - (N^+ + E_+^0)$.

Since $E/M$ is ordered by the cone $\tau(E_+)$, for any $y, z$ with $\tau(z) \leq \tau(y)$ there is $y'$ such that $z \leq y'$ and $\tau(y) = \tau(y')$. The next task is to treat the case $\tau(z) \leq \tau(y) \leq \tau(x)$ and $z \leq x$ and to find a condition of existence $y''$ such that $z \leq y'' \leq x$ and $\tau(y) = \tau(y'')$.

**LEMMA 5.** Let $S_1$ and $S_2$ be closed convex subsets containing 0. If for any $0 < \lambda < 1$, $f \in S_1$ and $g \in S_2^0$ there are $f' \in S_1^0$ and $g' \in S_2^0$ such that

$$\lambda f + (1 - \lambda)g = \lambda f' + (1 - \lambda)g'$$

and

$$\lambda \|f'\|, (1 - \lambda) \|g'\| \leq \|\lambda f + (1 - \lambda)g\|$$

then $(S_1 \cap S_2)^{**}$ coincides with $S_1^* \cap S_2^*$ where $(\cdot)^*$ denotes the weak* closure.

**Proof.** In view of the Krein-Smulian theorem it suffices to prove that for any $\gamma > 0$ the weak* closure of $\text{conv} (S_1^0 \cup S_2^0) \cap \gamma U^0$ is contained in the norm closure of $\text{conv} (S_1^0 \cup S_2^0)$. Suppose that $0 < \lambda_a < 1, f_a \in S_1^0$, $g_a \in S_2^0$ and $\|\lambda_a f_a + (1 - \lambda_a)g_a\| \leq \gamma$ and that the net $\{\lambda_a f_a +
(1 - λα)gα weak* converges to h and the net {λα} converges to λ. By hypothesis {λαfα} and ((1 - λα)gα) can be assumed to be bounded, hence to weak* converge to f' and g' respectively. If 0 < λ < 1, {fα} and {gα} can be assumed to weak* converge to f'' ∈ S_i and g'' ∈ S_i respectively. Then h = λf'' + (1 - λ)g'' belongs to conv (S_i ∪ S_2). In case λ = 0, h = f' + g'' and n f' belongs to S for any n > 0. Therefore h, as the norm limit of 1/n(n f') + (1 - 1/n)g'', belongs to the norm closure of conv (S_i ∪ S_2). The case λ = 1 is treated similarly.

**Corollary 6.** \((\bigcap_{i=1}^{n} (x_i - E_+))^\sim = \bigcap_{i=1}^{n} (x_i - E_+)^\sim\) whenever \(x_i \geq 0\) i = 1, 2, ..., n.

**Proof.** E, hence E*, can be assumed to be regular. \((x_i - E_+)^\circ\) consists of all \(0 \leq f\) with \(f(x_i) \leq 1\). Suppose that \(λ_i \geq 0, \sum_{i=1}^{n} λ_i = 1\) and \(f_i ∈ (x_i - E_+)^\circ\). Since the norm is monotone on the dual positive cone by regularity, it follows that \(λ_i\|f_j\| \leq \|\sum_{i=1}^{n} λ_i f_i\| j = 1, 2, \cdots, n\). Now inductive application of Lemma 5 yields the assertion.

The following theorem was proved by Perdrizet [9]. Let us give a proof based on Theorem 1.

**Theorem 4.** Let E be an ordered Banach space with positive cone E+. Suppose that M is a hypostrict ideal and E/M is ordered by the cone τ(E+) where τ is the canonical map from E to E/M. If \(z_i \leq 0 \leq x_i\) and \(τ(z_i) \leq τ(y) \leq τ(x_i) i = 1, 2, \cdots, n\), then for any \(ε > 0\) there is \(y'\) such that \(z_i \leq y' ≤ x_i + ε i = 1, 2, \cdots, n\) and \(τ(y) = τ(y')\). Further \(ε\) can be made 0 if every \(x_i\) is an interior point of E+ or if E has the Riesz interpolation property.

