# A TWO SIDED APPROXIMATION THEOREM FOR 2-SPHERES

WILLIAM THOMAS EATON

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## W. T. EATON

The side approximation theorem proved by R. H. Bing and later improved by F. M. Lister states that a sphere $S$ topologically embedded in Euclidean three space can be $\varepsilon$-approximated with polyhedral spheres $g(S)$ and $h(S)$ such that $g(S - \cup G_i) \subset \operatorname{Int} S$, $g(G_i) \cap S \subset G_i$, $h(S - \cup H_i) \subset \operatorname{Ext} S$, and $h(H_i) \cap S \subset H_i$ where $\{G_i\}$ and $\{H_i\}$ are respectively finite collections of disjoint $\varepsilon$-disks in $S$. In this article the theorem is strengthened by showing that the sets $\cup G_i$ and $\cup H_i$ may also be taken to be disjoint.

The theorem is a generalization of R. H. Bing's side approximation theorem [2], [3] and is used [7], [8] to distinguish certain decomposition spaces from 3-manifolds.

The proof is divided into two main parts. In §2, Theorem 2.1 is reduced to Corollary 6.1 of §6 using techniques that are consequences of Bing's side approximation theorem. The proof of Corollary 6.1 is the main topic of this paper. The proof depends on Lemma 5.15 and Theorem 5.16 of §5 which give the combinatorial structure of collections of components associated with the general position of two 2-spheres in a 3-manifold. The proofs of Lemma 5.15 and Theorem 5.16 depend in turn on the concepts of separation complexes and winding functions introduced, respectively, in §§3 and 4.

**2. Reduction to geometry.** The main result of this paper is Theorem 2.1 stated below. The purpose of this section is to reduce the proof of Theorem 2.1 to Corollary 6.1 of §6.

THEOREM 2.1. *If $S$ is a 2-sphere in $E^3$ and $\varepsilon$ is a positive number then there exists a finite collection $\{G_1, \cdots, G_n, H_1, \cdots, H_n\}$ of disjoint $\varepsilon$-disks in $S$ and $\varepsilon$-homeomorphisms $g, h \colon S \to E^3$ such that*
  (2.1.1)  $g(S - \cup \operatorname{Int} G_i) \subset \operatorname{Int} S$,
  (2.1.2)  $g(G_i) \cap S \subset \operatorname{Int} G_i$,
  (2.1.3)  $h(S - \cup \operatorname{Int} H_i) \subset \operatorname{Ext} S$, *and*
  (2.1.4)  $h(H_i) \cap S \subset \operatorname{Int} H_i$.
*Furthermore, $g(S)$ and $h(S)$ may be chosen to be polyhedral and disjoint.*

*Proof.* It follows from F. M. Lister's approximation theorem [9] that there exists a disjoint collection of $(\varepsilon/3)$-disks $C_1, \cdots, C_n$ in $S$ and $(\varepsilon/3)$-homeomorphisms $g_0, h_0 \colon S \to E^3$ such that $g_0(S - \cup \operatorname{Int} C_i) \subset$

Int $S$, $h_0(S - \cup \text{Int } C_i) \subset \text{Ext } S$, $g_0(C_i) \cap S \subset \text{Int } C_i$, $h_0(C_i) \cap S \subset \text{Int } C_i$, and $g_0(S)$, $h_0(S)$ are polyhedral. Let $V_1, \cdots, V_n$ be a collection of disjoint open $\varepsilon$-subsets of $E^3$ such that $C_i \cup g_0(C_i) \cup h_0(C_i) \subset V_i$  There is no loss in generality [4] in assuming that $\text{Bd} C_i$ is tame.

The required disks $G_i$, $H_i$ will be disjoint subdisks of $C_i$ and the required homeomorphisms $g, h$ will have the properties that

$$g\,|\,S - \cup C_i = g_0\,|\,S - \cup C_i,\; g(C_i) \subset V_i,\; h\,|\,S - \cup C_i = h_0\,|\,S - \cup C_i$$

and $h(C_i) \subset V_i$. All of the adjusting is done in the $V_i$'s one-at-a-time and independently. Let $i$ be fixed and to simplify notation let $C = C_i$, and $V = V_i$. Let $E_0 = h_0(C)$ and $E_1 = g_0(C)$. By standard scissor-and-paste techniques we may assume that $E_0 \cap E_1 = \varnothing$.

By the polyhedral approximation theorem [1] we may polyhedrally approximate $C$ to obtain a disk $E'$ such that $E' \cap (S - \text{Int } C) = \text{Bd} E' = \text{Bd} C$, $E' \subset V$, $E' \cap h_0(S) \subset \text{Int } E_0$, and $E' \cap g_0(S) \subset \text{Int } E_1$. We may assume that $E'$ and $E_0 \cup E_1$ are in general position. Let $E''$ be the component of $E' - (E_0 \cup E_1)$ that contains $\text{Bd} E'$. By filling in the holes of $E''$ with disks in $E_0 \cup E_1$ pushed slightly to the sides of $E_0 \cup E_1$, we obtain a polyhedral disk $E$ such that $E \cap (S - \text{Int } C) = \text{Bd} E = \text{Bd} C$, $E \subset V$, and $E \cap h_0(S) = E \cap g_0(S) = \varnothing$.

Let $\delta = (1/9)\rho(E, E_0 \cup E_1)$. By [5, Theorem 9.1] there exists a tame Sierpinski curve $X \subset C$ such that $\text{Bd} C \subset X$ and the components $X_1, X_2, \cdots$ of $C - X$ each have diameter less than $\delta$. By the polyhedral approximation theorem [1], there exist disks $Y_i$ such that $\text{Bd} Y_i = \text{Bd} X_i$, $D = X \cup (\cup Y_i)$ is a disk, $\text{Diam } Y_i < \delta$, $Y_i \subset V$, $X_j \cap Y_i = \varnothing$ if $i \neq j$, and $Y_i$ is locally polyhedral modulo $\text{Bd} Y_i$. It follows from [5] that $D$ is tame. Keeping points of $E^3 - V$, $g_0(S) - \text{Int } E_1$, $S - \text{Int } E$, and $h_0(S) - \text{Int } E_0$ fixed and moving no point as far as $\delta$, we first move $E_0 \cup E \cup E_1$ into general position with respect to $D$ with $\text{Bd} E = \text{Bd} D$ and then with a move "parallel" to $D$, we push $((E_0 \cup E \cup E_1) \cap D) - \text{Bd} E$ to the inaccessible part of $X$ (see [6] for more details).

The preceding adjustments enable us assume without loss of generality that $E_0$, $E$, $E_1$ are polyhedral and disjoint, $E_0 \cup E \cup E_1$ is in general position with respect to $D$ with $((E_0 \cup E \cup E_1) \cap D) - \text{Bd} E$ in the inaccessible part of $X$ and

(2.2)              no $\text{Cl } X_i$ intersects more than one of $E_0$, $E$, $E_1$.

In order to apply Corollary 6.1 we add the ideal point $\infty$ to $E^3$, let $T$ be the 2-sphere $(S - C) \cup D$, $U_i$ be the component of $S^3 - T$ containing $\text{Bd} E_i$, and let $E$, $E_0$, $E_1$ of Corollary 6.1 be as above. Let $D_0$, $D_1$ be the singular disks of the conclusion of Corollary 6.1.

Since $(E_0 \cup E_1) \cap D$ lies in the inaccessible part of $X$ there is a

natural open disk-with-holes $B^*$ in $S$ associated with each $B \in \mathscr{D}_i$; namely, let $B^*$ be obtained from $B$ by replacing each $Y_j \subset B$ with $\mathrm{Cl}\, X_j$. If $\mathscr{D}_i^* = \{B^* \,|\, B \in \mathscr{D}_i\}$ then $D_i^* = \cup \{\mathrm{Cl}\, B \,|\, B \in \mathscr{D}_i^* \cup \mathscr{E}_i\}$ are clearly singular disks, and $D_0^* \cap D_1^* = \varnothing$ since $D_0 \cap D_1 = \varnothing$ and each $X_j$ intersects at most one of $E_0$ and $E_1$. Some of the $X_j$'s may pass through elements of $\mathscr{E}_i$; thus, $D_i^*$ may not lie in the closure of a complementary domain of $S$. In the next paragraph we adjust the $D_i^*$'s so that each does lie in the closure of a complementary domain of $S$ while retaining the property $D_0^* \cap D_1^* = \varnothing$. We first observe that no element of $\mathscr{D}_{1-i}^*$ intersects an element of $\mathscr{E}_i$. For suppose that $X_j \subset B^* \in \mathscr{D}_{1-i}^*$, $A \in \mathscr{E}_i$ and $X_j \cap A \neq \varnothing$ then, since $X_j$ is connected and $X_j \cap Y_k = \varnothing$ if $k \neq j$, there exists a component $K$ of $X_j \cap U_i$ such that $K \cap A \neq \varnothing$ and $Y_j \cap \mathrm{Cl}\, K \neq \varnothing$. Since $Y_j \subset B$ and $E$ separates $A$ from $B$ in $\mathrm{Cl}\, U_i$ by (6.1.2), $\mathrm{Cl}\, K$ intersects $E$. Thus, $\mathrm{Cl}\, X_j$ intersects both $E$ and $E_i$ and this contradicts (2.2).

For $i = 0, 1$, let $f_i^*$ be a map on $E_i$ with the property that $f_i^*(E_i) = D_i^*$ and $f_i^* \,|\, \mathrm{Bd}\, E_i = 1$. Let $E_0^*$ be the component of

$$E_0 \cap (f_0^*)^{-1}(\mathrm{Cl}\,(\mathrm{Ext}\, S) \cap D_0^*)$$

that contains $\mathrm{Bd}\, E_0$ and let $E_1^*$ be the component of

$$E_1 \cap (f_0^*)^{-1}(\mathrm{Cl}\,(\mathrm{Int}\, S) \cap D_1^*)$$

that contains $\mathrm{Bd}\, E_1$. By the observations of the previous paragraph there exist subsequences $\{X_k^0\}, \{X_j^1\}$, of $\{X_i\}$ such that $\mathrm{Cl}\,(\cup X_k^0) \cap \mathrm{Cl}\,(\cup X_j^1) = \varnothing$ and $f_0^*((\mathrm{Bd}\, E_0^*) - \mathrm{Bd}\, E_0) \subset \mathrm{Cl}\,(\cup X_k^0)$, $f_1^*((\mathrm{Bd}\, E_1^* - \mathrm{Bd}\, E_0) \subset \mathrm{Cl}\,(\cup X_j^1)$. Also, if $C_0$ is a component of $E_0 - E_0^*$ then there exists a unique $X_k^0$ such that $f_0^*(\mathrm{Bd}\, C_0) \subset X_k^0$. Let $E_0^k = \mathrm{Cl}\,(\cup \{C_0 \,|\, f_0^*(\mathrm{Bd}\, C_0) \subset X_k^0\})$. We apply the Tietze-Extension-Theorem to normal space $E_0^k$, closed set $\mathrm{Bd}\, E_0^k$, map $f_0^* \,|\, \mathrm{Bd}\, E_0^k$, and disk $\mathrm{Cl}\, X_k^0$ to extend $f_0^* \,|\, E_0^*$ to $E_0^* \cup E_0^k$. Since $\{\mathrm{Cl}\, X_k^0\}$ is a null-sequence, extending $f_0^* \,|\, E_0^*$ to each $E_0^k$ in the above manner and piecing together the resulting maps yeilds a map $f_0': E_0 \to \mathrm{Cl}\,(\mathrm{Ext}\, S)$. In the same way, we construct a map $f_1': E_1 \to \mathrm{Cl}\,(\mathrm{Int}\, S)$. The maps $f_0': E_0 \to \mathrm{Cl}\,(\mathrm{Ext}\, S)$, $f_1': E_1 \to \mathrm{Cl}\,(\mathrm{Int}\, S)$ have the following properties $f_i' \,|\, \mathrm{Bd}\, E_i = 1$, $f_0'(E_0 - \mathrm{Bd}\, E_0) \cap h_0(S - \mathrm{Int}\, C) = \varnothing$, $f_1'(E_1 - \mathrm{Bd}\, E_1) \cap g_0(S - \mathrm{Int}\, C) = \varnothing$, $f_i'(E_i) \subset V$, and most importantly $f_0'(E_0) \cap f_1'(E_1) = \varnothing$.

By the results of [5], for each positive number $\delta$ there exist $\delta$-maps $\alpha: \mathrm{Cl}\,(\mathrm{Int}\, S) \to \mathrm{Cl}\,(\mathrm{Int}\, S)$, $\beta: \mathrm{Cl}\,(\mathrm{Ext}\, \delta) \to \mathrm{Cl}\,(\mathrm{Ext}\, S)$ such that $S \cap \alpha(\mathrm{Cl}\,(\mathrm{Int}\, S))$, $S \cap \beta(\mathrm{Cl}\,(\mathrm{Int}\, S))$ are 0-dimensional and $\alpha \,|\, (\mathrm{Int}\, S) - N = 1$, $\beta \,|\, (\mathrm{Ext}\, S) - N = 1$ where $N$ is an arbitrary neighborhood of $S$. By choosing $\delta$ and $N$ appropriately, in particular $\delta$ less than $(1/2)\rho(f_0'(E_0)$, $f_1'(E_1))$, we select maps $\alpha$, $\beta$ so that $\alpha f_1'(E_1) \subset \mathrm{Cl}\,(\mathrm{Int}\, S)$, $\alpha f_1' \,|\, \mathrm{Bd}\, E_1 = 1$, $\alpha f_1'(E_1 - \mathrm{Bd}\, E_1) \cap g_0(S - \mathrm{Int}\, C) = \varnothing$, $\alpha f_1'(E_1) \subset V$, $\beta f_0'(E_0) \subset \mathrm{Cl}\,(\mathrm{Ext}\, S)$,

$\beta f_0' \,|\, \mathrm{Bd}\,E_0 = 1$, $\beta f_0'(E_0 - \mathrm{Bd}\,E_0) \cap h_0(S - \mathrm{Int}\,C) = \varnothing$, $\beta f_0'(E_0) \subset V$, and $\alpha f_0'(E_1) \cap S$, $\beta f_0'(E_0) \cap S$ are disjoint compact 0-dimensional subsets of Int $C$.

