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FUNDAMENTAL GROUPS OF COMPACT COMPLETE LOCALLY AFFINE COMPLEX SURFACES

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The fundamental group of a compact complete locally affine complex manifold of two complex dimensions is a solvable group which is a finite cyclic extension of a nilpotent or abelian group. Such a manifold has vanishing Euler characteristic and is finitely covered by a nilmanifold. A description of these manifolds and their fundamental groups is obtained in the course of the proofs of these facts.

1. Introduction. A locally affine manifold is a manifold with an affine connection having zero curvature and torsion. A complete locally affine real manifold is of the form $\boldsymbol{R}^n/\Gamma$ ([3]) and a complete locally affine complex manifold is of the form $\boldsymbol{C}^n/\Gamma$([7]); Γ denotes a freely-acting properly discontinuous group of real or complex affine transformations, and the connection is induced from the usual one on $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$. This representation allows a group-theoretic study of complete locally affine spaces, the most difficult aspect of which is determining which abstract groups can be embedded in the group of affine transformations of $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$ to give a Γ as described above. Such groups are of course the fundamental groups of complete locally affine spaces.

Kuiper ([5]) has studied compact locally affine real surfaces, benefiting from the knowledge of fundamental groups of compact real surfaces in general. Auslander ([1]) has studied compact locally Hermitian complex surfaces, benefiting from the fact that these are finitely covered by tori, a fact which is a consequence of Bieberbach's theorems on crystallographic groups. Vitter ([8]) has studied arbitrary compact locally affine complex surfaces using the results of Kodaira on general complex surfaces.

In this paper, we prove several results about the fundamental group Γ of a compact complete locally affine complex surface $\boldsymbol{C}^2/\Gamma$ which are necessary for a detailed study of such structures. We show: Γ is a finite cyclic extension of a subgroup Γ_0; Γ_0 is either abelian or nilpotent and its structure can be described precisely. Furthermore: $\boldsymbol{C}^2/\Gamma_0$ is a nilmanifold, and the Euler characteristic of $\boldsymbol{C}^2/\Gamma$ vanishes. The methods used here are in the spirit of Auslander and Kuiper and are quite different from those of Kodaira and Vitter.

2. Algebraic preliminaries. In this section, we derive several facts about subgroups Γ of the group $A(2, \boldsymbol{C})$ of complex affine transformations of C^2, using only the assumption that Γ acts freely on C^2.

A transformation $A \in A(2, C)$ may be identified with a nonsingular complex matrix $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$. The action of A, on the left of C^2, sends (x, y) to (x', y'), where

$$x' = ax + by + r$$

$$y' = cx + dy + s\ .$$

If $A = \begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix} \in A(2, C)$, we denote by $h(A)$ the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, C)$, the "holonomy part" of A.

LEMMA 2.1. *If $\Gamma \subset A(2, C)$ acts freely on C^2, then each element of $h(\Gamma)$ has 1 as an eigenvalue.*

Proof. The point (x, y) is a fixed point of $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix} \in A(2, C)$ exactly if

$$(a - 1)x + by = -r$$

$$cx + (d - 1)y = -s\ .$$

These equations have a solution unless 1 is an eigenvalue of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let G_1 denote the group of all complex matrices of the form $\begin{pmatrix} a & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ with $a \neq 0$; let G_2 denote the group of all complex matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ with $d \neq 0$.

