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**BOUNDED ENTIRE SOLUTIONS OF ELLIPTIC EQUATIONS**

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Let

$$(1.1) \quad Lu = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}.$$

Consider the equation

$$(1.2) \quad Lu(x) = f(x).$$

It is shown, under some general conditions on the coefficients of  $L$ , that if  $f(x)$  is locally Hölder continuous and

$$(1.3) \quad f(x) = O(|x|^{-2-\mu}) \quad \text{as } |x| \longrightarrow \infty \quad (\mu > 0)$$

then there exists a bounded solution of (1.2) in  $R^n$  when  $n \geq 3$ . If  $n = 2$  then bounded entire solutions may not exist, but there exists a nonnegative solution of (1.2) in  $R^2$  which is bounded above by  $O(\log |x|)$ . An application of these results to the Cauchy problem is given in the final section of the paper.

If in (1.3)  $\mu = 0$  then already the equation  $Lu = f$  ( $n \geq 3$ ) may not have an entire bounded solution; an example is given by Meyers and Serrin [4].

2. Existence of a bounded solution. We shall need the following conditions:

$$(2.1) \quad \sum_{i,j=l}^n a_{ij}(x) \xi_j \xi_i > 0 \quad \text{if } x \in R^n, \xi \in R^n, \xi \neq 0,$$

$$(2.2) \quad a_{ij}(x), b_i(x) \text{ are bounded, locally Hölder continuous in } R^n \quad (1 \leq i, j \leq n),$$

$$(2.3) \quad \text{For some } \delta > 0, R > 0, 0 < \rho < 1,$$

$$(2 + \delta) |x|^{-2} \sum_{i,i=l}^n a_{ij}(x) x_i x_j \leq \rho \sum_{i=1}^n a_{ii}(x) + \sum_{i=1}^n x_i b_i(x) \quad \text{if } |x| > R,$$

$$(2.4) \quad \sum_{i=1}^n a_{ii}(x) \geq \gamma > 0 \quad \text{for all } x \in R^n \quad (\gamma \text{ constant}).$$

Notice that (2.1) and (2.4) both follow from the condition of uniform ellipticity

$$(2.5) \quad \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \gamma_0 |\xi|^2 \quad \text{for all } x \in R^n, \xi \in R^n$$

( $\gamma_0$  positive constant).

Denote the eigenvalues of  $(a_{ij}(x))$  by  $\lambda_1(x) \leq \dots \leq \lambda_n(x)$ . Then the condition in (2.3) means that

$$(2.6) \quad (2 + \delta)\tilde{\lambda}(x) \leq \rho [\lambda_1(x) + \dots + \lambda_n(x)] + \sum_{i=1}^n x_i b_i(x)$$

for some  $\lambda_1(x) \leq \tilde{\lambda}(x) \leq \lambda_n(x)$ .

We finally impose on  $f(x)$  the condition:

$$(2.7) \quad f(x) = O(|x|^{-2-\nu}) \quad \text{as } |x| \rightarrow \infty \quad (\nu > 0).$$

**THEOREM 1.** *Suppose that either the conditions (2.1)–(2.4) or the conditions (2.5), (2.2) and (2.3) with  $\rho = 1$  hold. Then for any locally Hölder continuous function  $f(x)$  satisfying (2.7) there exists a unique bounded solution  $u(x)$  of (1.2) in  $R^n$  satisfying  $u(x) \rightarrow 0$  if  $|x| \rightarrow \infty$ .*

*Proof.* We shall construct a function  $v(r)$  for  $r > R$  such that

$$(2.8) \quad Lv(r) \leq -|f(x)| \quad \text{if } r = |x| > R,$$

$$(2.9) \quad v'(r) < 0 \quad \text{if } r > R.$$

It is easily seen that

$$Lv(r) = \frac{1}{r^2} \left[ \sum_{i,j} a_{ij}(x) x_i x_j \right] v''(r) + \frac{v'(r)}{r} \left[ \sum_i a_{ii}(x) - \frac{1}{r^2} \sum_{i,j} a_{ij}(x) x_i x_j + \sum_i x_i b_i(x) \right].$$