**Proof.** E is assumed to be regular, hence Q is of norm one. \(z_i + E_+\) is a closed convex set containing 0. It is splittable because both \(P^*\) and \(Q^*\) are positive and \(z_i\) is negative. Similiary \(x_i - E_+\) is splittable. Let \(S_1 = \bigcap_{i=1}^{n} (z_i + E_+)\) and \(S_2 = \bigcap_{i=1}^{n} (x_i - E_+)\). Then by Corollaries 2, 6, and Theorem 1 both \(S_1\) and \(S_2\) are splittable and

\[
\bigcap_{i=1}^{n} τ(z_i + E_+) \cap \bigcap_{i=1}^{n} τ(x_i - E_+) = τ(S_1) \cap τ(S_2) \subseteq τ(S_1 \cap (S_2 + ε U))
\]

which is just the first assertion.

If every \(x_i\) is an interior point of E+, \(S_2\) contains 0 in its interior and by Corollary 2 and Theorem 1

\[τ(S_1) ∩ τ(S_2) = τ(S_1 ∩ S_2)\]
Suppose finally that $E$ has the Riesz interpolation property. Since $E^{**}$ becomes a lattice as stated in §1, $S_i$ consists of all $w \in E^{**}$ with $\bigvee_{i=1}^n z_i \leq w$, where $\bigvee_{i=1}^n z_i$ denotes the supremum of $z_i, \ldots, z_n$ in $E^{**}$. Then $S_0^i$ consists of all $0 \leq f$ with $f(\bigvee_{i=1}^n z_i) \leq 1$. Similarly $S_0^2$ consists of all $0 \leq g$ with $g(\bigwedge_{i=1}^n x_i) \leq 1$ where $\bigwedge_{i=1}^n x_i$ denotes the infimum of $x_1, \ldots, x_n$, in $E^{**}$. Take $0 < \lambda < 1$, $f \in S_0^i$ and $g \in S_0^2$ and let $h = \lambda f + (1 - \lambda) g$. Since $E^*$ is a Banach lattice as stated in §1, and since both $-f$ and $g$ are positive, it follows that $0 \leq h \wedge 0 \leq \lambda f$ and $0 \leq h \vee 0 \leq (1 - \lambda) g$. Let $f' = (1/\lambda)(h \wedge 0)$ and $g' = (1)/(1 - \lambda) h \vee 0$. Then it follows from the above characterization of $S_i^0$ that $f' \in S_i^0$, $g' \in S_0^2$ and $h = \lambda f' + (1 - \lambda) g'$. Now since $\|h \wedge 0\|, \|h \vee 0\| \leq \|h\|$, Lemma 5 yields $(S_i \cap S_0^2) = S_i^0 \cap S_2^0$ and the assertion follows from Theorem 1.

4. Linear lifting. Let $E$ be a Banach space with unit ball $U$ and $M$ a closed subspace. The canonical map from $E$ to $E/M$ is denoted by $\tau$. A continuous linear map $\varphi$ from $E/M$ to $E$ is called a linear lifting if $\tau \circ \varphi = 1$. If $\varphi$ is a linear lifting, $\varphi \circ \tau$ is a projection with $M$ as its kernel. Conversely, a linear lifting exists if $M$ is the kernel of a continuous projection.

In this section it is assumed:

There is a projection $P$ from $E^*$ to $M^*$ such that

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*)$$

and $Q$ stands for $1 - P$.

Let $F$ be a finite dimensional Banach space with unit ball $V$. Consider the dual system $\langle F^* \otimes E, F \otimes E^* \rangle$ of tensor products. When $F^* \otimes E$ is provided with the Minkowski functional of $(V \otimes U)^0$ as norm, it is called the inductive tensor product of $F^*$ and $E$ and is denoted by $F^* \otimes E$. When $F \otimes E^*$ is provided with the Minkowski functional of conv $(V \otimes U)^*$ as norm, it is called the projective tensor product of $F$ and $E^*$ and is denoted by $F \otimes E^*$. Let $B = B(F, E)$ denote the Banach space of all continuous linear maps from $F$ to $E$, provided with operator-norm. Since $F$ is finite dimensional, $B$ is isometric to the inductive tensor product $F^* \otimes E$ under the canonical correspondence. The following lemma, whose proof is found in [10] Chap. IV §9, is basic in the subsequent development.