PROPOSITION 2.2. (Cf. [5].) *If $\Gamma \subset A(2, C)$ acts freely on C^2, then Γ is conjugate in $A(2, C)$ to a subgroup of G_1 or a subgroup of G_2.*

Proof. Suppose first that Γ contains an element A such that $h(A)$ has an eigenvalue $\lambda \neq 1$. Put $h(A)$ in diagonal form $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ by conjugating by $P \in GL(2, C)$. Suppose $B \in \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1}$. Write $h(B) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $h(A)h(B) = \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix}$. Since both $h(A)$ and $h(AB)$ have 1 as an eigenvalue, we get $(a-1)(d-1) - bc = 0$ and $(\lambda a - 1)(d - 1) -$

$\lambda bc = 0$. Multiply the first equation by λ and subtract it from the second to obtain $(\lambda - 1)(d - 1) = 0$. Hence $d = 1$, and then $bc = 0$. That is, $h(B) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ or $h(B) = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}$. We cannot have both kinds of $h(B)$ occurring; for if both $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a' & 0 \\ c' & 1 \end{pmatrix}$ were in $Ph(\Gamma)P^{-1}$, with $b \neq 0$ and $c' \neq 0$, we would have $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & 0 \\ c' & 1 \end{pmatrix} = \begin{pmatrix} aa' + bc' & b \\ c' & 1 \end{pmatrix} \in Ph(\Gamma)P^{-1}$, but this matrix does not have 1 as an eigenvalue. Hence we have, in this case, that $\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}\Gamma\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1}$ is contained in G_1 or in the group of all complex matrices of the form $\begin{pmatrix} a & 0 & r \\ c & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$; the latter is conjugate to G_2 via $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and we are done.

Now suppose every element of $h(\Gamma)$ has both eigenvalues 1. If $h(\Gamma)$ consists only of the identity, we are done. Otherwise, some conjugate of Γ contains an element of the form $\begin{pmatrix} 1 & 1 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$. Let $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$ be an arbitrary element of this conjugate of Γ. Then both $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ c & d \end{pmatrix}$ have both their eigenvalues 1. Thus $(a-1)(d-1) - bc = 0$ and $(a + c - 1)(d - 1) - (b + d)c = 0$. Subtracting these equations gives $c = 0$. Hence $a = d = 1$, and we are done.

COROLLARY 2.3. *If $\Gamma \subset A(2, \boldsymbol{C})$ acts freely on $\boldsymbol{C}^2$, then Γ is solvable.*

Proof. The third derived group of Γ is trivial since this is true of G_1 and G_2.

LEMMA 2.4. *If $\Gamma \subset A(2, \boldsymbol{C})$ acts freely on $\boldsymbol{C}^2$ and $h(\Gamma)$ is abelian, then Γ is conjugate in $A(2, \boldsymbol{C})$ to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & 0 & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ $(d \neq 0)$ or to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$.*

Proof. If $h(\Gamma)$ consists only of the identity, we are done. Suppose $h(\Gamma)$ contains a non-identity element A which is diagonalizable. Conjugate A in $GL(2, \boldsymbol{C})$ to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 1$. If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the corresponding conjugate of $h(\Gamma)$, the fact that $AB = BA$ implies $b = c = 0$, and the fact that Γ acts freely implies $a = 1$ or $d = 1$.

If a were $\neq 1$ we would have $AB = \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix}$, in contradiction to 2.1. Hence every element of $h(\Gamma)$ is simultaneously conjugate to a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, and we are done in this case.

Suppose now that no element of $h(\Gamma)$ is diagonalizable. Let A be a non-identity element of $h(\Gamma)$. Conjugate A in $GL(2, C)$ to $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda \neq 0$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in the corresponding conjugate of $h(\Gamma)$. The fact that $AB = BA$ implies $c = 0$ and $a = d$, so necessarily $a = d = 1$. As before, we are done.

LEMMA 2.5. *If* $\begin{pmatrix} a & b & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *has no fixed points in* C^2, *then* $b = 0$.

Proof. If $b \neq 0$, $(0, -r/b)$ is a fixed point.

LEMMA 2.6. *If* $\Gamma \subset G_1$ *acts freely on* C^2, *then* $h(\Gamma)$ *is abelian.*

Proof. Let $A = \begin{pmatrix} a & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' & r' \\ 0 & 1 & s' \\ 0 & 0 & 1 \end{pmatrix}$ be elements of Γ. Then $ABA^{-1}B^{-1} = \begin{pmatrix} 1 & f & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By 2.5, $f = 0$.