If (2.9) holds then, by (2.3),

$$(2.10) \quad Lv(r) \leq \left[ v''(r) + (1 + \delta) \frac{v'(r)}{r} \right] \frac{1}{r^2} \sum_{i,j} a_{ij}(x) x_i x_j + \frac{(1 - \rho)v'(r)}{r} \sum_i a_{ii}(x).$$

Take  $\mu > 0$  such that  $\mu < 1, \mu < \nu, \mu \leq \delta$  and take  $0 < R_0 < R$ . Consider the function

$$v(r) = B \int_r^\infty \frac{ds}{s^{1+\mu}} \int_{R_0}^\infty \frac{\tau^{1+\mu}}{\tau^{2+\nu}} d\tau$$

for any constant  $B > 0$ . Then  $v(r)$  satisfies (2.9), and

$$(2.11) \quad \begin{aligned} v''(r) + (1 + \mu) \frac{v'(r)}{r} &= -\frac{B}{r^{2+\nu}}, \\ v'(r) &< -\frac{BC'}{r^{1+\mu}}, \\ 0 < v(r) &< \frac{BC}{r^\mu} \end{aligned}$$

if  $r > R$ , where  $C', C$  are positive constants independent of  $B$ . Recalling (2.10) and assuming that (2.3), (2.4) hold, we get

$$Lv(r) \leq -\frac{BC'(1-\rho)}{r^{2+\mu}} \sum_i a_{ii}(x) \leq -|f(x)| \quad \text{if } |x| = r > R$$

provided  $B$  is sufficiently large. If instead of (2.3), (2.4) one assumes that (2.5) and (2.3) with  $\rho = 1$  hold, then again one derives from (2.10) the inequality  $Lv(r) \leq -|f(x)|$ .

Consider the exterior Dirichlet problem

$$(2.12) \quad \begin{aligned} L\phi_0(x) &= f(x) & \text{in } |x| > R, \\ \phi_0 &= 0 & \text{on } |x| = R, \\ \phi_0(x) &\rightarrow 0 & \text{if } |x| \rightarrow \infty. \end{aligned}$$

In Meyers-Serrin [4] it is proved that there is a unique solution  $\phi_0$  of (2.12) if (2.7), (2.2) and (2.3) with  $\rho = 1$  hold, and if  $\sum a_{ij}(x)x_i x_j \geq |x|^2$ . The last condition is equivalent to the condition (2.5). The crucial step in the proof in [4] is the construction of  $v(r)$  for which  $Lv(r) \leq -|f(x)|$  and (2.11) holds. Since we have constructed such a  $v(r)$  also when the assumptions (2.5), (2.3) with  $\rho = 1$  are replaced by (2.3), (2.4), the proof of [4] shows that the problem (2.12) has a unique solution  $\phi_0$ .

Consider next the Dirichlet problem

$$(2.13) \quad \begin{cases} L\phi = 0 & \text{in } |x| > R, \\ \phi = h & \text{on } |x| = R, \\ \phi(x) \rightarrow 0 & \text{if } |x| \rightarrow \infty \end{cases}$$

where  $h$  is a continuous function. This again has a unique solution  $\phi$ .

Take  $R' > R$  and let  $w$  be the solution of

$$(2.14) \quad \begin{cases} Lw = 0 & \text{in } |x| < R', \\ w = \phi & \text{on } |x| = R'. \end{cases}$$

Finally let  $w_0$  be the solution of

$$(2.15) \quad \begin{cases} Lw_0(x) = f(x) & \text{in } |x| < R', \\ w_0 = \phi_0 & \text{on } |x| = R'. \end{cases}$$

Then  $\phi + \phi_0$  and  $w + w_0$  are solutions of  $Lu = f$  in  $|x| > R$  and  $|x| < R'$  respectively, and they coincide on  $|x| = R'$ . If there exists a function  $h$  such that

$$(2.16) \quad \phi + \phi_0 = w + w_0 \quad \text{on } |x| = R,$$

then  $\phi + \phi_0 = w + w_0$  in  $R < |x| < R'$ , so that

$$u(x) = \begin{cases} \phi + \phi_0 & \text{in } |x| > R, \\ w + w_0 & \text{in } |x| < R' \end{cases}$$

defines a bounded solution of (1.2) in  $R^n$  which tends to zero as  $|x| \rightarrow \infty$ .