Let $G$, $H$ be disjoint disks in Int $C$ such that $\alpha f_1'(E_1) \cap S \subset \mathrm{Int}\,G$ and $\beta f_0'(E_0) \cap S \subset \mathrm{Int}\,H$. We apply Dehn's lemma [10] to each of $\alpha f_1'(E_1)$ and $\beta f_0'(E_0)$ in a small neighborhood and each to obtain real disks $E_1'$ and $E_0'$, respectively. We obtain the maps $g$, $h$ of the conclusion by adjusting $g_0$, $h_0$ in $V$ so that $g_0(G) = E_1'$ and $h_0(H) = E_0'$.

**3. Separation complexes.** In §2 we reduced Theorem 2.1 to Corollary 6.1 of Theorem 5.16 in §§6 and 5, respectively. In this section we develop the concept of a separation complex which is used extensively in §§4, 5, and 6. We begin by reminding the reader of the definition of a collar and then defining a separation complex.

DEFINITION 3.1. If $M$ is a $PL$ manifold and $N$ is a $PL$ manifold in $\mathrm{Bd}\,M$ then a *collar of $N$ in $M$* is a $PL$ embedding $h$ of $N \times I$ into $M$ such that $h(y \times 0) = y$, and $h(N \times (0, 1]) \subset \mathrm{Int}\,M$.

DEFINITION 3.2. If $N$ is a compact, connected, orientable 2-manifold in $S^3$ with nonempty boundary and $U$ is a side of $N$ (i.e., $U$ is the interior of a compact, connected 3-manifold $M$ in $S^3$ such that $N \subset \mathrm{Bd}\,M$) then a *collar of $N$ to the $U$ side of $N$* is a $PL$ embedding $h$ of $N \times I$ into $\mathrm{Cl}\,U$ such that $h(y \times 0) = y$, and $h(N \times (0, 1]) \subset U$.

DEFINITION 3.3. A *separation complex* is a finite collection $K = K_1 \cup K_2 \cup K_3$ of sets in $S^3$ such that
  (3.3.1)  $K_1$ is a collection of disjoint $PL$ simple closed curves, called *separation 1-simplices*,
  (3.3.2)  $K_2$ is a collection of compact, connected, orientable, $PL$ 2-manifolds with nonempty boundary, call *separation 2-simplices*, such that
      (3.3.2.1)  if $\sigma \in K_2$ and $\dot\sigma$ is the collection of separation 1-simplices of $K$ intersecting $\sigma$, then $\mathrm{Bd}\,\sigma = \cup \{\tau \,|\, \tau \in \dot\sigma\}$ and each element of $\dot\sigma$ is called a *1-face* of $\sigma$,
      (3.3.2.2)  if $\sigma_1, \sigma_2 \in K_2$ then $\sigma_1 \cap \sigma_2$ is a union of faces of each,
  (3.3.3)  $K_3$ is a collection of compact, connected, $PL$ 3-manifolds with nonempty boundary, called *separation 3-simplices*, such that
      (3.3.3.1)  if $\sigma \in K_3$ and $\dot\sigma$ is the collection of separation 2-simplices of $K$ intersecting $\sigma$, then $\mathrm{Bd}\,\sigma = \cup \{\tau \,|\, \tau \in \dot\sigma\}$ and each element $\tau$ of $\dot\sigma$ is called a *2-face* of $\sigma$; each separation 1-simplex which intersects $\sigma$ lies in $\mathrm{Bd}\,\sigma$

and is called a 1-*face* of $\sigma$,

(3.3.3.2)   if $\sigma_1, \sigma_2 \in K_3$ then $\sigma_1 \cap \sigma_2$ is a union of 1 and 2 faces of each.

**DEFINITION 3.4.** A *subcomplex* of a separation complex $K$ is a subcollection $L \subset K$ that is also a separation complex.

**LEMMA 3.5.** *If $K$ is a separation complex then $L \subset K$ is a subcomplex of $K$ if and only if $\sigma \in L$ implies $\dot{\sigma} \subset L$.*

NOTATION. If $K$ is a separation complex then $|K| = \bigcup_{\sigma \in K} \sigma$, and if $\sigma \in K$ then $\bar{\sigma}$ denotes the subcomplex of $K$ consisting of $\sigma$ and its faces.

**DEFINITION 3.6.** If $K = K_1 \cup K_2 \cup K_3$, $L = L_1 \cup L_2 \cup L_3$ are separation complexes, then a function $f: |K| \to |L|$ is a *separation isomorphism from $K$ to $L$* if $f$ is one-to-one onto, $\sigma \in K_i$ implies $f(\sigma) \in L_i$, and $\sigma \in K$ implies $f(\mathrm{Bd}\sigma) = \mathrm{Bd}f(\sigma)$. Note that the function $f \mid \sigma \in K$ need not be a homeomorphism or even continuous. In general $\sigma$ and $f(\sigma)$ are necessarily homeomorphic only if $\sigma \in K_1$, in which case they are both simple closed curves.

**DEFINITION 3.7.** If $K, K'$ are separation complexes then $K'$ is a *subdivision of $K$* if $|K'| = |K|$ and each element of $K$ is a union of elements of $K'$.

Suppose $K, L$ are separation complexes, $f$ is a separation isomorphism from $K$ to $L$, and $K'$ is a subdivision of $K$. Does there exist a subdivision $L'$ of $L$ such that $K'$ and $L'$ are separation isomorphic? In general the answer is no as Figure 1 indicates. However, for suitably restricted subdivisions $K'$ the answer is yes.

**DEFINITION 3.8.** If $K = K_1 \cup K_2 \cup K_3$ is a separation complex then a subdivision $K' = K_1' \cup K_2' \cup K_3'$ of $K$ is a *separation subdivision of $K$* if

(3.8.1)   if $\sigma \in K_1'$ then there exists $\tau \in K_2$ such that $\sigma \in \dot{\tau}$ or $\sigma \subset \mathrm{Int}\,\tau$ and separates $\tau$,

(3.8.2)   if $\sigma \in K_2'$ then (a) there exists a $\tau \in K_2$ such that $\sigma \subset \tau$ or (b) there exists a $\tau \in K_3$ such that $\sigma \subset \tau$, $\sigma \cap \mathrm{Bd}\tau = \mathrm{Bd}\sigma$, and $\sigma$ separates $\tau$.

One easily verifies the equivalence of Definition 3.8 with the following:

**DEFINITION 3.9.** If $K = K_1 \cup K_2 \cup K_3$ is a separation complex, then a subdivision $K' = K_1' \cup K_2' \cup K_3'$ is an *elementary separation subdivision of $K$* if there are separation simplices $\sigma \in K_i'$ ($i = 1$ or 2),

$$K' = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$$
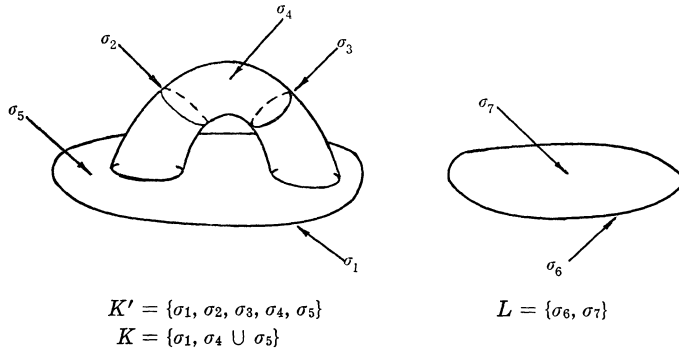$$K = \{\sigma_1, \sigma_4 \cup \sigma_5\}$$

$$L = \{\sigma_6, \sigma_7\}$$

FIGURE 1

$\tau_1, \tau_2 \in K'_{i+1}$, and $\tau \in K_{i+1}$ such that

(3.9.1)   $K' = (K - \{\tau\}) \cup \{\tau_1, \tau_2, \sigma\}$

(3.9.2)   $\tau = \tau_1 \cup \tau_2$, and

(3.9.3)   $\sigma = \tau_1 \cap \tau_2$.

(Note that if $i = 2$, then each face of $\sigma$ lies not only in $K'_1$ but also in $K_1$.) A subdivision $K''$ of $K$ is a *separation subdivision* if there is a sequence $K = K_0, K_1, \cdots, K_n = K''$ such that, for each $i > 0$, $K_i$ is an elementary separation subdivision of $K_{i-1}$.

REMARK. The conditions of Definition 3.8 are often easier to check then those of Definition 3.9. On the other hand, the conditions of Definition 3.9 are easier to use in proofs about separation subdivisions (e.g., see the proof of Lemma 3.10). Separation subdivisions will be obtained in this paper by the methods outlined in Lemmas 3.10, 3.13, and 3.14.

LEMMA 3.10. *Suppose* $K = K_1 \cup K_2 \cup K_3$, $L = L_1 \cup L_2 \cup L_3$ *are separation complexes,* $f$ *is a separation isomorphism from* $K$ *to* $L$, *and* $K' = K'_1 \cup K'_2 \cup K'_3$ *is a separation subdivision of* $K$. *Then there exists a separation subdivision* $L' = L'_1 \cup L'_2 \cup L'_3$ *of* $L$ *and a separation isomorphism* $f'$ *of* $K'$ *to* $L'$ *such that* $f'(\sigma) = f(\sigma)$ *if* $\sigma \in K$.

*Proof.* By induction on the number of elementary subdivisions required to change $K$ to $K'$, it suffices to prove Lemma 3.10 when $K'$ is an elementary separation subdivision of $K$. Let $\sigma,, \tau_1, \tau_2$, and $\tau$ be as (3.9) with (Case A) $\sigma \in K'_1$ or (Case B) $\sigma \in K'_2$.

*Case* A. Let $M$ be a collar of $f((\text{Bd}\tau_1) - \sigma)$ in $f(\tau)$ and let $M'$ be a 2-manifold-with-boundary in $f(\tau)$ obtained by connecting up the components of $M$ with 2-dimensional tubes in $\text{Int} f(\tau)$. The boundary of $M'$ is $f((\text{Bd}\tau_1) - \sigma) \cup J$ where $J$ is a *PL* simple closed curve in $\text{Int} f(\tau)$. Let $L' = L'_1 \cup L'_2 \cup L'_3$ be defined by $L'_1 = L_1 \cup \{J\}, L'_2 =$

$(L_2 - \{f(\tau)\}) \cup \{M', \mathrm{Cl}\,(f(\tau) - M')\}$, and $L_3' = L_3$. Let $f'$ be defined by $f' \| K | - \mathrm{Int}\,\tau = f \| K | - \mathrm{Int}\,\tau$, $f' | \sigma$ is any one-to-one function onto $J$, $f' | \mathrm{Int}\,\tau_1$ is any one-to-one function onto $\mathrm{Int}\,M'$, and $f' | \mathrm{Int}\,\tau_2$ is any one-to-one function onto $(\mathrm{Int}\,f(\tau)) - M'$.

*Case* B. This case is handled in much the same way as Case A. Let $M$ be a collar of $f((\mathrm{Bd}\tau_1) - \mathrm{Int}\,\sigma)$ in $f(\tau)$ and let $M'$ be a 3-manifold-with-boundary in $f(\tau)$ obtained by connecting up the components of $M$ with 3-dimensional tubes in $\mathrm{Int}\,f(\tau)$. The boundary of $M'$ is $f((\mathrm{Bd}\tau_1) - \sigma) \cup J$ where $J$ is a connected *PL* 2-manifold in $f(\tau)$ such that $J \cap \mathrm{Bd}f(\tau) = \mathrm{Bd}J = f(\mathrm{Bd}\sigma)$. Let $L' = L_1' \cup L_2' \cup L_3'$ be defined by $L_1' = L_1$, $L_2' = L_2 \cup \{J\}$, and $L_3' = (L_3 - \{f(\tau)\}) \cup \{M', \mathrm{Cl}\,(f(\tau) - M')\}$. Let $f'$ be defined by $f' \| K | - \mathrm{Int}\,\tau = f \| K | - \mathrm{Int}\,\tau$, $f' | \mathrm{Int}\,\sigma$ is any one-to-one function onto $\mathrm{Int}\,J$, $f' | \mathrm{Int}\,\tau_1$ is any one-to-one function onto $\mathrm{Int}\,M'$, and $f' | \mathrm{Int}\,\tau_2$ is any one-to-one function onto $(\mathrm{Int}\,f(\tau)) - M'$.

NOTATION. If $K$ is a separation complex, $L$ subcomplex of $K$ and $K'$ a subdivision of $K$ then $L'$ denotes a subdivision of $L$ induced by $K'$ given by $L' = \{\sigma \in K' | \sigma \subset |L|\}$. Note that if $K'$ is a separation subdivision of $K$ then $L'$ is separation subdivision of $L$.

We are greatly indebted to the referee for the notation $K \bmod W$ in definition 3.12. This concept has simplified our earlier exposition considerably.

DEFINITION 3.11. Let $K = K_1 \cup K_2 \cup K_3$ denote a separation complex in $S^3$ and $W$ a compact *PL* 2-manifold-with-boundary in $S^3$. We shall say that $K$ and $W$ are in general position if the following conditions are satisfied:

(3.11.1)  $W \cap | K_1 | = \varnothing$.

(3.11.2)  If $\tau \in K_2$ and $J$ is a component of $W \cap |\tau|$, then either $J$ is a component of $\mathrm{Bd}\,W$, or $J$ is a simple closed curve in $\mathrm{Int}\,W$ and $\tau$ and $W$ locally cross each other at $J$.

(3.11.3)  If $\sigma \in K_3$, then $\mathrm{Bd}\,W \cap \mathrm{Int}\,\sigma = \varnothing$ and each component of $W \cap |\sigma|$ intersects $\mathrm{Bd}\sigma$.

DEFINITION 3.12. If $K$ and $W$ are in general position, then we define $K \bmod W = K_1' \cup K_2' \cup K_3'$ as follows:

(3.12.1)  $K_1' = K_1 \cup \{t | \text{there exists a } \tau \in K_2 \text{ such that } t \text{ is a component of } W \cap |\tau|\}$.