COROLLARY 2.7. *If* $\Gamma \subset A(2, C)$ *acts freely on* C^2, *then* Γ *is conjugate in* $A(2, C)$ *to a subgroup of* G_2.

Proof. By 2.2, Γ is conjugate to a subgroup of G_1 or G_2. If Γ is conjugate to a subgroup of G_1, then $h(\Gamma)$ is abelian by 2.6. Then, by 2.4, Γ is conjugate to a subgroup of G_2.

LEMMA 2.8. *If* $\Gamma \subset A(2, C)$ *is abelian and acts freely on* C^2, *then* Γ *is conjugate in* $A(2, C)$ *to a subgroup of the group of all matrices of the form* $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ *or to a subgroup of the group of all matrices of the form* $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $(d \neq 0)$.

Proof. $h(\Gamma)$ is abelian, so by 2.4 we can conjugate Γ into the group of all $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$, in which case we are done, or into the group of all $\begin{pmatrix} 1 & 0 & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$. In the latter case: If all entries d which occur are 1, we are done. If some element has $d \neq 1$, further conjugation by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s/(d-1) \\ 0 & 0 & 1 \end{pmatrix}$ takes this element to $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and other elements to $\begin{pmatrix} 1 & 0 & r' \\ 0 & d' & s' \\ 0 & 0 & 1 \end{pmatrix}$. Since Γ is abelian, we must have all $s' = 0$.

3. Topological preliminaries.

The hypotheses that Γ acts properly discontinuously on $\boldsymbol{C}^2$ and that $\boldsymbol{C}^2/\Gamma$ is compact are brought into play in this section.

We note the following important fact ([4]), p. 357): The dimension of a real Euclidean space on which a group Γ acts freely, properly discontinuously, and with compact orbit space is determined by Γ itself, namely as the projective dimension of the integer group ring of Γ.

As a first application of this remark, we prove the following from Auslander ([2]).

LEMMA 3.1. *Suppose that $\Gamma \subset A(2, \boldsymbol{C})$ acts freely and properly discontinuously, and that $\boldsymbol{C}^2/\Gamma$ is compact. Then the set of translational parts (r, s) of elements $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$ of Γ contains a basis for $\boldsymbol{C}^2$ as a real vector space.*

Proof. Let V be the real subspaces of $\boldsymbol{C}^2$ spanned by the translational parts of elements of Γ. Then the action of Γ on $\boldsymbol{C}^2$ sends V to itself. Further, Γ acts freely and properly discontinuously on V, and V/Γ is compact. By the remark above, V and $\boldsymbol{C}^2$ have the same dimension, so $V = \boldsymbol{C}^2$.

COROLLARY 3.2. *If $\Gamma \subset A(2, \boldsymbol{C})$ is abelian, acts freely and properly discontinuously, and $\boldsymbol{C}^2/\Gamma$ is compact, then Γ is conjugate in $A(2, \boldsymbol{C})$ to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$.*

Proof. By 2.8, the only alternative is that Γ can be conjugated to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By 3.1, this cannot happen.

LEMMA 3.3. *Suppose $\Gamma \subset A(2, \boldsymbol{C})$ acts properly discontinuously on $\boldsymbol{C}^2$ and contains elements $A = \begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & f & u \\ 0 & h & v \\ 0 & 0 & 1 \end{pmatrix}$ such that $d \neq 1$ and $AB \neq BA$. Then d is a root of unity.*