**Lemma 6.** The dual of $B(F, E)$ is isometric to the projective tensor product $F^* \otimes E^*$ while the second dual is isometric to the inductive tensor product $F^* \otimes E^{**}$, hence to $B(F, E^{**})$.

In view of Lemma 6 $B^{**}$ is always identified with $B(F, E^{**})$. 
In this case the imbedding of $B$ to $B^{**}$ is just the natural imbedding of $B(F, E)$ to $B(F, E^{**})$. In accordance with the terminology in §1 the weak** closure of a subset $G$ of $B$ is formed in $B(F, E^{**})$ and is denoted by $G^{\sim}$. When $K$ and $S$ are a subset of $F$ and a closed convex subset of $E$ containing $0$ respectively, $G(K, S)$ and $\mathcal{E}(K, S^{\sim})$ denote the set of all $\varphi \in B$ with $\varphi(K) \subseteq S$ and the set of all $\psi \in B^{**}$ with $\psi(K) \subseteq S^{\sim}$. Obviously $G(K, S)$ is a closed convex subset of $B$ containing $0$ and its weak** closure is contained in $\mathcal{E}(K, S^{\sim})$.

**Corollary 6.** (a) $G(V, U)^{\sim} = \mathcal{E}(V, U^{\sim})$. (b) $\{G(H, 0) \cap G(F, N)^{\sim}\} = \mathcal{E}(H, 0) \cap \mathcal{E}(F, N^{\sim})$ if $H$ and $N$ are closed subspaces of $F$ and $E$ respectively. (c) $G(K, S)^{\sim} = \mathcal{E}(K, S^{\sim})$ if $K$ is a cone generated by a linearly independent basis $\{x_1, \cdots, x_n\}$ of $F$ and $S$ is a cone.

**Proof.** (a) is an immediate consequence of Lemma 6. (b) $G(F, N)^{\sim} = \mathcal{E}(F, N^{\sim})$ follows from Lemma 6 applied to $N$ instead of $E$. Since $F$ is finite dimensional, $H$ is the kernel of a projection $\sigma$. Then

$$\mathcal{E}(H, 0) \cap \mathcal{E}(F, N^{\sim}) = \mathcal{E}(F, N^{\sim}) \circ \sigma = G(F, N)^{\sim} \circ \sigma \subseteq \{G(F, N) \circ \sigma \}^{\sim} = \{G(H, 0) \cap G(F, N)\}^{\sim},$$

while the reverse inclusion is obvious. (c) Take any $\varphi$ in $\mathcal{E}(K, S^{\sim})$ and let $y_i = \varphi(x_i)$ \(i = 1, 2, \cdots, n\). Since each $y_i$ belongs to $S^{\sim}$, there are nets $(y_i)_{\alpha}$ in $S$, weak** converging to $y_i \ i = 1, 2, \cdots, n$. Consider a net $(\varphi_{\alpha})$ in $B$ defined by $\varphi_{\alpha}(x_i) = y_{i, \alpha} i = 1, 2, \cdots, n$. By hypothesis it is contained in $G(K, S)$ and weak** converges to $\varphi$. Thus $\mathcal{E}(K, S^{\sim})$ is contained in $G(K, S)^{\sim}$ with the reverse inclusion is obvious.

Since $B^*$ is identified with the projective tensor product $F \hat{\otimes} E^*$ by Lemma 6, the operators $1 \otimes P$ and $1 \otimes Q$ are considered to define projections in $B^*$. The adjoints of $1 \otimes P$ and $1 \otimes Q$ are realized in $B(F, E^{**})$ according to the following formula:

$$(5) \quad (1 \otimes P)^* \varphi = P^* \circ \varphi \quad \text{and} \quad (1 \otimes Q)^* \varphi = Q^* \circ \varphi \quad (\varphi \in B(F, E^{**})).$$

**Lemma 7.** The annihilator $G(F, M)^{\perp}$ is the range of $1 \otimes P$.

**Proof.** Since $Q^*$ is a projection onto $M^{\sim}$, by (5) $(1 \otimes Q)^*$ projects $B^{**}$ onto $\mathcal{E}(F, M^{\sim})$, which coincides with $G(F, M)^{\sim}$ by Corollary 6. Then $1 \otimes P$ is obviously a projection from $B^*$ to $G(F, M)^{\perp}$.