Denote by  $X$  the Banach space of continuous functions on  $|x| = R$  with the sup norm, and denote by  $\| \cdot \|$  the norm of operators in  $X$ . Denote by  $Wh$  the restriction of  $w$  to  $|x| = R$ . Then (2.16) reduces to

$$(2.17) \quad h - Wh = w_0 - \phi_0.$$

If we show that

$$(2.18) \quad \|W\| < 1$$

then the existence of a unique solution  $h$  of (2.17) follows, and the existence part of the theorem is proved.

The function

$$\tilde{\phi}(x) = \|h\| \frac{v(r)}{v(R)} \quad (\|h\| = \sup_{|x|=R} |h(x)|)$$

satisfies:

$$L\tilde{\phi} \leq 0 \text{ if } |x| > R, \tilde{\phi} \geq \phi \text{ if } |x| = R, \tilde{\phi}(x) - \phi(x) \rightarrow 0 \text{ if } |x| \rightarrow \infty.$$

By the maximum principle it follows that  $\tilde{\phi} \geq \phi$  if  $|x| > R$ . Similarly  $\tilde{\phi} \geq -\phi$ . Hence

$$|\phi(x)| \leq \|h\| \frac{v(R')}{v(R)} = \sigma \|h\| \quad \text{if } |x| = R',$$

where  $\sigma < 1$  by (2.9). Since, by the maximum principle,

$$\sup_{|x|=R} |w(x)| \leq \sup_{|x|=R'} |\phi(x)|,$$

we conclude that

$$\sup_{|x|=R} |w(x)| \leq \sigma \|h\|.$$

This gives (2.18).

Suppose now that  $\tilde{u}(x)$  is another solution of (1.2) in  $R^n$  which tends to zero as  $|x| \rightarrow \infty$ . We shall prove that  $\tilde{u} \equiv u$ . Let  $z = u - \tilde{u}$  and denote by  $h$  the restriction of  $z$  to  $|x| = R$ . Then  $Wh = h$ . Since  $\|W\| < 1$ ,  $h = 0$ . It follows that  $z \equiv 0$  in  $R^n$ .

From the proof of Theorem 1 we obtain the estimate

$$(2.19) \quad u(x) = O(|x|^{-\mu})$$

on the solution. Hence:

**COROLLARY 1.** *Let the assumptions of Theorem 1 hold. Then for any number  $N$  there is a unique solution of (1.2) in  $R^n$  satisfying:  $u(x) \rightarrow N$  if  $|x| \rightarrow \infty$ ; further,*

$$u(x) = N + O(|x|^{-\mu}) \quad \text{as } |x| \rightarrow \infty$$

for any  $\mu \leq \delta, \mu < \nu, \mu < 1$ .

**COROLLARY 2.** *Suppose (2.1), (2.2) hold and suppose*

$$(2.20) \quad |x| \sum_{i=1}^n |b_i(x)| \rightarrow 0 \quad \text{if } |x| \rightarrow \infty,$$

$$(2.21) \quad \bar{a}_{ij} = \lim_{|x| \rightarrow \infty} a_{ij}(x) \text{ exists for } 1 \leq i, j \leq n.$$

If the matrix  $(\bar{a}_{ij})$  has at least three positive eigenvalues then the assertion of Theorem 1 and Corollary 1 are valid.

*Proof.* A nonsingular affine transformation  $x \rightarrow Tx$  does not change the assumptions and assertions of the corollary. Such a transformation changes  $(a_{ij})$  into  $T(a_{ij})T^*$ . Thus, without loss of generality one may assume that

$$\bar{a}_{ij} = 0 \quad \text{if } i \neq j, \bar{a}_{ii} = 1 \quad \text{if } i = 1, 2, 3, a_{ii} = 0 \quad \text{or } 1 \quad \text{if } i \geq 3.$$

But then the conditions (2.4), (2.3) (with  $\rho = 1$ ) are satisfied, so that Theorem 1 and Corollary 1 can be applied.