Proof. By direct computation we verify that

$$A^n = \begin{pmatrix} 1 & \dfrac{d^n - 1}{d - 1} b & \dfrac{d^n - 1}{d - 1}\dfrac{bs}{d - 1} + n\left(r - \dfrac{bs}{d - 1}\right) \\ 0 & d^n & \dfrac{d^n - 1}{d - 1} s \\ 0 & 0 & 1 \end{pmatrix}$$

for all integers n, and that the matrix $C_n = A^{-n}BA^nB^{-1} = \begin{pmatrix} 1 & f_n & u_n \\ 0 & 1 & v_n \\ 0 & 0 & 1 \end{pmatrix}$ has entries given by

$$f_n = (d^n - 1)\left(\frac{f}{h} - \frac{b}{d - 1} + \frac{d}{(d - 1)h}\right)$$

$$u_n = (d^n - 1)\left(-\frac{fv}{h} - \frac{bv}{(d - 1)h} + \frac{bv}{d - 1} + \frac{bs}{(d - 1)^2} + \frac{sf}{d - 1} - \frac{bsh}{(d - 1)^2}\right)$$
$$+ (d^{-n} - 1)\left(-\frac{bsh}{(d - 1)^2} + \frac{bv}{d - 1} + \frac{bs}{(d - 1)^2}\right)$$

$$v_n = (d^{-n} - 1)\left(v + \frac{s}{d - 1} - \frac{sh}{d - 1}\right).$$

We claim that if d is not a root of unity, then the matrices C_n are distinct. For suppose that $C_m = C_n$ with $m \neq n$. This would give $f/h - b/(d - 1) + b/((d - 1)h) = 0$ and $v + s/(d - 1) - sh/(d - 1) = 0$; that is, $(d - 1)f + b(1 - h) = 0$ and $(d - 1)v + s(1 - h) = 0$. Multiply the first of the latter two equations by s and the second by b; subtract to obtain $(d - 1)(sf - bv) = 0$. The equations $(d - 1)f = b(h - 1)$, $(d - 1)v = s(h - 1)$, and $sf = bv$ imply $AB = BA$; a contradiction.

Assume that d is not a root of unity and consider the points (x_n, y_n) of $\boldsymbol{C}^2$ obtained by applying the distinct transformations C_n of Γ to the point $(0, v - sh/(d - 1))$. We have

$$x_n = f_n\left(v - \frac{sh}{d - 1}\right) + u_n = (d^{-n} - 1)\left(-\frac{bsh}{(d - 1)^2} + \frac{bv}{d - 1} + \frac{bs}{(d - 1)^2}\right)$$

$$y_n = v - \frac{sh}{d - 1} + v_n = d^{-n}\left(v + \frac{s}{d - 1} - \frac{sh}{d - 1}\right) - \frac{s}{d - 1}.$$

If $|d| = 1$, we may find a sequence n_i tending to infinity such that

$\lim d^{n_i} = 1$. Then the sequence of points (x_{n_i}, y_{n_i}) has a limit in $\boldsymbol{C}^2$, contradicting the assumption that Γ acts properly discontinuously. If $|d| \neq 1$, we may assume $|d| > 1$. Again this sequence of points has a limit point and we obtain a contradiction.

PROPOSITION 3.4. *If $\Gamma \subset A(2, \boldsymbol{C})$ acts freely and properly discontinuously on $\boldsymbol{C}^2$, and $\boldsymbol{C}^2/\Gamma$ is compact, then Γ contains a unipotent normal subgroup Γ_0 of finite index with Γ/Γ_0 cyclic.*

Proof. By 2.7, we may assume $\Gamma \subset G_2$. Assume that Γ contains a central element $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ with $d \neq 1$. Conjugate Γ by

$$\begin{pmatrix} 1 & \dfrac{-b}{d-1} & 0 \\ 0 & 1 & \dfrac{s}{d-1} \\ 0 & 0 & 1 \end{pmatrix} \in G_2$$

to obtains a subgroup of G_2 containing the central element $\begin{pmatrix} 1 & 0 & r' \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since this element is central, all the elements of this new subgroup have the form $\begin{pmatrix} 1 & 0 & u \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix}$. But, according to 3.1, this cannot happen.