(3.12.2)  $K_2' = \{\mathrm{Cl}\,Y | \text{there exists a } \sigma \in K_3 \text{ such that } Y \text{ is a component of } W \cap \mathrm{Int}\,\sigma\} \cup \{\mathrm{Cl}\,Y | \text{there exists a } \tau \in K_2 \text{ such that } Y \text{ is a component of } |\tau| - W\}$.

(3.12.3)  $K_3' = \{\mathrm{Cl}\,Z | \text{there exists a } \sigma \in K_3 \text{ such that } Z \text{ is a component of } |\sigma| - W\}$.

It is not necessarily true that $K \bmod W$ is even a separation com-

plex (a 2-simplex might intersect the interior of a 3-simplex or a 1-simplex the interior of a 2-simplex). However, we do have the following.

LEMMA 3.13. *Let $K$ denote a separation complex in $S^3$ and $W$ a compact PL 2-manifold-with-boundary in $S^3$ such that $K$ and $W$ are in general position. Then $K$ mod $W$ is a separation subdivision of $K$ if and only if the following two conditions are satisfied:*

> (3.13.1)   If $\tau \in K_2$ and $J$ is a (nonempty) component of $W \cap |\tau|$, then $J$ separates $\tau$.
>
> (3.13.2)   If $\sigma \in K_3$ and $Y$ is a (nonempty) component of $W \cap \operatorname{Int}\sigma$, then $\operatorname{Cl} Y$ separates $\sigma$.

*Proof.* First check that $K$ mod $W$ is a separation complex. Conditions (3.8.1) and (3.8.2) follow respectively from (3.13.1) and (3.13.2).

LEMMA 3.14. *Suppose the following are given:*
(3.14.1)   *$Q$ is a PL 2-sphere in $S^3$.*
(3.14.2)   *$U$ is a component of $S^3 - Q$.*
(3.14.3)   *$K$ is a separation complex such that $|K_1| \subset Q$ and $\cap \{\operatorname{Int}\sigma \,|\, \sigma \in K_3\} \subset U$.*
(3.14.4)   *$W$ is a compact PL 2-manifold-with-boundary such that $W \subset U$ and $W$ and $K$ are in general position (3.12).*
*Then $K$ mod $W$ is a separation subdivision of $K$ if condition (3.13.1) is satisfied.*

*Proof.* We check that (3.13.2) is also satisfied. Let $\sigma \in K_3$, $Y$ be a nonempty component of $W \cap \operatorname{Int}\sigma$, and $J$ be a nonempty component of $\operatorname{Bd} Y$. By (3.13.1) $J$ separates the member $\tau$ of $K_2$ that includes $J$. Thus, let $\tau(J)$ be the closure of a component of $\tau - J$. By (3.14.3), $(\operatorname{Bd}\tau(J)) - J \subset Q$. For each component $K$ of $(\operatorname{Bd}\tau(J)) - J$ we add to $\tau(J)$ a disk in $Q$ bounded by $K$, thus we obtain a singular 2-manifold $M(J)$ with boundary $J$. By (3.14.3), $M(J) \cap \operatorname{Int}\sigma = \varnothing$ for each component $J$ of $\operatorname{Bd} Y$. Thus, since the singular 2-manifold $(\cup \{M(J) \,|\, J$ is a component of $\operatorname{Bd} Y\}) \cup Y$ separates $\operatorname{Int}\sigma$, $Y$ separates $\operatorname{Int}\sigma$.

4. **Winding functions.** In this section we discuss the concept of a winding function which is used extensively in §§5 and 6. We begin by introducding notation that we use later for labeling abstract trees.

DEFINITION 4.1. A *tree labeling system* is a collection $\Gamma$ of finite sequences $(a_1, a_2, \cdots, a_n)$ of nonnegative integers such that

(4.1.1)   $(0) \in \Gamma$ and is the only one element sequence in $\Gamma$,

(4.1.2)   if $(a_1, a_2, \cdots, a_n) \in \Gamma$ then $(a_1, a_2, \cdots, a_{n-1}) \in \Gamma$ if $n \neq 1$, and $(a_1, a_2, \cdots a_{n-1}, a_n - 1) \in \Gamma$ if $a_n \neq 0$.

(4.1.3)   A *maximal* element of $\Gamma$ is finite sequence $(a_1, a_2, \cdots, a_n) \in \Gamma$ such that there does not exist a nonnegative integer $a$ such that $(a_1, a_2, \cdots a_n, a) \in \Gamma$.

(4.1.4)   the *lexicographical ordering* of $\Gamma$ is 0, 00, 01, 02, $\cdots$, 000, 001, $\cdots$, etc.

(4.1.5)   a *tree labeling system* for set $B$ is a one-to-one function from $\Gamma$ onto $B$.

NOTATION. If $\Gamma$ is a tree labeling system for set $B$ then we use lower case greek letters to refer to elements of $\Gamma$ and denote the image of $\alpha \in \Gamma$ in $B$ by $b_\alpha \in B$. If $\alpha = (a_1, a_2 \cdots, a_n) \in \Gamma$ and $(i_1, \cdots, i_j)$ is a sequence of nonnegative integers such that $(a_1, \cdots, a_n, i_1, \cdots, i_j) \in \Gamma$ then we write $a_1 a_2 \cdots a_n \in \Gamma$, $\alpha i_1 i_2 \cdots i_j \in \Gamma$, respectively. If $\alpha = (a_1, a_2, \cdots, a_n) \in \Gamma$ then $|\alpha| = 0$ if $n$ is an even integer and $|\alpha| = 1$ if $n$ is an odd integer.

In items (4.2) through (4.9), $R$ is a *PL* 2-sphere in $S^3$, $H$ is a disk-with-holes in $R$, $V$ is a component of $S^3 - R$, and $M = h(H \times I)$ is a collar of $H$ in Cl $V$.

DEFINITION 4.2.   A *corner* of $M$ is $h(E \times [0, t])$ where $E$ is a collar of Bd$H$ in $H$ and $0 < t < 1$.

DEFINITION 4.3.   A compact, orientable, *PL* 2-manifold $S$ in $M$ *partitions* $M$ if for each component $Y$ of $S$ we have $Y \cap \text{Bd}M = Y \cap ((\text{Bd}M) - H) = \text{Bd}\,Y \neq \varnothing$.

The next lemma shows that a partition of $M$ induces a special type of separation subdivision of the separation complex $\bar{M} = \{J \mid J$ is a component Bd$H\} \cup \{H, \text{Bd}M - \text{Int } H\} \cup \{M\}$.

LEMMA 4.4.   *If $S$ partitions $M$ then $\bar{M} \bmod S = K_1 \cup K_2 \cup K_3$ is a separation complex having the property that the intersection of each pair of elements of $K_3$ is either empty or a single component of $S$. Furthermore, there exists a tree labeling system $\Gamma$ for $K_3$ such that $H \subset \text{Bd}\sigma_0$ and $\sigma_\alpha \cap \sigma_\beta$ is nonempty if and only if either $\beta = \alpha i$ or $\alpha = \beta i$ for some nonnegative integer $i$.*

*Proof.*   Let $Y$ be component of $S$ and $J$ a component of Bd$Y$. Since Bd$M -$ Int $H$ is a disk-with-holes and $J \subset (\text{Bd}M) - H$, $J$ separates Bd$M -$ Int $H$. Thus, by Lemma 3.14, $\bar{M} \bmod S$ is a separation complex. The other properties of $\bar{M} \bmod S$ follow by induction on the number of components of $S$.

DEFINITION 4.5. If $S$ partitions $M$ then we call $\bar{M} \bmod S$ the *separation complex supported by $S$ in $M$.*

DEFINITION 4.6. If $R$ is a *PL* 2-sphere in $S^3$, $V$ is a component of $S^3 - R$, $H$ is a disk-with-holes in $R$, $M$ is a collar of $H$ in $\mathrm{Cl}\, V$, $S$ partitions $M$, and $K = K_1 \cup K_2 \cup K_3$ is the separation complex supported by $S$ in $M$ then function $P: M \to S^3$ is a *winding function with respect to* $(M, S, H, R)$ if

(4.6.1)  $P|H = 1$ and $P$ is the identity on a corner of $M$,

(4.6.2)  if $\sigma \in K$ then $\{P(\tau) | \tau \in \bar{\sigma}\} = \overline{P(\sigma)}$ is a separation complex and $P|\sigma$ is a separation isomorphism from $\bar{\sigma}$ to $\overline{P(\sigma)}$,

(4.6.3)  if $\sigma \in K_3$ then $P(\sigma - (S \cup H)) \subset S^3 - R$ and $P(\sigma \cap (S \cup H)) \subset R$,

(4.6.4)  if $\sigma_0, \sigma_1 \in K_3$ and $\sigma_0 \cap \sigma_1$ is a component of $S$ then $P(\sigma_0 - (S \cup H))$, $P(\sigma_1 - (S \cup H))$ lie in different components of $S^3 - R$, and

(4.6.5)  if $\sigma_0, \sigma_1 \in K_3$, $\sigma_0 \neq \sigma_1$, but $P(\sigma_0 - (S \cup H)) \cap P(\sigma_1 - (S \cup H)) \neq \varnothing$ then for $i = 0$, or 1, $P(\sigma_i - (S \cup H)) \subset \mathrm{Int}\, P(\sigma_{1-i})$ and $P(\sigma_i \cap (S \cup H)) \subset \mathrm{Int}\, P(\sigma_{1-i} \cap (S \cup H))$.

We now establish several properties of a winding function.

LEMMA 4.7. *If $P: M \to S^3$ is a winding function with respect to $(M, S, H, R)$, $K = K_1 \cup K_2 \cup K_3$ is the separation complex supported by $S$ in $M$, and $\Gamma$ is a tree labeling system for $K_3$ satisfying Lemma 4.4. Then it is impossible to have $P(\sigma_\beta) \subset P(\sigma_\alpha)$ where $\alpha = \beta i_1 \cdots i_n$ for some positive integer $n$. In particular, $P(\sigma_0) \subset P(\sigma_\alpha)$ implies $\alpha = 0$.*

*Proof.* Suppose the contrary, then there exist a smallest positive integer $n$ with the property that there exist $\alpha, \beta \in \Gamma$ such that $\alpha = \beta i_1 \cdots i_n$ and $P(\sigma_\beta) \subset P(\sigma_\alpha)$. By (4.6.3), (4.6.4), and (4.6.5) either $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n-1}})$ or there exists a nonnegative integer $i_{n+1}$ such that $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_n i_{n+1}})$. Suppose $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n-1}})$ then since $n$ is minimal, we must have $n = 2$ and $\sigma_{\beta i_1} = \sigma_{\beta i_1 \cdots i_{n-1}}$. But by (4.6.5) $P(\sigma_\beta \cap \sigma_{\beta i_1}) \subset \mathrm{Int}\, P(\sigma_{\beta i_1 i_2} \cap \sigma_{\beta i_1})$, thus we have distinct 2-faces of $\sigma_{\beta i_1}$ pinched together by $P$ contradicting (4.6.2). Hence we must have $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n+1}})$.

The above argument is repeated to $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n+1}})$ to find a nonnegative integer $i_{n+2}$ such that $P(\sigma_{\beta i_1 i_2}) \subset P(\sigma_{\beta i_1 \cdots i_{n+1} i_{n+2}})$, and then repeated over again. Eventually a nonnegative integer $i_{n+p}$ is found such that $P(\sigma_{\beta i_1 i_2 \cdots i_p}) \subset P(\sigma_{\beta i_1 \cdots i_{n+p}})$ and $\beta i_1 \cdots i_{p+n}$ is maximal in $\Gamma$. One more application of the argument yields a contradiction.

LEMMA 4.8. *If $P$ is a winding function with respect to $(M, S, H, R)$ then*

(4.8.1)   $H - P(M - H)$ separates $P(M) - P(\mathrm{Bd}M)$ from $S^3 - P(M)$ in $S^3 - P(\mathrm{Bd}M - H)$, and

(4.8.2)   $P(M)$ contains a collar of $P(\mathrm{Bd}M - \mathrm{Int}\, H)$ to the side of $P(\mathrm{Bd}M - \mathrm{Int}\, H)$ that contains a corner of $M$.

*Proof.* Suppose $pq$ is an arc from $p \in S^3 - P(M)$ to $q \in P(M) - P(\mathrm{Bd}M)$ that lies in $S^3 - P(\mathrm{Bd}M - H)$. Let $r$ be the first point of $pq$ that intersect the compact set $P(M)$. Since $r$ is a boundary point of the 3-manifold $P(\sigma)$ for some $\sigma \in K_3$ and $pq \cap P(\mathrm{Bd}M - H) = \varnothing$ it follows that $r \in \mathrm{Int}\, P(S \cup H) \subset R$. By (4.6.2) and (4.6.4) since $r$ is the first point of intersection with $P(M)$, $r \notin P(S)$. Hence $r \in P(H) - P(S) = H - P(M - H)$ by (4.6.1) and (4.6.3), and the proof of (4.8.1) is complete.

Let $\Gamma$ be the tree labeling system for $K_3$ in Lemma 4.4. The collar required in (4.8.2) is constructed in $P(M)$ by induction on the elements of $\Gamma$. The set $P(\sigma_0) - (\cup P(\sigma_{0\alpha}))$ contains a collar $N_0$ of $\mathrm{Cl}((\mathrm{Bd}P(\sigma_0)) - R)$ in $P(\sigma_0)$ such that $N_0$ contains a corner of $M$ by (4.6.1), (4.6.2) and Lemma 4.7. We may assume that $N_0 \cap P(\sigma_0 \cap S)$ is a collar of $\mathrm{Bd}P(\sigma_0 \cap S)$ in $P(\sigma_0 \cap S)$. Let $N_{0i}$ be a collar of $\mathrm{Cl}((\mathrm{Bd}P(\sigma_{0i})) - R)$ in $P(\sigma_{0i}) - (\bigcup_{\alpha \in \Gamma_{0i}} P(\sigma_\alpha))$ where $\alpha \in \Gamma_{0i}$ if and only if $P(\sigma_\alpha) - R \subset \mathrm{Int}\, P(\sigma_{0i})$ by (4.6.2) and (4.6.5). We may assume that $N_{0i} \cap P(\sigma_{0i} \cap S)$ is a collar of $\mathrm{Bd}P(\sigma_{0i} \cap S)$ in $P(\sigma_0 \cap S)$ and $N_0 \cap P(\sigma_0 \cap \sigma_{0i}) = N_{0i} \cap P(\sigma_0 \cap \sigma_{0i})$. The induction step from $\sigma_\alpha$ to $\sigma_{\alpha i}$ is similar. The required collar is $\cup \{N_\alpha \mid \alpha \in \Gamma\}$.