On the basis of Lemma 7, a sentence "$G(K, S)$ is splittable" will always mean that $G(K, S)$ is $1 \otimes P$-splittable.

**Corollary 7.** If $S$ is splittable and $G(K, S)^{\sim} = \mathcal{E}(K, S^{\sim})$ then
$G(K, S)$ is splittable.

Proof. This follows from (5) by Lemma 1.

The following lemma can be considered a development of a basic device in Michael and Pełczynski [8], treating linear lifting in a special case. The crucial requirement for $P$ plays a decisive role in the proof.

**Lemma 8.** Suppose that $S$ is splittable and $G(K, S)^\sim = \mathcal{C}(K, S^\sim)$ for a subset $K$ of $F$. If $\varphi$ belongs to

$$G(\pi(K), S) \cap G(\pi(V), U) \cap G(K, S + M) \cap G(V, U + M)$$

where $\pi$ is a projection of $F$ to a subspace $H$, then for any $\varepsilon > 0$ there is $\varphi$ in $G(K, S) \cap G(V, U)$ such that

$$\tau \circ \varphi = \tau \circ \varphi \quad \text{and} \quad ||(\varphi - \psi)|_H|| < \varepsilon.$$  

Proof. Remark first of all that the requirement for $P$ means by Corollary 1 that the unit ball $U$ is splittable.

Let $\psi_1 = \psi - Q^* \circ \psi \circ (1 - \pi)$. Since

$$Q^* \circ \psi_1(K) \subseteq Q^* \circ \psi \circ \pi(K) \subseteq Q^*(S) \subseteq S^\sim$$

and

$$P^* \circ \psi_1(K) \subseteq P^* \circ \psi(K) \subseteq P^*(S + M) \subseteq S^\sim$$

by splitability of $S$, $\psi_1$ belongs to $\mathcal{C}(K, S^\sim)$ by Lemma 1, hence to $G(K, S^\sim)$ by hypothesis. Since $U$ is splittable and $G(V, U)^\sim = \mathcal{C}(V, U^\sim)$ by Corollary 6, the same argument shows that $\psi_1$ belongs also to $G(V, U)^\sim$. Moreover it belongs to $(G(K, S) \cap G(V, U))^\sim$ because $G(V, U)$ is the unit ball of $B$. On the other hand, $Q^* \circ \psi \circ (1 - \pi)$ belongs to $\mathcal{C}(H, 0) \cap \mathcal{C}(F, M^\sim)$, hence to $(G(H, 0) \cap G(F, M))^\sim$ by Corollary 6. Thus $\psi$ belongs to

$$(G(K, S) \cap G(V, U) + G(H, 0) \cap G(F, M))^\sim.$$  

It follows that $\psi$ must be contained in the norm closure of

$$G(K, S) \cap G(V, U) + G(H, 0) \cap G(F, M).$$

Therefore there is $\psi_2 \in B$ such that $\psi - \psi_2 \in G(H, 0) \cap G(F, M)$ and

$$||\psi_2 - G(K, S) \cap G(V, U)|| < \varepsilon.$$  

Since $G(K, S) \cap G(V, U)$ is splittable by hypothesis and Corollary 7, Lemma 3 guarantees that there is $\varphi \in G(K, S) \cap G(V, U)$ such that $\varphi - \psi_2 \in G(F, M)$ and $||\varphi - \psi_2|| < \varepsilon$. Now $\psi_2 - \psi \in G(H, 0) \cap G(F, M)$
implies that \( \tau \circ \varphi = \tau \circ \psi \) and
\[
|| |(\varphi - \psi)|H|| = |||(\varphi - \psi)|H|| \leq ||\varphi - \psi|| < \varepsilon.
\]

Let \( S \) be a closed splittable subset of \( E \) and \( L \) a subset of \( \tau(S) \). Suppose that there is a sequence of projections \( \{\pi_n\} \) in \( E/M \) such that
(1) the range \( F_n \) of \( \pi_n \) is of finite dimension, (2) \( ||\pi_n|| \leq 1 \), (3) \( \pi_n \cdot \pi_m = \pi_m \) for \( n \leq m \), (4) \( \pi_n(L) \subseteq L \) and (5) \( \pi_n \) converges strongly to the identity as \( n \to \infty \).