Now in general, Γ is the fundamental group of the compact manifold $\boldsymbol{C}^2/\Gamma$, so it is finitely generated. Let $A_i = \begin{pmatrix} 1 & b_i & r_i \\ 0 & d_i & s_i \\ 0 & 0 & 0 \end{pmatrix} (1 \leqq i \leqq k)$ be a set of generators of Γ. If A_i is central, $d_i = 1$ by the preceding paragraph. If A_i is not central, d_i is a root of unity by 3.3. Hence the image of the homomorphism $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix} \to d$ of Γ into the unit circle group in $\boldsymbol{C}$ is a finite group. This finite group, being a subgroup of the multiplicative group of a field is cyclic. The kernel of this homomorphism is the desired subgroup Γ_0.

4. The main theorem.

In this section, we sharpen the statement of Proposition 3.4 and interpret our results terms of compact complete locally affine complex surfaces.

Let D_k $(k \geqq 1)$ denote the torsion free nilpotent group with generators: A, B, C, and D; and relations: $ABA^{-1}B^{-1} = C^k$, C and D central.

THEOREM 4.1. *Let $\Gamma \subset A(2, \boldsymbol{C})$ act freely and properly discon-*

tinuously on $\boldsymbol{C}^2$, *and let* $\boldsymbol{C}^2/\Gamma$ *be compact. Then* Γ *contains a unipotent normal subgroup* Γ_0 *of finite index such that* Γ_0 *is isomorphic to* $\boldsymbol{Z}^4$ *or* D_k *(for some* $k \geqq 1$*) and* Γ/Γ_0 *is cyclic of order* 1, 2, 3, 4, 5, 6, 8, 10, *or* 12.

Proof. Let Γ_0 be the subgroup of Γ obtained in 3.4. By results of Malcev ([6]) on nilmanifolds, Γ_0 may be considered as a discrete subgroup of a unique connected simply-connected nilpotent Lie group N such that N/Γ_0 is compact. Then Γ_0 acts freely and properly discontinuously on the space N and the orbit space is compact. N is topologically Euclidean, so by the remark preceding 3.1, the real dimension of N is four. Now, there are only two four-dimensional connected simply-connected Lie groups, namely $\boldsymbol{R}^4$ and $\left\{\text{all real matrices of the form } \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\right\} \times \boldsymbol{R}$. For these two groups, the discrete subgroups with compact quotients are known to be isomorphic to $\boldsymbol{Z}^4$ in the first case and the D_k in the second case.

Let $G \subset \boldsymbol{C}$ be the additive group of complex numbers b such that $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_0$. Let $S \in \Gamma$ be an element whose image in Γ/Γ_0 generates this cyclic group. Then $S = \begin{pmatrix} 1 & f & u \\ 0 & \lambda & v \\ 0 & 0 & 1 \end{pmatrix}$ with $\lambda^n = 1$, where n is the index of Γ_0 in Γ. $S^{-1}\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}S = \begin{pmatrix} 1 & \lambda b & r' \\ 0 & 0 & \lambda^{-1}s \\ 0 & 0 & 1 \end{pmatrix}$ shows that $\lambda b \in G$ for $b \in G$. If G is the trivial subgroup (0) of $\boldsymbol{C}$, then the "holonomy group" $h(\Gamma)$ is finite cyclic of order n and we are in the case studied by Auslander in [1]. In this case, $n = 1, 2, 3, 4$, or 6. Assume now that G is not trivial. Then G is a free abelian group of rank r, with $1 \leqq r \leqq 4$, since Γ_0 can be generated by four elements. Let $b_1 \cdots, b_r$ be a basis of G. Expressing λb_i in terms of this basis and taking a determinant, we obtain a polynomial of degree r with integer coefficients which is satisfied by λ. Hence the field generated by λ over the rationals is of degree at most r. This field is the field generated by a primitive n^{th} root of unity, so it has degree $\varphi(n)$, where φ is Euler's totient. Thus $\varphi(n) \leqq r$. The only solutions of $\varphi(n) \leqq 4$ are those listed in the statement of the theorem.