The next lemma gives a lifting property for winding functions.

**LEMMA 4.9.** *If* $P \colon M \to S^3$ *is a winding function with respect to* $(M, S, H, R)$, $W$ *is a compact 2-manifold such that each component of* $W$ *has nonempty boundary,* $W$ *is in general position with respect to* $R$, $\mathrm{Int}\, W \subset P(M) - P(\mathrm{Bd}M - \mathrm{Int}\, H)$, *and* $\mathrm{Bd}\, W \subset P(H) - P(S)$. *Then there exists a compact PL 2-manifold* $\overline{W}$ *in* $M$ *and a function* $P' \colon M \to S^3$ *such that*

(4.9.1)   $P' \mid H = P \mid H = 1$ and if $K = K_1 \cup K_2 \cup K_3$ is the separation complex supported by $S$ in $M$ then $P'(\sigma) = P(\sigma)$ if $\sigma \in K$,

(4.9.2)   $\overline{W}$ and $K$ are in general position and if $\overline{W}_0$ is a component of $\overline{W}$ then $\overline{W}_0 \cap \mathrm{Bd}M = \overline{W}_0 \cap H = \mathrm{Bd}\overline{W}_0 \neq \varnothing$,

(4.9.3)   $K' = K \bmod \overline{W} = K_1' \cup K_2' \cup K_3'$ is a separation subdivision of $K$,

(4.9.4)   if $\sigma \in K_3$ then $P' \mid \sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P'(\tau) \mid \tau \in \bar{\sigma}'\}$,

(4.9.5)   $P' \mid \overline{W}$ is a separation isomorphism from $\overline{W} \bmod (S \cup H)$ to $W \bmod R$,

(4.9.6)   if $\sigma_0, \sigma_1 \in K_3$ and $\sigma_0 \neq \sigma_1$ but $P(\sigma_0) \subset P(\sigma_1)$ then there exists $\sigma_2 \in K_3'$ such that $\sigma_2 \subset \sigma_1$ and $P(\sigma_0) \subset P'(\sigma_2)$.

*Proof.* Let $\Gamma$ be the tree labeling system for $K_3$ of Lemma 4.4. For $\sigma \in K_3$ let $\Gamma(\sigma) = \{\alpha \in \Gamma \,|\, P(\sigma_\alpha)$ is a proper subset of $P(\sigma)\}$ and let $W(\sigma) = W \cap (P(\sigma) - \cup \{P(\sigma_\alpha) \,|\, \alpha \in \Gamma(\sigma)\})$. It is clear that $W = \cup \{W(\sigma) \,|\, \sigma \in K_3\}$ and $W(\sigma_\alpha) \cap W(\sigma_\beta) = \varnothing$ if $\sigma_\alpha \cap \sigma_\beta = \varnothing$ by (4.6.5). Also, $W(\sigma_\alpha) \cap W(\sigma_{\alpha i}) = W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \,|\, \beta \in \Gamma(\sigma_\alpha)\}) = W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \,|\, \beta \in \Gamma(\sigma_{\alpha i})\})$ by (4.6.4) and the fact that $W$ is in general position with respect to $R$. By hypothesis on $W$, each component $W_i(\sigma)$ of $W(\sigma)$ is a compact 2-manifold with $\varnothing \neq \mathrm{Bd}\, W_i(\sigma) \subset \mathrm{Int}\, P(\sigma \cap (S \cup H)) \subset R$. The 2-manifold $W_i(\sigma)$ separates the 3-cell bounded by $R$ and containing $W_i(\sigma)$: consequently, $W_i(\sigma)$ separates $P(\sigma)$. If $t$ is a boundary component of $W_i(\sigma)$ then there exists a component $Y$ of $\sigma \cap (S \cup H)$ such that $t \subset \mathrm{Int}\, P(Y) \subset R$. Since $P(Y)$ is a disk-with-holes, $t$ separates $P(Y)$. It now follows by Lemma 3.13 that $L(\sigma) = L_1(\sigma) \cup L_2(\sigma) \cup L_3(\sigma) = \overline{P(\sigma)}$ mod $W(\sigma)$ is a separation subdivision of $\{P(\tau) \,|\, \tau \in \bar\sigma\} = \overline{P(\sigma)}$.

We use Lemma 3.10 to lift the separation subdivision $L(\sigma)$ of $\overline{P(\sigma)}$ to $\bar\sigma$. By Lemma 3.10 there exists a subdivision $\bar\sigma_0'$ of $\bar\sigma_0$ and a separation isomorphism $P' | \sigma_0$ from $\bar\sigma_0'$ to $L(\sigma_0)$ such that $(P' | \sigma_0)(\tau) = P(\tau)$ if $\tau \in \bar\sigma_0$, and $(P' | \sigma_0) | H = P | H$. Proceeding inductively, we assume that separation subdivision $\bar\sigma_\alpha'$ of $\bar\sigma_\alpha$ and separation isomorphism $P' | \sigma_\alpha$ from $\bar\sigma_\alpha'$ to $L(\sigma_\alpha)$ have been defined such that $(P' | \sigma_\alpha)(\tau) = P(\tau)$ if $\tau \in \bar\sigma_\alpha$. Let $i$ be a nonnegative integer such that $\alpha i \in \Gamma$, as we have observed before $W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \,|\, \beta \in \Gamma(\sigma_\alpha)\} = W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \,|\, \beta \in \Gamma(\sigma_{\alpha i})\})$ so by Lemma 3.10 there exists a separation subdivision $\bar\sigma_{\alpha i}'$ of $\bar\sigma_{\alpha i}$ and a separation isomorphism $P' | \sigma_{\alpha i}$ from $\bar\sigma_{\alpha i}'$ to $L(\sigma_{\alpha i})$ such that $(P' | \sigma_{\alpha i})(\tau) = P(\tau)$ if $\tau \in \bar\sigma_{\alpha i}$ and $(P' | \sigma_{\alpha i}) | \sigma_\alpha \cap \sigma_{\alpha i} = (P' | \sigma_\alpha) | \sigma_\alpha \cap \sigma_{\alpha i}$.

Let $P' = \bigcup_{\alpha \in K_3} (P' | \sigma)$, $\overline{W} = \bigcup_{\alpha \in K_3} (P' | \sigma)^{-1}(W(\sigma))$, and $K' = \bigcup_{\alpha \in K_3} \bar\sigma'$. Properties (4.9.1), $\cdots$, (4.9.5) are straight forward consequences of the construction. Property (4.9.6) follows since $P(\sigma_0) \subset P(\sigma_1)$ implies that $P(\sigma_0) \cap W(\sigma_1) = \varnothing$ and the latter implies $P(\sigma_0)$ is included in an element of $L_3(\sigma_1)$ since $P(\sigma_0)$ is connected.

**5. Structure theorems.** The purpose of this section is to prove the combinatorial structure theorems (Lemma 5.15 and Theorem 5.16) for components associated with the general position of two arbitrary $PL$ 2-spheres in $E^3$. Throughout this section an arbitrary $PL$ 2-sphere $R$ in $S^3$ is fixed. Also, a choice function $\mu$ on the collection of nonempty sets of simple closed curves in $R$ is fixed. Some of the objects defined below depend on $R$ or $\mu$, but to save notation no special attention is called to this fact.

DEFINITION 5.1. Let $\mathscr{S}$ be the collection of all 4-tuples $(X, U, q, m)$ such that

(5.1.1)   $X$ is a compact, connected, $PL$ 2-manifold in $S^3$ without
          boundary and in general position with respect to $R$ such
          that if $t$ is a component (simple closed curve) of $R \cap X$
          then $t$ separates $X$,

(5.1.2)   U is one of the two components of $S^3 - X$,

(5.1.3)   $q$ is a point of $X - R$, and

(5.1.4)   $m$ is a component (simple closed curve) of $R \cap X$.

For $a = (X, U, q, m) \in \mathscr{S}$ the existence of the following objects
is clear and we use the following notation.

(5.2)   $A_a$ is the closure of the component of $X - m$ that does not
        contain $q$,

(5.3)   $V_a$ is the component of $S^3 - R$ that contains the interior
        of a collar of $m$ in $A_a$,

(5.4)   $B_a$ is the disk in $R$ bounded by $m$ that contains a collar
        of $m$ in $R \cap \mathrm{Cl}\, U$,

(5.5)   $C_a$ is the closure of the component of $A_a - R$ that contains
        $m$ in its boundary,

(5.6)   $D_a$ is the closure of the component of $B_a - C_a$ that contains
        $m$ in its boundary,

(5.7)   $|a|$ is the number of components of $A_a - R$, and

(5.8)   $L_a$ is the collection of all components $t$ of $\mathrm{Bd}C_a - \mathrm{Bd}D_a$
        such that $B_b \cap D_a = \varnothing$ where $b = (X, U, q, t)$.

Since $C_a$ separates $\mathrm{Cl}\, V_a$ and $R$ is a 2-sphere, the following lemma
is easily established.

LEMMA 5.9  *The sets $C_a$, $D_a$, and $L_a$ have the following properties*:

(5.9.1)   $\mathrm{Bd}D_a \subset \mathrm{Bd}C_a$,

(5.9.2)   *there exists a collar of* $\mathrm{Bd}D_a$ *in* $D_a \cap \mathrm{Cl}\, U$, *and*

(5.9.3)   *if* $\mathrm{Bd}C_a - \mathrm{Bd}D_a \neq \varnothing$ *then* $L_a \neq \varnothing$.

The existence of the following objects associated with $a = (X, U, q, m) \in \mathscr{S}$ will be established by induction on $|a|$ in Lemma
5.15.  See Figure 2.

(5.10a)   $H_a$ is a disk-with-holes in $D_a$ such that

(5.10.1a)   $m \subset \mathrm{Bd}H_a$, and $\mathrm{Bd}H_a \subset X \cap R$,

(5.10.2a)   there exists a collar of $\mathrm{Bd}H_a$ in $H_a \cap \mathrm{Cl}\, U$,

(5.10.3a)   $\mathrm{Bd}H_a$ is the boundary of a compact, connected, 2-
            manifold $J_a$ in $A_a$.

(5.11a)   $M_a = h(H_a \times I)$ is a collar of $H_a$ in $\mathrm{Cl}\, V_a$ such that

(5.11.1a)   $h(\mathrm{Bd}H_a \times I)$ is a collar of $\mathrm{Bd}J_a$ in $J_a$,

(5.11.2a)   $M_a$ intersects a component $Y$ of $X \cap V_a$ if and only
            if $\mathrm{Cl}\, Y$ intersects $H_a$,

(5.12a)   $S_a$ is a partition of $M_a$ and $K_a = K_1(a) \cup K_2(a) \cup K_3(a)$ is
          the separation complex supported by $S_a$ in $M_a$,
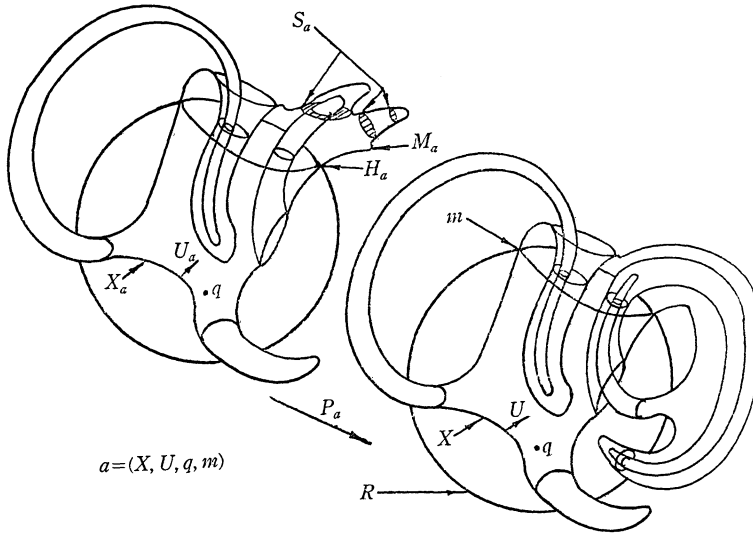
FIGURE 2

(5.13a)    $X_a$ is compact, connected, *PL* 2-manifold in $S^3$ without boundary and in general position with respect to $R \cup S_a$ such that

    (5.13.1a)    $X_a$ contains $G_a = (\mathrm{Bd} M_a) - \mathrm{Int}\, H_a$,

    (5.13.2a)    if $t$ is a component of $(R \cup S_a) \cap X_a$ then $t$ separates $X_a$,

    (5.13.3a)    if $t$ is a component of $(X_a - G_a) \cap S_a$ then $t$ separates the component of $S_a$ containing $t$.

From (5.13.3a), the fact that $(R - H_a) \cup G_a$ is a 2-sphere, and Lemma 3.14 it follows that $K_a' = K_a \bmod (X_a - \mathrm{Int}\, G_a) = K_1'(a) \cup K_2'(a) \cup K_3'(a)$ is a separation subdivision of $K_a$.

(5.14a)    $P_a$ is a function from $S^3$ to $S^3$ such that

    (5.14.1a)    $P_a \,|\, S^3 - (M_a - H_a) = 1$,

    (5.14.2a)    $P_a \,|\, M_a$ is a winding function with respect to $(M_a, S_a, H_a, R)$,

    (5.14.3a)    $P_a \,|\, X_a$ is a separation isomorphism from $\bar{X}_a = X_a \bmod (R \cup S_a)$ to $\bar{X} = X \bmod R$.