Let \( \mathcal{G}_n \) denote the set of all \( \varphi \in B(F_n, E) \) with \( \varphi \circ \pi_n(L) \subseteq S \) while \( \mathcal{G}_n \) is the set of all \( \psi \in B(F_n, E^{**}) \) with \( \psi \circ \pi_n(L) \subseteq S^{**} \). As before, the second dual of \( B(F_n, E) \) is identified with \( B(F_n, E^{**}) \).

**Lemma 9.** If the weak** closure of \( \mathcal{G}_n \) coincides with \( \mathcal{G}_n \), \( n = 1, 2, \cdots \), then there is a linear lifting \( \varphi \) from \( E/M \) to \( E \) such that \( \varphi(L) \subseteq S \) and \( ||\varphi|| \leq 1 \).

**Proof.** Let \( \pi_0 = 0 \) and \( \varphi_0 = 0 \). Assume that linear maps \( \varphi_j \in B(F_j, E) \) \( j = 0, 1, \cdots, n \) have been found in such a way that \( \tau \circ \varphi_j = 1 \) on \( F_j \), \( ||\varphi_j|| \leq 1 \), \( \varphi_j \circ \pi_j(L) \subseteq S \) and \( ||(\varphi_j - \varphi_i)|F_{j-1}|| < 1/2^{j-1} \) \( j = 0, 1, \cdots, n \). Since \( F_{n+1} \) is finite dimensional by hypothesis, there is \( \psi \in B(F_{n+1}, E) \) such that \( \tau \circ \psi = 1 \) on \( F_{n+1} \). Consider the map \( \psi' = \varphi_n \circ \pi_n + \psi(1 - \pi_n) \) from \( F_{n+1} \) to \( E \). Then by assumption
\[
\psi'(\pi_n(L)) = \varphi_n \circ \pi_n(L) \subseteq S
\]
and in view of \( ||\pi_n|| \leq 1 \)
\[
\psi' \circ \pi_n(V_{n+1}) = \varphi_n(V_n) \subseteq U
\]
where \( V_i \) denotes the unit ball of \( F_i \). Since \( V_{n+1} \subseteq \tau(U) \) by Theorem 1 and \( \pi_{n+1}(L) \subseteq \tau(S) \),
\[
\psi'(V_{n+1}) \subseteq U + M \text{ and } \psi'(\pi_{n+1}(L)) \subseteq S + M.
\]
Since the weak** closure of \( \mathcal{G}_n \) coincides with \( \mathcal{G}_n \) by hypothesis, Lemma 8, applied to \( F_{n+1}, \pi_{n+1}(L) \) and \( \pi_n \) instead of \( F, K \) and \( \pi \), yields that there is \( \varphi_{n+1} \in \mathcal{G}_{n+1} \) such that \( ||\varphi_{n+1}|| \leq 1, \tau \circ \varphi_{n+1} = 1 \) on \( F_{n+1} \) and \( ||(\varphi_{n+1} - \varphi_n)|F_n|| < 1/2^n \), completing induction. Now the sequence \( \{\varphi \circ \pi_n\} \) is uniformly bounded and
\[
\sum_{k=n}^{\infty} ||(\varphi_{k+1} - \varphi_k)|F_n|| \leq \sum_{j=n}^{\infty} 1/2^k < \infty
\]
guaranteeing convergence of \( \varphi_k(x) \) for every \( x \in F_n \) as \( k \to \infty \). Then \( \{\varphi \circ \pi_n\} \) converges strongly to some map \( \varphi \) from \( E/M \) to \( E \). Obviously \( \varphi \) is a required linear lifting.

It is better to introduce some terminology before stating the main
result on linear lifting. A Banach space $E$ is called a $\pi$-space if there is a sequence $\{F_n\}$ of finite dimensional subspaces such that $F_1 \subseteq F_2 \subseteq \cdots$ with $\bigcup_{n=1}^{\infty} F_n = E$ and each $F_n$ is the range of a projection of norm one. Here projections $\pi_n$ can be assumed to have the property that $\pi_n \pi_m = \pi_n$ for $n \leq m$ and that $\pi_n$ converges strongly to the identity as $n \to \infty$. An ordered Banach space is called a $\Pi$-space if, in addition, projections can be chosen positive and if each $F_n$ has the positive cone generated by a linearly independent basis.