The groups $\boldsymbol{Z}^4$ and D_k of 4.1 do occur as subgroups of $A(2, \boldsymbol{C})$ which act freely and properly discontinuously on $\boldsymbol{C}^2$ with compact orpbit space. Which cyclic extensions of these groups can occur is a more delicate question.

EXAMPLE 4.2. Let A, B, C, and D be the matrices

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & \sqrt{-1}\\ 0 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & \sqrt{-1}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

respectively. These matrices generate a subgroup Γ_0 of $A(2, \boldsymbol{C})$ which is isomorphic to D_1. Γ_0 acts freely and properly discontinuously on $\boldsymbol{C}^2$ and $\boldsymbol{C}^2/\Gamma_0$ is a compact complete locally affine complex surface which is even a nilmanifold. Let S be the matrix $\begin{pmatrix}1 & 0 & 1/2\sqrt{-1}\\ 0 & -1 & \sqrt{-1}\\ 0 & 0 & 1\end{pmatrix}$. Then $S^2 = D$ and conjugation by S sends A, B, C, D to A^{-1}, B^{-1}, C, D respectively. S and the group Γ_0 generate a subgroup Γ of $A(2, \boldsymbol{C})$ containing Γ_0 as a normal subgroup of index two. Γ acts freely and properly dicontinuously on $\boldsymbol{C}^2$ and $\boldsymbol{C}^2/\Gamma$ is a compact complete locally affine complex surface. The fundamental group Γ of this surface is solvable but not nilpotent. The commutator subgroup of Γ is generated by A^2, B^2, and C, from which we obtain the first Betti number of $\boldsymbol{C}^2/\Gamma$ as $b_1 = 1$. Using Poincaré duality and the vanishing of the Euler characteristic, proved below, we find all Betti numbers of $\boldsymbol{C}^2/\Gamma$ are given by 1, 1, 0, 1, 1. $\boldsymbol{C}^2/\Gamma$ is an example of a compact complete locally affine complex surface with non-abelian "holonomy group" $h(\Gamma)$.

A complete locally affine surface may be represented in two ways as $\boldsymbol{C}^2/\Gamma$ and $\boldsymbol{C}^2/\Gamma'$. This corresponds to Γ and Γ' being conjugate in $A(2, \boldsymbol{C})$; the element of $A(2, \boldsymbol{C})$ effecting the conjugation amounts to a change of coordinates on the surface. This allows us to interpret 4.1 as follows.

Consequences 4.3. 1. A compact complete locally affine complex surface has a fundamental group which is solvable and is an extension of $\boldsymbol{Z}^4$ or some D_k by a finite cyclic group of order 1, 2, 3, 4, 5, 6, 8, 10, or 12.

2. Such a surface is finitely covered by another such surface which is a nilmanifold with fundamental group $\boldsymbol{Z}^4$ or some D_k; the cover is normal with deck transformation group cyclic of one of the above orders.

We conclude with a proof of the following theorem. This result was also obtained by Vitter ([8]) using the fact that the presence of a locally affine structure on a compact complex manifold implies that all its Chern classes, excepting the zeroth, are zero.

THEOREM 4.4. *A compact complete locally affine complex surface has Euler characteristic zero.*

Proof. It suffices to prove this for a finite cover of the surface and this cover may be represented as C^2/Γ_0 with

$$\Gamma_0 \subset \left\{ \begin{matrix} \text{all complex} \\ \text{matrices of} \\ \text{the form} \end{matrix} \quad \begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

With coordinates (x, y) on C^2, the vector field $\partial/\partial x$ has non-vanishing real part on C^2 which is invariant under the action of Γ_0. This gives a non-vanishing vector field on C^2/Γ_0, and hence the Euler characteristic of this surface is zero.

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