    (5.14.4a)    $P_a(G_a) = J_a$,

    (5.14.5a)    $P_a(M_a) \cap (X_a - M_a) = \varnothing$,

    (5.14.6a)    if $\sigma \in K_3(a)$ then $P_a \,|\, \sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P_a(\tau) \,|\, \tau \in \bar{\sigma}'\}$, and

    (5.14.7a)    if $\sigma_0, \sigma_1 \in K_3(\alpha)$ and $\sigma_0 \neq \sigma_1$ but $P_a(\sigma_0) \subset P_a(\sigma_1)$ then there exists $\sigma_2 \in K_3'(a)$ such that $\sigma_2 \subset \sigma_1$ and $P_a(\sigma_0) \subset P_a(\sigma_2)$.

LEMMA 5.15.    *If* $a = (X, U, q, m) \in \mathscr{S}$ *where* $\mathscr{S}$ *is given by* (5.1)

*then there exists a 5-tuple* $(H_a, M_a, S_a, X_a, P_a)$ *that satisfies* (5.10a), (5.11a), $\cdots$, (5.14a).

*Proof.* The proof is by induction on $|a|$ given by (5.7). We use the notation given by (5.1), $\cdots$, (5.8).

Suppose $a = (X, U, q, m) \in \mathscr{S}$ and $|a| = 1$. It is clear that $C_a = A_a$, $D_a = B_a$ and $\mathrm{Bd}C_a = \mathrm{Bd}D_a$. Let $H_a = D_a$, $J_a = C_a$ and let $h$ be a $PL$ embedding of $H_a \times I$ into $\mathrm{Cl}\,V_a$ such that $h(y \times 0) = y$, $h(H_a \times (0, 1]) \subset V_a$, $h(\mathrm{Bd}H_a \times I)$ is a collar of $\mathrm{Bd}J_a$ in $J_a$, and $h(H_a \times I)$ intersects a component $Y$ of $X \cap V_a$ if and only if $\mathrm{Cl}\,Y$ intersects $H_a$. Let $M_a = h(H_a \times I)$ and let $S_a = \varnothing$. A one-to-one function $P$, given by $P|H_a = 1$, $P(G_a) = J_a$, $P(\mathrm{Int}\,M_a) = \mathrm{Int}\,L$ where $L$ is the 3-manifold bounded by $H_a \cup J_a$ and containing a collar of $J_a$ in $\mathrm{Cl}\,U$, is a trivial winding function with respect to $(M_a, S_a, H_a, R)$. Let $W = X \cap (L - J_a)$ and apply Lemma 4.9 to $P$ and $W$ and let $X_a = \bar{W} \cup (X - L) \cup G_a$, $P_a|M_a = P'$ where $\bar{W}$, $P'$ are as in the conclusion of Lemma 4.9. It is straight forward to check that $(H_a, M_a, S_a, X_a, P_a)$ satisfies (5.10a), $\cdots$, (5.14a).

Inductively, we suppose that the 5-tuple $(H_b, M_b, S_b, X_b, P_b)$ exists with properties (5.10b), $\cdots$, (5.14b) provided that $|b| < k$. Let $a = (X, U, q, m) \in \mathscr{S}$ with $|a| = k$. By Lemma 5.9.1 we have that $\mathrm{Bd}D_a = \mathrm{Bd}C_a$ or $\mathrm{Bd}C_a - \mathrm{Bd}D_a \neq \varnothing$ and we accordingly, break the induction step into two cases.

*Case* A. $\mathrm{Bd}D_a = \mathrm{Bd}C_a$. In this case we do not need the induction hypothesis and we proceed formally as above for $|a| = 1$. That is, we let $H_a = D_a$, $J_a = C_a$, $M_a$ be a collar of $H_a$ in $\mathrm{Cl}\,V_a$, $S_a = \varnothing$, $L$ be the 3-manifold bounded by $H_a \cup J_a$ and containing a collar of $J_a$ in $\mathrm{Cl}\,U$, and let $X_a$, $P_a$ be obtained by the same application of Lemma 4.9 as was used above.

*Case* B. $\mathrm{Bd}C_a - \mathrm{Bd}D_a \neq \varnothing$. See Figure 3. By Lemma 5.9.3, $L_a \neq \varnothing$ and we let $m_a = \mu(L_a)$ where $\mu$ is the choice function mentioned in the first paragraph of §5. Let $b = (X, U, q, m_a)$. Since $m_a \subset A_a$ we have that $A_b \subset A_a$ and $|b| < |a| = k$. By the induction hypothesis a 5-tuple $(H_b, M_b, S_b, X_b, P_b)$ exists with properties (5.10b), $\cdots$, (5.14b). By (5.6), $m \subset D_a$. by (5.10b), (5.6), (5.9), $H_b \subset D_b \subset B_b$: and by (5.8) $B_b \cap D_a = \varnothing$. Hence, $D_a \cap H_b = \varnothing$, $m \cap H_b = \varnothing$ and $M_b \cap m = \varnothing$ by (5.11b). Also, by (5.14.1b) and (5.14.3b) we have that $m \subset X_b$. Let $g$ be a $PL$ homeomorphism of $S^3$ onto $S^3$ that is fixed outside a thin shell neighborhood of $M_b$ and pushes $M_b$ across $R$ to $S^3 - \mathrm{Cl}\,V_b$. Since $D_a \cap H_b = \varnothing$, $m \cap H_b = \varnothing$ it follows that no point near $D_a$ is moved by $g$ and $g(m) = m$. Let $c = (g(X_b), g(U_b), g(q_b), m)$ where $U_b$ is the component of $S^3 - X_b$ such that $H_b \cap \mathrm{Cl}\,U_b$ contains the collar of $\mathrm{Bd}H_b$
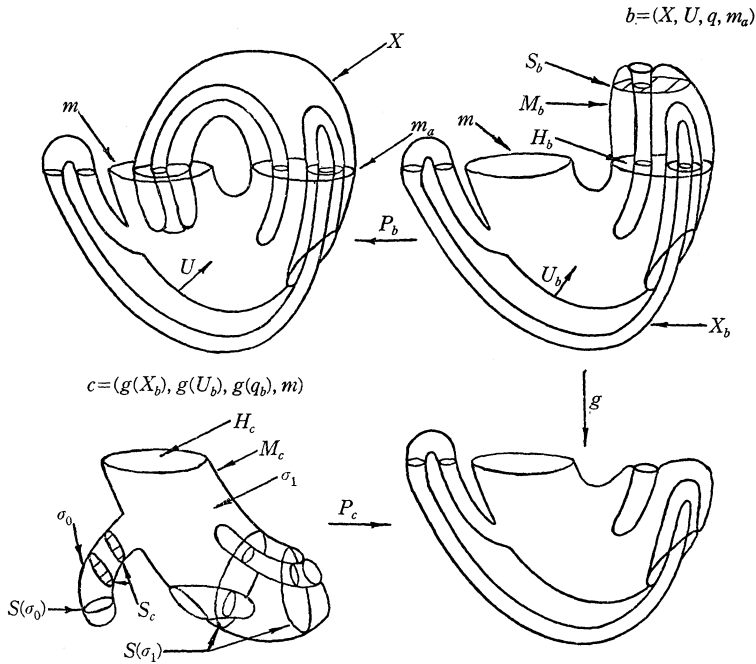
$b = (X, U, q, m_a)$

$c = (g(X_b), g(U_b), g(q_b), m)$

FIGURE 3

given in (5.10.2b), and $q_b = (P_b | X_b)^{-1}(q)$. Since $g$ removes components of $R \cap X_b$, in particular $m_a$, and by property (5.14.3b), it follows that $|c| < |b| < |a| = k$. By the induction hypothesis, a 5-tuple $(H_c, M_c, S_c, X_c, P_c)$ exists with properties (5.10c), $\cdots$, (5.14c).

Let $H_a = H_c$, $M_a = M_c$, and $J_a = P_b g^{-1}(J_c)$. By pushing $M_a$ into a smaller neighborhood of $H_a$ if necessary, we may assume that (5.11.2a) is satisfied and $g^{-1} | M_a = 1$. Properties (5.10a), (5.10.1a), (5.10.2a), (5.10.3a), (5.11a), (5.11.1a) are consequences of the corresponding properties for $c$ and the facts that $P_b | M_a = 1$ and $g^{-1} | M_a = 1$.

It follows from (4.7) that $P_c(M_c)$ misses the interior of a collar of $m$ in $\mathrm{Cl}(S^3 - V_c)$. Since $H_b$ is in the same component of $S^3 - P_c(\mathrm{Bd} M_c - H_c)$ that contains the interior of such a collar of $m$ in $\mathrm{Cl}(S^3 - V_c)$, it follows from (4.8.1) and (4.7) that $H_b \cap P_c(M_c) = \varnothing$.

Let $\sigma \in K_3(c)$. Since $g^{-1}$ is a homeomorphism that only moves points near $H_b$ into $M_b$ and $H_b \cap P_c(M_c) = \varnothing$ we have by (5.14.2c), (4.6.2), (4.6.3) that $g^{-1} P_c | \sigma$ is a separation isomorphism from $\bar{\sigma}$ to $\{g^{-1} P_c(\tau) | \tau \in \bar{\sigma}\} = \overline{g^{-1} P_c(\sigma)}$ such that $g^{-1} P_c(\sigma \cap S_c) \subset R - H_b$. Let $W = S_b \cup H_b$ and let $Q$ be the 2-sphere obtained from $(R - H_b) \cup G_b$ by pushing $G_b$ slightly away from $M_b$. Since $g^{-1} P_c((\mathrm{Bd}\sigma) - S_c) \subset X_b$ and $X_b$ satisfies (5.13b), (5.13.2b), (5.13.3b) it follows that each component $t$ of $g^{-1} P_c((\mathrm{Bd}\sigma) - S_c) \cap W$ separates the component of $W$ containing $t$ or $t \subset \mathrm{Bd} W$, and $t$ separates the 2-face of $\overline{g^{-1} P_c(\sigma)}$ containing $t$. By Lemma 3.14 applied to $Q$ and $W$, $\overline{g^{-1} P_c(\sigma)} \bmod W$ is a separation subdivision of

$\overline{g^{-1}P_c(\sigma)}$. By Lemma 3.10 we assume without loss of generality that $S(\sigma)$ is a compact $PL$ 2-manifold in $\sigma$ such that if $Y$ is a component of $S(\sigma)$ then $Y \cap \mathrm{Bd}\,\sigma = Y \cap ((\mathrm{Bd}\,\sigma) - S_c) = \mathrm{Bd}\,Y \neq \varnothing$ and $g^{-1}P_c(Y)$ is a component of $W \cap g^{-1}P_c(\sigma)$.

Let $S_a = S_c \cup (\cup \{S(\sigma)\,|\,\sigma \in K_3(c)\})$. It is clear that $S_a$ satisfies (5.12) and $g^{-1}P_c\,|\,\sigma \in K_3(a)$ is a separation isomorphism from $\bar{\sigma}$ to $\{g^{-1}P_c(\tau)\,|\,\tau \in \bar{\sigma}\}$ that satisfies property (4.6.5).

Let $P = P_b g^{-1} P_c\,|\,M_a$. We now show that $P$ is a winding function with respect to $(M_a, S_a, H_a, R)$. Property (4.6.1) is satisfied since $P_c$ satisfies (4.6.1), and $g^{-1}$, $P_b$, as pointed out previously, are the identity on a neighborhood of $H_a$. Property (4.6.2) is established by observing that, by the way $K_a$ was constructed in the previous two paragraphs, if $\sigma \in K_3(a)$ then $M_b \cap \mathrm{Int}\,g^{-1}P_c(\sigma) = \varnothing$ or $g^{-1}P_c(\sigma) = |L|$ where $L$ is a subcomplex of $\bar{\sigma}_0'$ for some $\sigma_0 \in K_3(b)$ ($\bar{\sigma}_0'$ is the induced subdivision of $\bar{\sigma}_0$ by $K_b'$). In the former case, (4.6.2) follows since $P_b\,|\,S^3 - (M_b - H_b) = 1$. In the latter case, (4.6.2) follows by (5.14.6b). Properties (4.6.3) and (4.6.4) are easy consequences of the corresponding propertis for $P_b$ and $P_c$ and the way $S_a$ was constructed in the previous two paragraphs. We now establish property (4.6.5). Suppose $\sigma_0, \sigma_1 \in K_3(a)$, $\sigma_0 \neq \sigma_1$ and $P(\sigma_0 - S_a) \cap P(\sigma_1 - S_b) \neq \varnothing$, then (a) $M_b \cap \mathrm{Int}\,g^{-1}P_c(\sigma_i) = \varnothing$ for $i = 0, 1$ or (b) $g^{-1}P_c(\sigma_i) \subset M_b$, $M_b \cap \mathrm{Int}\,g^{-1}P_c(\sigma_{1-i}) = \varnothing$ for $i = 0, 1$ or (c) $g^{-1}P_c(\sigma_i) \subset M_b$, $i = 0, 1$. In Case (a), (4.6.5) follows from the corresponding property for $P_c$ and the fact that $g^{-1}$ is a homeomorphism. In Case (b), $P(\sigma_i - S_a) \subset \mathrm{Int}\,P(\sigma_{1-i})$ since $P(\sigma_i - S_a)$ is connected and fails to contain any point in $\mathrm{Bd}\,P(\sigma_{1-i})$ by (5.14.5b). In Case (c), either (c.1) there exists a $\sigma^* \in K_3(b)$ such that $g^{-1}P_c(\sigma_i) \subset \sigma^*$ for $i = 0, 1$ or (c.2) there exists distinct $\sigma_0^*, \sigma_1^* \in K_3(b)$ such that $g^{-1}P_c(\sigma_i) \subset \sigma_i^*$. In Case (c.1), (4.6.5) follows from the corresponding property for $P_c$ and the fact that $g^{-1}$ is a homeomorphism and (5.14.6b). In Case (c.2), (4.6.5) follows from (5.14.7b) since we must have $P_b(\sigma_1^*) \subset P_b(\sigma_{1-i}^*)$ for $i = 0$ or 1, by property (4.6.5) for $P_b$.