We recall a result of Gilbarg-Serrin [2; Theorem 3] asserting that if (2.2), (2.5), (2.21) hold, and if  $n \geq 3$  and

$$\sum_i |b_i(x)| = O\left(\frac{1}{|x|}\right) \quad \text{as } |x| \rightarrow \infty,$$

then any bounded solution of  $Lu = 0$  in  $R^n$  has a limit at infinity. By the maximum principle, this yields a Liouville theorem: Any entire bounded solution of  $Lu = 0$  is a constant. Hence:

**COROLLARY 3.** *Suppose (2.1), (2.2), (2.20), (2.21) hold, and let the matrix  $(\bar{a}_{ij})$  be nonsingular. Then, any bounded solution of (1.2) in  $R^n, n \geq 3$ , has the form  $N + u(x)$  where  $u(x)$  is the solution asserted in Theorem 1. (Recall that  $u(x)$  satisfies (2.19).)*

3. The case  $n = 2$ . If (2.20), (2.21) hold and  $n = 2$ , then the condition (2.3) with  $\rho = 1$  is not satisfied. We shall now study this situation. The following conditions will be imposed:

$$(3.1) \quad n = 2 \text{ and for all } x \in R^2, \xi \in R^2,$$

$$\sum_{i,j=1}^2 a_{ij}(x) \xi_i \xi_j \geq \nu_0 |\xi|^2 \quad (\nu_0 \text{ positive constant}),$$

$$(3.2) \quad \sum_{i,j} |a_{ij}(x) - \bar{a}_{ij}| \leq \frac{C}{(1 + |x|)^\kappa} \quad (C > 0, \kappa > 0),$$

$$(3.3) \quad \sum_j |b_j(x)| \leq \frac{C}{(1 + |x|)^{1+\kappa}} \quad (C > 0, \kappa > 0),$$

**THEOREM 2.** *Let the conditions (2.2), (3.1)–(3.3) hold. Then for any locally Hölder continuous function  $f(x)$  satisfying (2.7) there exists a solution  $u(x)$  of (1.2) in  $R^2$  satisfying*

$$(3.4) \quad 0 \leq u(x) \leq K \log(2 + |x|) \quad (K \text{ constant}).$$

*Proof.* Without loss of generality we may assume that  $\bar{a}_{ij} = \delta_{ij}$ ,  $1 \leq i, j \leq 2$ . We shall construct functions  $v_1(r), v_2(r)$  for  $r > R_0$  ( $R_0$  and fixed positive number) satisfying:

$$(3.5) \quad \begin{cases} Lv_1(r) \leq 0 & \text{if } r \geq R_0, \\ v_1(R_0) = 0, v_1'(r) > 0 & \text{if } r > R_0, \end{cases}$$

$$(3.6) \quad \begin{cases} Lv_2(r) \geq 0 & \text{if } r \geq R_0, \\ v_2(R_0) = 0, v_2'(r) > 0 & \text{if } r > R_0. \end{cases}$$

The inequality  $Lv_1 \leq 0$  is satisfied if

$$(3.7) \quad v_1'' + \frac{1}{r} \left(1 + \frac{c}{r^\kappa}\right) v_1' = 0, \quad v_1' > 0$$

where  $c$  is a sufficiently large positive constant. A solution of (3.7) which vanishes at  $r = R_0$  is given by

$$(3.8) \quad \begin{aligned} v_1(r) &= \int_{R_0}^r \exp \left\{ - \int_{R_0}^t \frac{c}{s^{1+\kappa}} ds \right\} \frac{dt}{t} \\ &= \int_{R_0}^r \exp \left\{ \frac{c}{\kappa} (t^{-\kappa} - R_0^{-\kappa}) \right\} \frac{dt}{t}. \end{aligned}$$

This function then satisfies (3.5). Similarly,

$$(3.9) \quad v_2(r) = \int_{R_0}^r \exp \left\{ - \frac{c}{\kappa} (t^{-\kappa} - R_0^{-\kappa}) \right\} \frac{dt}{t}$$

is a solution of (3.6). From (3.8), (3.9) it is clear that

$$(3.10) \quad c_1 \log(1 + r) \leq v_1(r) \leq v_2(r) \leq c_2 \log(1 + r) \quad (c_1 > 0, c_2 > 0)$$

for all  $r \geq R_0 + 1$ .