**Theorem 5.** Suppose that the annihilator of a closed subspace $M$ is the range of a projection $P$ such that

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*).$$

If the quotient space $E/M$ becomes a $\pi$-space then there is a linear lifting of norm one, or equivalently, $M$ is the kernel of a projection of norm one.

**Proof.** Since the unit ball $U$ is splittable by Corollary 1, all requirements in Lemma 9 are fulfilled with $S = U$ and $L = \tau(U)$ by Corollary 7.

**Corollary 8.** Let $N$ and $M$ be closed subspaces and suppose that $M^\perp$ is the range of a projection $P$ such that $P(N^\perp) \subseteq N^\perp$ and

$$\|f\| = \|Pf\| + \|f - Pf\| \quad (f \in E^*).$$

If the quotient space $N/N \cap M$ is a $\pi$-space, there is a linear lifting of norm one from $N/N \cap M$ to $N$.

**Proof.** In view of Theorem 5 it suffices to prove that the annihilator of $N \cap M$ in $N^*$ is the range of a projection $\mathcal{P}$ such that

$$\|g\| = \|\mathcal{P}g\| + \|g - \mathcal{P}g\| \quad (g \in N^*).$$

When $N^*$ is identified with $E^*/N^\perp$, the annihilator of $N \cap M$ becomes the image of $(N \cap M)^\perp$ under the canonical map from $E^*$ to $E^*/N^\perp$. Since hypothesis implies splittability of $N$ by Corollary 3, $N + M$ is closed by Theorem 1 so that $(N \cap M)^\perp$ is weak* closed and coincides with $(N \cap M)^\perp$. Therefore the annihilator of $N \cap M$ in $N^*$ becomes the image of $M^\perp$ in $E^*/N^\perp$. Since $N^\perp$ is invariant under $P$, there arises a natural projection $\mathcal{P}$ from $N^*$ to the annihilator $N \cap M$. Since by hypothesis

$$\|f - N^\perp\| \geq \|Pf - N^\perp\| + \|f - Pf - N^\perp\|,$$

$\mathcal{P}$ is easily seen to have the required property.
When $E$ is the space of continuous functions on a compact set and $M$ consists of functions vanishing on a fixed closed subset, Corollary 8 was proved by Michael and Pełczyński [8].

**Theorem 6.** Let $M$ be a closed subspace of an ordered Banach space $E$. Suppose that $M$ is the range of a projection $P$ such that $f \geq Pf \geq 0 \ (f \geq 0)$ and

$$\|f\| = \|Pf\| + \|f - Pf\| \ (f \in E^*).$$

If $E/M$ is a $\Pi$-space under the canonical ordering, there is a positive linear lifting of norm one, or equivalently, $M$ is the kernel of a positive projection of norm one.

**Proof.** Since the positive cone $E_+$ is splittable by Corollary 3, all requirements in Lemma 9 are fulfilled with $S = E_+$ and $L = \tau(E_+)$ by definition of a $\Pi$-space and Corollary 6.

To be a $\pi$-space or a $\Pi$-space is not so severe restriction. Let us prove:

*Separable complex (resp. real) $L_p(1 \leq p < \infty)$ and complex (resp. real) $C(X)$ on compact metrizable $X$ are $\pi$-spaces (resp. $\Pi$-spaces).*

In fact, it suffices for the first part to treat a $L_p$ space on a finite measure space $(\mathcal{B}, \mu)$. Since the Borel field is separable with respect to $\mu$, there is an increasing sequence $\{\mathcal{B}_n\}$ of finite Borel subfields such that $\bigcup_{n=1}^{\infty} L_p(\mathcal{B}_n)$ is dense in $L_p$ where $L_p(\mathcal{B}_n)$ is the subspace of $\mathcal{B}_n$-measurable functions. Each $L_p(\mathcal{B}_n)$ is finite dimensional and the conditional expection relative to $\mathcal{B}_n$ becomes a (positive) projection of norm one from $L_p$ to $L_p(\mathcal{B}_n)$ (cf. [3]). The assertion for $C(X)$ is proved in [7] by using peaked partition.

**References**


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