Let $W = X \cap P(M_a) - P(\mathrm{Bd}\,M_a - \mathrm{Int}\,H_a))$. Apply Lemma 4.9 to $P$ and $W$ to obtain $\bar{W}$ and $P'$ satisfying (4.9.1), $\cdots$, (4.9.6).

Let $X_a = \bar{W} \cup G_a \cup (X - P'(M_a))$ and let $P_a$ be given by $P_a\,|\,M_a = P'$ and $P_a\,|\,S^3 - (M_a - H_a) = 1$. We now varify the remaining properties of the conclusion of the Lemma 5.15 to complete the induction step. Properties (5.14.1a), (5.14.2a) are clear by (4.9.1) applied to $P'$. Property (5.14.3a) follows from (4.9.5) applied to $P'$, the definition given previously of $J_a$, and the fact that $P\,|\,S^3 - (M_a - H_a) = 1$. Property (5.14.4a) is a consequence of the definition of $J_a$. Property (5.14.5a) follows since $X_a - M_a = X - P'(M_a)$. Properties (5.14.6a) and (5.14.7a) are consequences of (4.9.4) and (4.9.6) respectively, applied to $P'$. Properties (5.13.1a), (5.13.2a), and (5.13.3a) follow from the definition of $X_a$ and (4.9.3), (4.9.5).
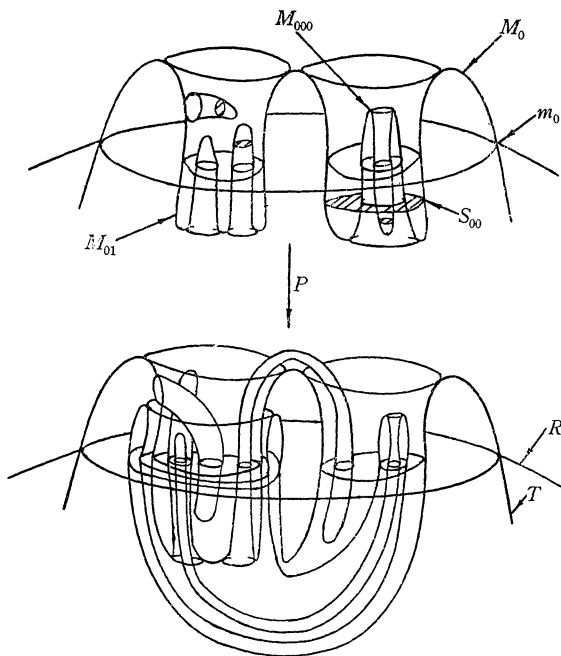
FIGURE 4

The following theorem gives the combinatorial structure of the collection of components of $R \cap T$, $R - T$, $T - R$, $S^3 - (T \cup R)$ where $R$ and $T$ are $PL$ 2-spheres in general position in $S^3$. The reader is referred to Definition 4.1 for some of the notation used. See Figure 4 for an illustration.

Theorem 5.16. *Suppose $R$, $T$ are $PL$ 2-spheres in $S^3$ that are in general position and $m$ is a simple closed curve in $R \cap T$ such that $m$ bounds a disk $D$ in $T$ and a disk $E$ in $R$ such that $R \cap T \subset D \cap E$. Let $B$ be the 3-cell bounded by $\mathrm{Cl}\,(T-D) \cup \mathrm{Cl}\,(R-E)$ such that $\mathrm{Int}\,B \cap \mathrm{Int}\,D = \mathrm{Int}\,B \cap \mathrm{Int}\,E = \varnothing$ and let $U_0$, $U_1$ and $V_0$, $V_1$ be respectively the components of $S^3 - T$, and $S^3 - R$ and such that $\mathrm{Int}\,B \subset U_0$, $\mathrm{Int}\,B \subset V_0$. Then there exists a finite tree labeling system $\Gamma$, a 4-tuple $(m_\alpha, H_\alpha, M_\alpha, S_\alpha)$ for each $\alpha \in \Gamma$, and a function $P$ from $S^3$ to $S^3$ such that*

(5.16.1)   $m_\alpha$ *is a component (a simple closed curve) of $D \cap E$ with*
   $m_0 = m = \mathrm{Bd}D = \mathrm{Bd}E$,

(5.16.2)   $H_\alpha$ *is a disk-with-holes in $E$ such that*
   (5.16.2.1)   $\mathrm{Bd}H_\alpha = m_\alpha \cup m_{\alpha 0} \cup m_{\alpha 1} \cup m_{\alpha 2} \cup \cdots$,
   (5.16.2.2)   $E = \cup \{H_\alpha | \alpha \in \Gamma\}$, $H_\alpha \cap H_{\alpha i} = m_{\alpha i}$,
   (5.16.2.3)   $H_\alpha$ *contains a collar of $\mathrm{Bd}H_\alpha$ in $\mathrm{Cl}\,(U_{|\alpha|})$*,

(5.16.3)   $M_\alpha$ *is a collar of $H_\alpha$ in $\mathrm{Cl}\,V_{|\alpha|}$ such that if $|\alpha| = |\beta|$ and $\alpha \neq \beta$ then $M_\alpha \cap M_\beta = \varnothing$,*

(5.16.4)   $S_\alpha$ *is a partition of $M_\alpha$ that supports the separation*

*complex* $K(\alpha) = K_1(\alpha) \cup K_2(\alpha) \cup K_3(\alpha)$, *and*
*if* $D^* = \bigcup_{\alpha \in \Gamma} (\mathrm{Bd}\, M_\alpha - \mathrm{Int}\, H_\alpha)$, $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, *and* $K = K_1 \cup K_2 \cup K_3$, $\bar{D}^*$,
$\bar{D}$ *are the separation complexes given respectively by* $K_i = \bigcup_{\alpha \in \Gamma} K_i(\alpha)$,
$\bar{D}^* = D^* \bmod (S \cup E)$ *and* $\bar{D} = D \bmod E$ *then*

(5.16.5)  $P \mid S^3 - (|K| - E) = 1$, *and* $P$ *is the identity on a corner of* $M_\alpha$ *for each* $\alpha \in \Gamma$,

(5.16.6)  $P \mid D^*$ *is a separation isomorphism from* $\bar{D}^*$ *to* $\bar{D}$, *and a homeomorphism from* $D^*$ *onto* $D$,

(5.16.7)  *if* $\sigma \in K$ *then* $\{P(\tau) \mid \tau \in \bar{\sigma}\} = \overline{P(\sigma)}$ *is a separation complex and* $P \mid \sigma$ *is a separation isomorphism from* $\bar{\sigma}$ *to* $\overline{P(\sigma)}$,

(5.16.8)  *if* $\sigma \in K_3$ *then* $P(\sigma - (S \cup E)) \subset S^3 - R$ *and* $P(\sigma \cap (S \cup E)) \subset E$,

(5.16.9)  *if* $\sigma_0, \sigma_1 \in K_3$ *and* $\sigma_0 \cap \sigma_1$ *is a component of* $S$ *then* $P(\sigma_0 - (S \cup E))$, $P(\sigma_1 - (S \cup E))$ *lie in different components of* $S^3 - R$, *and*

(5.16.10)  *if* $\sigma_0, \sigma_1 \in K_3$, $\sigma_0 \neq \sigma_1$, *but* $P(\sigma_0 - (S \cup E)) \cap P(\sigma_1 - (S \cup E)) \neq \varnothing$ *then for* $i = 0$, *or* $1$, $P(\sigma_i - (S \cup E)) \subset \mathrm{Int}\, P(\sigma_{1-i})$ *and* $P(\sigma_i \cap (S \cup E)) \subset \mathrm{Int}\, P(\sigma_{1-i} \cap (S \cup E))$.

*Proof.* The sets $m_\alpha$, $H_\alpha$, $M_\alpha$, $S_\alpha$, and function $P$ are constructed inductively using Lemma 5.15, Lemma 3.14 and the properties of winding functions given in (4.6), (4.7), (4.8), and (4.9). Let $q$ be a point in $T - D$ and identify the 2-sphere $R$ of the hypothesis and the 2-sphere $R$ of §5.

The 4-tuple $a_0 = (T, U_1, q, m)$ is in $\mathscr{S}$ of Definition 5.1. Let $(H_0, M_0, S_0, X_0, P_0)$ be the 5-tuple of the conclusion of Lemma 5.15 with $K(0) = K_1(0) \cup K_2(0) \cup K_4(0)$ the separation complex supported by $S_0$ in $M_0$ with $K_1'(0) \cup K_2'(0) \cup K_3'(0)$ the separation subdivision of $K(0)$ of Lemma 5.15. Let $m_0 = m$, and let $m_{00}, m_{01}, m_{02}, \cdots$ be the other components of $\mathrm{Bd}\, H_0$. Since by (4.6.1), (4.8.1), (4.7) and (5.14.5) we have $B \cap P_0(M_0) = m$, we let $U_0^0$, $U_1^0$ be the components of $S^3 - X_0$ with $\mathrm{Int}\, B \subset U_0^0$ and let $\Gamma_0 = \{0\}$. From the conclusion of Lemma 5.15 it follows that $\Gamma_0$, 4-tuple $(m_0, H_0, M_0, S_0)$, and function $P_0$ satisfy (5.16.1), (5.16.3), (5.16.4), (5.16.5), (5.16.7), (5.16.8), (5.16.9), and (5.16.10). Condition (5.16.2) is satisfied with the exception of the requirement $E = \bigcup_{\alpha \in \Gamma_0} H_\alpha$ and condition (5.16.6) is satisfied for $P \mid D_0^*$ where $D_0^* = X_0 - (T - D)$ and $\bar{D}_0^* = D_0^* \bmod S_0 \cup E$. The conclusion of Lemma 5.15 shows that the following properties are also satisfied.

$(a_0)$  $P_0(M_0) \cap (X_0 - M_0) = \varnothing$,

$(b_0)$  if $\sigma \in K_3(0)$ then $P_0 \mid \sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P_0(\tau) \mid \tau \in \bar{\sigma}'\}$, and

$(c_0)$  if $\sigma_0, \sigma_1 \in K_3(0)$ and $\sigma_0 \neq \sigma_1$ but $P_0(\sigma_0) \subset P_0(\sigma_1)$ then there exists $\sigma_2 \in K_3'(0)$ such that $\sigma_2 \subset \sigma_1$ and $P_0(\sigma_0) \subset P_0(\sigma_2)$.

We proceed inductively up the lexicographical ordering of the $m_\alpha$'s. Suppose for each element $\alpha$ of a tree labeling system $\Gamma_k$, a 4-tuple

$(m_\alpha, H_\alpha, M_\alpha, S_\alpha)$, and function $P_k$ from $S^3$ to $S^3$ exist that satisfy (5.16.1), (5.16.2) as modified above, (5.16.3), $\cdots$, (5.16.5), (5.16.6) as modified above, (5.16.7), $\cdots$, (5.16.10), and,

$(a_k)$   $P_k(\cup \{M_\alpha | \alpha \in \Gamma_k\}) \cap (X_k - \cup \{M_\alpha | \alpha \in \Gamma_k\}) = \varnothing$

$(b_k)$   if $\sigma \in K_3(k)$ then $P_k | \sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P_k(\tau) | \tau \in \bar{\sigma}'\}$, and

$(c_k)$   if $\sigma_0, \sigma_1 \in K_3(k)$ and $\sigma_0 \neq \sigma_1$ but $P_k(\sigma_0) \subset P_k(\sigma_1)$ then there exists $\sigma_2 \in K_3'(k)$ such that $\sigma_2 \subset \sigma_1$ and $P_k(\sigma_0) \subset P_k(\sigma_2)$,

where $K(k) = K_1(k) \cup K_2(k) \cup K_3(k) = \cup \{K(\alpha) | \alpha \in \Gamma_k\}$, $K'(k) = K_1'(k) \cup K_2'(k) \cup K_3'(k)$ is the subdivision of $K(k)$ given by $K(k)$ mod $(X_k - \cup \{\mathrm{Bd}M_\alpha - \mathrm{Int}\, H_\alpha | \alpha \in \Gamma_k\})$, $D_k^* = X_k - (T - D)$, and $\bar{D}_k^* = D_k^*$ mod $(\cup \{S_\alpha | \alpha \in \Gamma_k\}) \cup E$. Let $U_0^k, U_1^k$ be the components of $S^3 - X_k$ where $\mathrm{Int}\, B \subset U_0^k$.

Let $\Gamma_{k+1} = \Gamma^k \cup \{\beta\}$ where $\beta$ is the next element in the lexicographical ordering of the $m_\alpha$'s. That is, if $\alpha = i_1 i_2 \cdots i_n$ is the last element of $\Gamma_k$ then $\beta = i_1 i_2 \cdots i_{n-1}(i_n + 1)$ or $\beta = i_1 i_2 \cdots i_n 0$ depending on which of these is the lowest subscript among the symbols labeling the boundary components of the desk-with-holes $\cup \{H_\alpha | \alpha \in \Gamma_k\}$. Let $a_{k+1} = (X_k, U_{|\beta|}^k, q, m_\beta)$ and apply Lemma 5.15 to obtain 5-tuple $(H_\beta, M_\beta, \hat{S}, \hat{X}_{k+1}, \hat{P})$ that satisfies the conclusion of Lemma 5.15 with $\hat{K} = \hat{K}_1 \cup \hat{K}_2 \cup \hat{K}_3$ the separation complex supported by $\hat{S}$ in $M_\beta$. Properties (5.16.1), (5.16.2) as modified above, and (5.16.3) and (5.16.3) are easily established and we leave this to the reader. Label the components (if there are any) of $(\mathrm{Bd}H_\beta) - m_\beta$ with $m_{\beta 0}, m_{\beta 1}, \cdots$.

It follows by (5.14.2), (4.6.1), (4.8.1), (4.7), (5.16.2), and (5.16.4) that $(B \cup (\cup \{G_\alpha | \alpha \in \Gamma_k\})) \cap \hat{P}(M_\beta) = m_\beta$ and $\hat{P} | (B \cup (\cup \{G_\alpha | \alpha \in \Gamma_k\})) = 1$ where $G_\alpha = \mathrm{Bd}M_\alpha - \mathrm{Int}\, H_\alpha$. Hence, $\hat{X}_{k+1}$ contains $(T - D) \cup (\cup \{G_\alpha | \alpha \in \Gamma_{k+1}\})$.