For each  $R > R_0 + 1$ , let  $u_R$  be the solution of

$$\begin{aligned} Lu_R &= 0 && \text{in } R_0 < |x| < R, \\ u_R &= 0 && \text{on } |x| = R_0, \\ u_R &= v_2(R) && \text{on } |x| = R. \end{aligned}$$

From the maximum principle it follows that  $u_R \geq v_2$  if  $R_0 < |x| < R$ . From (3.10) we have:

$$u_R \leq \frac{c_2}{c_1} v_1(R) \quad \text{on } |x| = R .$$

Hence, by the maximum principle,

$$u_R \leq \frac{c_2}{c_1} v_1 \quad \text{if } R_0 < |x| < R .$$

Using (3.10) once more we conclude that

$$c_1 \log(1+r) \leq u_R(x) \leq \frac{c_2}{c_1} c_2 \log(1+r) \quad \text{if } R_0 + 1 \leq |x| < R .$$

We can now take a subsequence  $\{u_{R_m}\}$ , with  $R_m \rightarrow \infty$  if  $m \rightarrow \infty$ , that is uniformly convergent in compact subsets of  $\{x; |x| \geq R_0\}$  to a solution  $w_2(x)$  of

$$(3.11) \quad \begin{cases} Lw_2 = 0 & \text{if } R_0 < |x| < \infty , \\ w_2 = 0 & \text{on } |x| = R_0 , \end{cases}$$

and

$$(3.12) \quad c_1 \log(1+r) \leq w_2(x) \leq \frac{c_2}{c_1} c_2 \log(1+r) \quad (r = |x| > R_0 + 1) .$$

Let  $R' = R_0$ ,  $R'' > R'$  and denote by  $S'$  and  $S''$  the circles given by  $|x| = R'$  and  $|x| = R''$  respectively. Let  $w_1$  be the unique solution (see [4]) of

$$(3.13) \quad \begin{cases} Lw_1 = f & \text{in } |x| > R' , \\ w_1 = 0 & \text{on } S' , \\ w_1 \text{ bounded in } |x| > R' . \end{cases}$$

Let  $z_1$  and  $z_2$  be the solutions of

$$(3.14) \quad \begin{cases} Lz_1 = f & \text{in } |x| < R'' , \\ z_1 = w_1 & \text{on } S'' , \end{cases}$$

$$(3.15) \quad \begin{cases} Lz_2 = 0 & \text{in } |x| < R'' , \\ z_2 = w_2 & \text{on } S'' . \end{cases}$$

Denote by  $z_1^*$ ,  $z_2^*$  the restriction to  $S'$  of  $z_1$  and  $z_2$ , respectively.

We shall introduce now an operator  $W$  similar to the operator  $W$  in the proof of Theorem 1. We denote by  $X$  the Banach space of the continuous functions  $h$  on  $S'$  provided with the uniform norm. For any  $h \in X$ , let  $w$  be the unique solution (see [4]) of



$$(3.16) \quad \begin{cases} Lw = 0 & \text{in } |x| > R' , \\ w = h & \text{on } S' , \\ w \text{ bounded in } |x| \leq R' , \end{cases}$$

and let  $z$  be the solution of

$$(3.17) \quad \begin{cases} Lz = 0 & \text{in } |x| < R'' , \\ z = w & \text{on } S'' . \end{cases}$$

Then  $Wh$  is defined as the restriction of  $z$  to  $S'$ .