For $\sigma \in \hat{K}_3$, $\hat{P}(\sigma)$ may intersect $\cup \{\mathrm{Int}\, M_\alpha | \alpha \in \Gamma_k\}$: however, $\hat{P}(\sigma)$ may intersect at most one of the sets $\mathrm{Int}\, M_\alpha, \alpha \in \Gamma_k$. Let $\alpha(\sigma)$ be the subscript such that $\hat{P}(\sigma) \cap \mathrm{Int}\, M_{\alpha(\sigma)} \neq \varnothing$. Proceeding as in Case B, paragraph 4, of the induction step in the proof of Lemma 5.15 we may apply Lemma 3.14 and Lemma 3.10 and assume without loss of generality that $S(\sigma)$ is a compact $PL$ 2-manifold in $\sigma$ such that if $Y$ is a component of $S(\sigma)$ then $Y \cap \mathrm{Bd}\sigma = Y \cap ((\mathrm{Bd}\sigma) - \hat{S}) = \mathrm{Bd}Y \neq \oslash$ and $\hat{P}(Y)$ is a component of $S_{\alpha(\sigma)} \cap \hat{P}(\sigma)$. Let $S_\beta = \hat{S} \cup (\cup \{S(\sigma) | \sigma \in \hat{K}_3\})$. It is clear that $S_\beta$ partitions $M_\beta$ and let $K(\beta) = K_1(\beta) \cup K_2(\beta) \cup K_3(\beta)$ be the separation complex that $S_\beta$ supports in $M_\beta$, thus (5.16.4) is satisfied. It is clear that $\hat{P}$ satisfies (5.16.7) and (5.16.10) for $\sigma, \sigma_0, \sigma_1 \in K_3(\beta)$ since $\hat{P}$ satisfied (4.6.2) and (4.6.5) for $\sigma, \sigma_0, \sigma_1 \in \hat{K}_3$ and the applications of Lemma 3.14 and Lemma 3.10 haven't altered these properties. Let $K(k + 1) = K_1(k + 1) \cup K_2(k + 1) \cup K_3(k+1) = \cup \{K(\alpha) | \alpha \in \Gamma_{k+1}\} = \hat{K}(k) \cup K(\beta)$, and Let $S_{k+1} = \cup \{S_\alpha | \alpha \in \Gamma_{k+1}\} = S_k \cup S_\beta$.

We now show that $P_k \cdot \hat{P} | M_\beta$ is winding function with respect to $(M_\beta, S_\beta, H_\beta, R)$. Property (4.6.1) is satisfied since by (5.14.2), and (5.16.5), respectively, $\hat{P}, P_k$ satisfy (4.6.1). Property (4.6.2) is estab-

lished by observing that, by the way $K(\beta)$ was constructed, if $\sigma \in K_3(\beta)$ then $(\cup \{M_\alpha | \alpha \in \Gamma_k\}) \cap \text{Int } \hat{P}(\sigma) = \varnothing$ or $\hat{P}(\sigma) = |L|$ where $L$ is a subcomplex of $\bar{\sigma}_0'$ for some $\sigma_0 \in K(k)$ ($\bar{\sigma}_0'$ is the induced subdivision of $\bar{\sigma}_0$ by $K'(k)$). In the former case, (4.6.2) follows by (5.16.5) and in the latter case by $(b_k)$. Properties (4.6.3) and (4.6.4) are straight forward consequences of the corresponding properties for $\hat{P}$, (5.16.8) and (5.16.9) for $P_k$, and the way $S_\beta$ was constructed. We now establish property (4.6.5). Suppose $\sigma_0, \sigma_1 \in K_3(\beta)$, $\sigma_0 \neq \sigma_1$, and $P_k \hat{P}(\sigma_0 - S_\beta) \cap P_k \hat{P}(\sigma_1 - S_\beta) \neq \varnothing$ then (a) $M_\alpha \cap \text{Int } \hat{P}(\sigma_i) = \varnothing$ for $i = 0, 1$ and all $\alpha \in \Gamma_k$, or (b) $\hat{P}(\sigma_i) \subset M_\alpha$, $M_\gamma \cap \text{Int } \hat{P}(\sigma_{1-i}) = \varnothing$ for some $\alpha$, and all $\gamma \in \Gamma_k$, or (c) $\hat{P}(\sigma_i) \subset M_\alpha$, $\hat{P}(\sigma_{1-i}) \subset M_\gamma$ for some $\alpha, \gamma \in \Gamma_k$. In Case (a), (4.6.5) follows from the corresponding properties for $\hat{P}$, the way $S_\beta$ was constructed, and (5.16.5). In Case (b), $P_k \hat{P}(\sigma_i - S_\beta) \subset \text{Int } P_k \hat{P}(\sigma_{1-i})$ since $P_k \hat{P}(\sigma_{1-i} - S_\beta) \subset X_k - (\cup \{M_\alpha | \alpha \in \Gamma_k\})$, $P_k \hat{P}(\sigma_i - S_\beta)$ is connected, and by $(a_k)$. In Case (c), either (c.1) there exists a $\sigma^* \in K_3(k)$ such that $\hat{P}(\sigma_i) \subset \sigma^*$ for $i = 0, 1$ or (c.2) there exist distinct $\sigma_0^*, \sigma_1^* \in K_3(k)$ such that $\hat{P}(\sigma_i) \subset \sigma_1^*$. In Case (c.1), (4.6.5) follows from the corresponding property for $\hat{P}$ and $(b_k)$. In Case (c.2), (4.6.5) follows from $(c_k)$, sice we must have $P_k(\sigma_i^*) \subset P_k(\sigma_{1-i}^*)$ for $i = 0$, or 1 by (5.16.10).

Let $\hat{P}_{k+1} = P_k \hat{P}$. We showed in the previous paragraph that $\hat{P}_{k+1} | M_\beta$ is a winding function with respect to $(M_\beta, S_\beta, H_\beta, R)$. This, combined with (5.16.5), (5.16.7), (5.16.8) and (5.16.9) for $P_k$ establishes (5.16.5), (5.16.7), (5.16.8), and (5.16.9) for $\hat{P}_{k+1}$. Property, (5.16.6) is valid for $\hat{P}_{k+1}$ since it is valid for $P_k$, $\hat{P}| \cup \{G_\alpha | \alpha \in \Gamma_k\} = 1$, and $\hat{P} | G_\beta$ is a separation isomorphism onto $J_{a_{k+1}}$ of Lemma 5.15.

Property (5.16.10) follows by the corresponding property for $P_k$ or $\hat{P}$ if either $\sigma_0, \sigma_1 \in K_3(k)$ or $\sigma_0, \sigma_1 \in K_3(\beta)$. The other cases are (a)$\sigma_0 \in K_3(k)$, $\sigma_1 \in K_3(\beta)$, and $\hat{P}(\sigma_1) \cap \text{Int } M_\alpha = \varnothing$ for all $\alpha \in \Gamma_k$ or (b) $\sigma_0 \in K_3(k)$, $\sigma_1 \in K_3(\beta)$, and $\hat{P}(\sigma_1) \subset M_\alpha$ for some $\alpha \in \Gamma_k$. In Case (a), $\hat{P}_{k+1}(\sigma_0 - S_{k+1}) \subset \text{Int } \hat{P}_{k+1}(\sigma_1)$, and $\hat{P}_{k+1}(\sigma_0 \cap (S_{k+1} \cup E)) \subset \text{Int } \hat{P}_{k+1}(\sigma_1 \cap (S_{k+1} \cup E))$ since $\hat{P}_{k+1}(\sigma_1 - S_{k+1}) \subset X_k - (\cup \{M_\alpha | \alpha \in \Gamma_k\})$, $\hat{P}_{k+1}(\sigma_0 - S_{k+1})$ is connected and by $(a_k)$. In Case (b) either (b.1) $\hat{P}(\sigma_1) \subset \sigma_0$ or (b.2) there exists a $\sigma^* \in K_3(k)$ such that $\hat{P}(\sigma_1) \subset \sigma^*$. In Case (b.1) (5.16.10) follows by $b_k$ since $\hat{P}(\sigma_1)$ must be the underlying set of a subcomplex of $K'(k)$ in $\sigma^*$. In Case (b.2), (5.16.10) follows from $c_k$, since we must have either $P_k(\sigma^*) \subset P_k(\sigma_0)$, or $P_k(\sigma_0) \subset P_k(\sigma^*)$ by (5.16.10) for $P_k$.

The properties remaining to be varified are $(a_{k+1})$, $(b_{k+1})$, and $(c_{k+1})$. We establish these by using the lifting property for winding functions in Lemma 4.9. For $\sigma \in K_3(k+1)$ let $N(\sigma) = \{\tau \in K_3(k+1) | \hat{P}_{k+1}(\tau) \subset \hat{P}_{k+1}(\sigma)$ and $\tau \neq \sigma\}$ and let $W(\sigma) = \hat{P}_{k+1}(\hat{X}_{k+1} - (\cup \{G_\alpha | \alpha \in \Gamma_{k+1}\})) \cap (\hat{P}_{k+1}(\sigma) - \cup \{\hat{P}_{k+1}(\tau) | \tau \in N(\sigma)\})$. It is clear that $(\hat{P}_{k+1}(\cup \{M_\alpha | \alpha \in \Gamma_{k+1}\}) - \hat{P}_{k+1}(\cup \{G_\alpha | \alpha \in \Gamma_{k+1}\})) \cap \hat{P}_{k+1}(\hat{X}_{k+1}) = \cup \{W(\sigma) | \sigma \in K_3(k+1)\}$ and if $\sigma_0 \neq \sigma_1$ then $\text{Int } W(\sigma_0) \cap \text{Int } W(\sigma_1) = \varnothing$. Also, by (5.16.8), (5.16.9), and (5.16.10), if $\sigma_0 \cap \sigma_1$ is a component of $S_{k+1}$ then $W(\sigma_0) \cap W(\sigma_1) = \hat{P}_{k+1}(\hat{X}_{k+1} - \cup \{G_\alpha | \alpha \in$

$\Gamma_{k+1}\}) \cap (\hat{P}_{k+1}(\sigma_0 \cap \sigma_1) - \cup \{\hat{P}_{k+1}(\tau) \,|\, \tau \in N(\sigma_i)\})$ for $i = 0$ or $1$. It follows that for each $\alpha \in \Gamma_{k+1}$, $W_\alpha = \cup \{W(\sigma) \,|\, \sigma \in K_3(\alpha)\}$ is a compact 2-manifold that satisfies the hypothesis of Lemma 4.9 along with winding function $\hat{P}_{k+1} \,|\, M_\alpha$ with respect to $(M_\alpha, S_\alpha, H_\alpha, R)$. Apply Lemma 4.9 for each $\alpha \in \Gamma_{k+1}$ and let $\bar{W}_\alpha$ be the resulting compact 2-manifold, $K'(\alpha)$ the resulting subdivision of $K(\alpha)$ satisfying 4.9.3 and $P_{k+1} \,|\, M_\alpha$ the resulting function that satisfies $(4.9.1), \cdots, (4.9.6)$.

Let $X_{k+1} = (\cup \{\bar{W}_\alpha \,|\, \alpha \in \Gamma_{k+1}\}) \cup (\cup \{G_\alpha \,|\, \alpha \in \Gamma_{k+1}\}) \cup (\hat{X}_{k+1} - \hat{P}_{k+1}(\cup \{M_\alpha \,|\, \alpha \in \Gamma_{k+1}\}))$, let $P_{k+1} \,|\, \cup \{M_\alpha \,|\, \alpha \in \Gamma_{k+1}\} = \cup \{(P_{k+1} \,|\, M_\alpha) \,|\, \alpha \in \Gamma_{k+1})\}$ and let $K'(k+1) = K_1'(k+1) \cup K_2'(k+1) \cup K_3'(k+1) = \cup \{K'(\alpha) \,|\, \alpha \in \Gamma_{k+1}\}$. By $(4.9.1)$, $P_{k+1}$ satisfies $(5.16.5), \cdots, (5.16.10)$, thus it is only necessary to varify $(a_{k+1})$, $(b_{k+1})$, and $(c_{k+1})$. Property $(a_{k+1})$ follows since $X_{k+1} - \cup \{M_\alpha \,|\, \alpha \in \Gamma_{k+1}\} = \hat{X}_{k+1} - \hat{P}_{k+1}(\cup \{M_\alpha \,|\, \alpha \in \Gamma_{k+1}\})$. Property $(b_{k+1})$ follows by $(4.9.4)$. Property $(c_k)$ is established by observing that if $P_{k+1}(\sigma_0) \subset P_{k+1}(\sigma)$ then $P_{k+1}(\sigma_0)$ is in a component $Z$ of $P_{k+1}(\sigma_1) - W(\sigma_1)$ since $P_{k+1}(\sigma_0)$ is connected and $W(\sigma_1) \cap P_{k+1}(\sigma_0) = \varnothing$ by definition of $W(\sigma_1)$. The lift of Cl $Z$ is the required $\sigma_2$, that is, $\sigma_2 = (P_{k+1} \,|\, \sigma_1)^{-1}(\text{Cl } Z)$.

The induction step is now complete. The above process terminates when $\cup \{H_\alpha \,|\, \alpha \in \Gamma_k\}$ is the disk $E$, thus completing property $(5.16.2)$. The reason for not including properties $(a_k)$, $(b_k)$, $(c_k)$ in the conclusion to Theorem 5.16 is that they are "empty" in the sense that $(\bar{D} \cup (T - D)) \cap (\cup \{\text{Int } M_\alpha \,|\, \alpha \in \Gamma\}) = \varnothing$.

## 6. Disjoint singular disks.

We now prove the following corollary to §§3, 4, and 5.

COROLLARY 6.1. *Suppose $T$ is a 2-sphere is $S^3$, $D$ is a PL subdisk of $T$, and $E$ is a PL disk in $S^3$ in general position with respect to $T$ such that $E \cap T \subset D$ and $\text{Bd} E = \text{Bd} D$. Let $U_0$, $U_1$ be the components of $S^3 - T$ and suppose that $E_i (i = 0, 1)$ is a PL disk in general position with respect to $D$ such that $\text{Bd} E_i \subset U_i$, $E_i$ fails to intersect the 2-sphere $E \cup (T - D)$, and $T - D$, $\text{Bd} E_0$, $\text{Bd} E_1$ are in the same component of $S^3 - (D \cup E)$. Let $F_i$ $(i = 0, 1)$ be the component of $E_i \cap U_i$ that contains $\text{Bd} E_i$. Then for $i = 0, 1$ there exist collections $\mathscr{D}_i$, $\mathscr{E}_i$ of components of $D - E_i$ and of $E_i \cap U_i$ respectively such that*

(6.1.1)  *if $A \in \mathscr{D}_0$, $B \in \mathscr{D}_1$ then Cl $A \cap$ Cl $B = \varnothing$,*

(6.1.2)  *if $A \in \mathscr{E}_i$ and $B \in \mathscr{D}_{1-i}$ then $E$ separates $A$ from $B$ in Cl $U_i$, and*

(6.1.3)  *the set $D_i = \cup \{\text{Cl } A \,|\, A \in \mathscr{D}_i \cup \mathscr{E}_i\}$ is a singular disk with $\text{Bd} D_i = \text{Bd} E_i$ obtained by pasting in the sense that there exists a map $f_i: E_i \to D_i$ such that $f_i \,|\, F_i = 1$ and if $K$ is any component of $f_i^{-1}(\text{Cl } A)$ for $A \in \mathscr{D}_i \cup \mathscr{E}_i$ then $f_i \,|\, K$ is a PL homeomorphism onto Cl $A$.*

*Proof.* It follows from the polyhedral approximation theorem of Bing [1] that we may assume that the 2-sphere $T$ is piecewise linear. In order to use Theorem 5.16, we assume that the component of $S^3 - T$ have been labeled so that $\mathrm{Cl}\,(U_1)$ contains a collar of $\mathrm{Bd}E$ in $E$ and $B$ is a collar of $\mathrm{Cl}\,(T - D)$ in $\mathrm{Cl}\,U_0$ such that $B \cap E = \mathrm{Bd}E$, and for $i = 0$, or 1, $B \cap E_i = \varnothing$. Let $R$ be the 2-sphere defined by $R = E \cup ((\mathrm{Bd}B) - (T - D))$ and let the components of $S^3 - R$ be $V_0$, $V_1$ where $\mathrm{Int}\,B \subset V_0$. It follows that $B - m$, $\mathrm{Bd}E_0$, $\mathrm{Bd}E_1$ are in the same component of $S^3 - (D \cup E)$. We use the notation of Theorem 5.16, in particular let $\Gamma$, $m_\alpha$, $H_\alpha$, $M_\alpha$, $S_\alpha$, $K(\alpha)$, $K$ and $P$ be as in the conclusion. The map $f_i$ is found by cutting $E_i$ off on $\cup \{P((\mathrm{Bd}M_\alpha) - H_\alpha)\,|\,\alpha \in \Gamma,\ |\alpha| = 1 - i\}$, but first we observe some useful facts.

It follows from (5.16.2), (5.16.3) that

(6.1.4)    $M_\alpha$ contains a corner in $\mathrm{Cl}\,(U_{|\alpha|} \cap V_{|\alpha|})$.

Also, by (5.16.5), (4.8.2) we have that

(6.1.5)    $P(M_\alpha)$ contains a collar of $P(\mathrm{Bd}M_\alpha - \mathrm{Int}\,H_\alpha)$ in $\mathrm{Cl}\,U_{|\alpha|}$.

By (5.16.5), (5.16.7), $\cdots$, (5.16.10), respectively, we have, respectively, (4.6.1), (4.6.2), $\cdots$, (4.6.5). Thus $P\,|\,M_\alpha$ is a winding function with respect to $(M_\alpha, S_\alpha, H_\alpha, R)$; consequently, by (4.8.1) we have

(6.1.6)    $((B - m) \cup \mathrm{Bd}E_0 \cup \mathrm{Bd}E_1) \cap P(M_\alpha) = \varnothing$ for all $\alpha \in \Gamma$.

We wish to establish that

(6.1.7)    $E_i \cap U_{1-i} \subset E_i \cap P(\cup \{M_\alpha\,|\,\alpha \in \Gamma\}) \subset E_i \cap P(\cup \{M_\alpha\,|\,\alpha \in \Gamma,\ |\alpha| = 1 - i\})$ for $i = 0$ or 1.

Let $A = \{Y\,|\,Y$ is a component of $E_i \cap U_{1-i}$ such that there does not exist a $\sigma \in K_3$ such that $Y \subset P(\sigma)\}$ and let $B = \{Z\,|\,Z$ is a component of $E_i \cap \mathrm{Int}\,P(\sigma)$ where $\sigma \in \cup \{K_3(\alpha)\,|\,\alpha \in \Gamma,\ |\alpha| = 1\}$ and there does not exist a $\tau \in \cup \{K_3(\alpha)\,|\,\alpha \in \Gamma,\ |\alpha| = 1 - i\}$ such that $P(\sigma) \subset P(\tau)\}$. Suppose $A \cup B \neq \varnothing$, then there exists an outermost element of $A \cup B$ in $E_i$. That is, there exists an element $W \in A \cup B$ such that $W'$ fails to separate $W$ from $\mathrm{Bd}E_i$ in $E_i$ for all $W' \in A \cup B$. Let $w$ be the outermost component of $\mathrm{Bd}W$ and let $X$ be the component of $E_i - T$ such that $w \subset \mathrm{Bd}X$ but $X \cap W = \varnothing$. If $W \in A$ then, since $w \subset P(\mathrm{Bd}\sigma - (S \cup E))$ for some $\sigma \in K_3$ and because of (6.1.5), it follows that there exists a $Z \in B$ such that $X \subset Z$ contradicting that $W$ is an outermost member of $A \cup B$ in $E_i$. If $W \in B$, then, by (6.1.5) since $W$ is also an outermost member of $B$, it follows that $X \in A$ contradicting that $W$ is an outermost member of $A \cup B$ in $E_i$. Hence we must have $A \cup B = \varnothing$ and (6.1.7) is established.

We wish now to establish that

(6.1.8)    $E$ separates $E_i - P(\cup \{M_\alpha\,|\,\alpha \in \Gamma,\ |\alpha| = 1 - i\})$ from $P(\cup \{(\mathrm{Bd}M_\alpha) - H_\alpha\,|\,\alpha \in \Gamma,\ |\alpha| = i\})$ in $\mathrm{Cl}\,U_i$.

Suppose (6.1.8) is false, then by (6.1.7) there exists an arc $ab$ from $a$ to $b$ such that $a \in E_i - P(\cup \{M_\alpha\,|\,\alpha \in \Gamma\})$, $b \in P((\mathrm{Bd}M_\alpha) - H_\alpha)$ for some $\alpha \in \Gamma$ with $|\alpha| = i$, and $\mathrm{Int}\,ab \subset U_i - E$. However, by (6.1.5)

there exists a point $c$ in $ab$ near $b$ such that $c \in (P(M_\alpha) - P(\mathrm{Bd}\,M_\alpha)) \cap U_i$. But this contradicts (4.8.1), thus (6.1.8) is established.

The collections $\mathscr{E}_i$, $\mathscr{D}_i$ $(i = 0, 1)$ of the conclusion will be chosen such that $\cup \{A \mid A \in \mathscr{E}_i\} \subset E_i - P(\cup \{M_\alpha \mid \alpha \in \Gamma, |\alpha| = 1 - i\})$ and $\cup \{\mathrm{Cl}\,A \mid A \in \mathscr{D}_i\} \subset P(\cup \{\mathrm{Bd}\,M_\alpha) - H \mid \alpha \in \Gamma, |\alpha| = 1 - i\})$. Properties (6.1.2) and (6.1.1) will then follow by (6.1.8) and the fact that $P(\cup \{(\mathrm{Bd}\,M_\alpha) - H_\alpha \mid \alpha \in \Gamma, |\alpha| = i\}) \cap P(\cup \{(\mathrm{Bd}\,M_\alpha) - H_\alpha \mid \alpha \in \Gamma, |\alpha| = 1 - i\}) = \varnothing$.

If $m$ is a component of $D \cap E_i$ then we let $E_i(m)$ be the disk $m$ bounds is $E_i$. Let $\mathscr{E}_i' = \{X \mid X$ is a component of $E_i - P(\cup \{M_\alpha \mid \alpha \in \Gamma, |\alpha| = 1 - i\})$. Each member of $\mathscr{E}_i'$ is an open disk-with-holes whose boundary consists of one "outer" simple closed curve and perhaps several "inner" simple closed curves. We accordingly let $\mathscr{O}_i = \{m \mid m$ is a component of $\mathrm{Bd}\,X$ for some $X \in \mathscr{E}_i'$, and $X \subset E_i(m)\}$ and $\mathscr{I}_i = \{m \mid m$ is a component of $\mathrm{Bd}\,X$ for some $X \in \mathscr{E}_i'$, and $X \cap E_i(m) = \varnothing\}$. For each $m \in \mathscr{I}_i$ we find a disk-with-holes $D(m) \subset P(\cup \{(\mathrm{Bd}\,M_\alpha) - H_\alpha \mid \alpha \in \Gamma, |\alpha| = 1 - i\})$ such that $m \subset \mathrm{Bd}\,D(m)$ and if $u$ is a component of $(\mathrm{Bd}\,D(m)) - m$ then $u \in \mathscr{O}_i$ and $u \subset E_i(m)$. The $D(m)$'s bridge the gaps between elements of $\mathscr{E}_i'$ and enable us to define the map $f_i$ of the conclusion (see Figure 5).
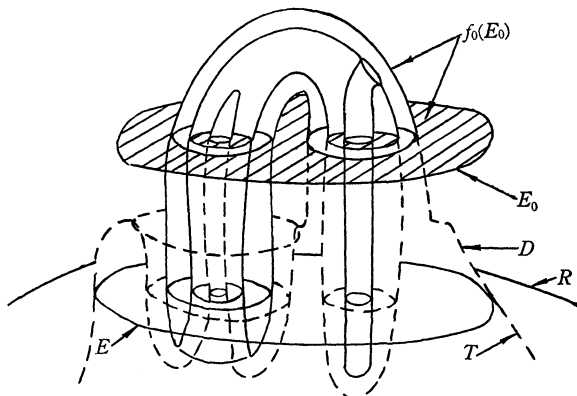


FIGURE 5

Let $m \in \mathscr{I}_i$; then $m \subset P(\mathrm{Bd}\,\sigma - (S \cup E))$ for a unique $\sigma \in K_3(\alpha)$ with $|\alpha| = 1 - i$. Let $Y$ be the component of $P(\sigma) \cap E_i$ that contains $m$. Clearly $Y \subset E_i(m)$, and since there does not exist a $\tau \in K_3$ such that $P(\sigma) \subset \mathrm{Int}\,P(\tau)$ it follows by (6.1.5) that each component of $(\mathrm{Bd}\,Y) - m$ is in $\mathscr{O}_i$. Since each component $t$ of $\mathrm{Bd}\,Y$ separates $T$, we have that $t$ separates the 2-face $\tau$ of $\{P(\tau) \mid \tau \in \bar{\sigma}\}$ that contains $t$. By Lemma 3.14, $Y$ separates $P(\sigma)$, and by Lemma 3.10 we may assume without loss of generality that there exists a compact, connected $PL$ 2-manifold $\sigma_0 \subset \sigma$ such that $\sigma_0 \cap \mathrm{Bd}\,\sigma = \sigma_0 \cap ((\mathrm{Bd}\,\sigma) - (S \cup E)) = \mathrm{Bd}\,\sigma_0$ and $P|\sigma_0$ is a separation isomorphism from $\bar{\sigma}_0$ to $\{t \mid t$ is a component of $\mathrm{Bd}\,Y\} \cup \{Y\}$. Let $m'$ be the component of $\mathrm{Bd}\,\sigma$ such that $P(m') = m$. The 2-mani-

fold $\sigma_0$ separates $M(\alpha)$ so let $D'(m)$ be the component of $(\mathrm{Bd}\,M(\alpha)) - \sigma_0$ that contains $m'$ in its boundary but does not contain $H(\alpha)$. It follows that $\mathrm{Bd}\,D'(m) \subset \mathrm{Bd}\,\sigma_0$. Let $D(m) = P(D'(m))$.

For $m_\alpha \in \mathscr{O}_i$, let $X(m_\alpha)$ be the element of $\mathscr{E}'_1$ that contains $m_\alpha$ in its boundary and let $m_{\alpha 0},\ m_{\alpha 1},\ \cdots$ be the remaining components of $\mathrm{Bd}\,X(m_\alpha)$. Also, let $m_{\beta 0},\ m_{\beta 1},\ \cdots$ be the components of $(\mathrm{Bd}\,D(m_\beta)) - m_\beta$. Let $m_0^i = \mathrm{Bd}\,E_i$, then for $i = 0, 1, f_i, \mathscr{E}_i, \mathscr{D}_i$ of conclusion are given respectively by

$$f_i(E_i) = X(m_0^i) \cup \left( \bigcup_{i_1} D(m_{0i_1}) \right) \cup \left( \bigcup_{i_1,i_2} X(m_{0i_1 i_2}) \right) \cup \left( \bigcup_{i_1,i_2,i_3} D(m_{0i_1,i_2,i_3}) \right) \cup \cdots$$

$$\mathscr{E}_i = \{X(m_0^i)\} \cup \{X(m_{0i_1 i_2})\} \cup \{X(m_{0i_1 i_2 i_3 i_4})\} \cup \cdots$$

$$\mathscr{D}_i = \{D(m_{0i_1})\} \cup \{D(m_{0i_1 i_2 i_3})\} \cup \{D(m_{0i_1 i_2 i_3 i_4 i_5})\} \cup \cdots.$$

That there exists no circularity in $f_i(E_i)$ follows from the fact that $\mathrm{Bd}\,D(m_\alpha) \subset E_i(m_\alpha)$ or, in other words, we proceed always from the "outside" to the "inside" along $E_i$.

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