By the maximum principle, for any  $\varepsilon > 0$ ,

$$\|h\| + \varepsilon v_1(r) \geq \pm w(x) \quad \text{in } |x| > R' .$$

This implies that

$$\sup_{|x|=R''} |w(x)| \leq \|h\| .$$

Again by the maximum principle,

$$\sup_{|x|=R'} |z(x)| \leq \sup_{|x|=R''} |z(x)| = \sup_{|x|=R''} |w(x)| .$$

Hence,  $\|Wh\| \leq \|h\|$ . Since, for  $h(x) \equiv 1$ ,  $Wh = h$ , it follows that  $\|W\| = 1$ .

Employing the function  $v_1(r)$  and using the maximum principle it can be shown (see [4, p. 523]) that Liouville's theorem is valid (under the assumptions of Theorem 2), that is, every bounded solution  $u$  of  $Lu = 0$  in  $R^2$  is a constant. Now,  $h$  satisfies  $Wh = h$  if and only if the corresponding  $w$  and  $z$  coincide on  $S'$ ,  $S''$  and, consequently, in the region  $R' < |x| < R''$ ; thus,  $Wh = h$  if and only if the pair  $w, z$  defines a bounded entire solution  $u$  of  $Lu = 0$ . By Liouville's theorem it follows that  $u \equiv \text{const.}$  and, in particular,  $h = \text{const.}$  Thus, 1 is an eigenvalue of  $W$  and the eigenspace is one dimensional.

From the interior Schauder estimates (see, for instance, [1]) one deduces that  $W$  maps bounded subsets of  $X$  into compact subsets. Hence the Fredholm-Riesz-Schauder theorem can be applied to solve equations of the form

$$(3.18) \quad \zeta + Wh = h .$$

Denoting by  $\hat{h}$  an eigenfunctional of the adjoint  $W^*$  of  $W$ , we can assert that the equation (3.18) has a solution if and only if

$$\hat{h}(\zeta) = 0 .$$

We wish to solve the equation

$$(3.19) \quad z_1^* + \lambda z_2^* + Wh = h$$

for some real number  $\lambda$ . We first show that

$$(3.20) \quad \hat{h}(z_2^*) \neq 0.$$

Suppose  $\hat{h}(z_2^*) = 0$ . Then the equation

$$(3.21) \quad z_2^* + Wh = h$$

has a solution  $h$ . Denote by  $w, z$  the corresponding solutions of (3.16), (3.17). Then the functions  $w + w_2$  and  $z + z_2$  coincide on  $S''$  and (by (3.21)) on  $S'$ . Since they both are solutions of  $Lu = 0$  in  $R' < |x| < R''$ , it follows that they coincide in this region. Consequently, the function

$$u_0(x) = \begin{cases} w(x) + w_2(x) & \text{if } |x| > R', \\ z(x) + z_2(x) & \text{if } |x| < R'' \end{cases}$$

is an entire solution of  $Lu_0 = 0$ . Since, by (3.12),  $u_0(x) \rightarrow \infty$  if  $|x| \rightarrow \infty$ ,  $u_0$  must attain a minimum at some point in  $R^2$ . But then, by the maximum principle,  $u_0(x) \equiv \text{const.}$ ; this is impossible since  $u_0(x) \rightarrow \infty$  if  $|x| \rightarrow \infty$ .

Having proved (3.20), we choose in (3.19)

$$\lambda = -\hat{h}(z_1^*)/\hat{h}(z_2^*).$$

Then

$$(3.22) \quad \hat{h}(z_1^* + \lambda z_2^*) = 0;$$

consequently (3.19) has a solution which we shall denote by  $h$ . Denote by  $w, z$  the solutions of (3.16), (3.17) corresponding to this  $h$ . The functions

$$w + w_1 + \lambda w_2, \quad z + z_1 + \lambda z_2$$

are solutions of  $Lu = f$  in  $|x| > R'$  and  $|x| < R''$  respectively. They coincide on  $S''$  and (by 3.19) on  $S'$ ; consequently, they coincide in  $R' < |x| < R''$ . The function

$$\hat{u}(x) = \begin{cases} w(x) + w_1(x) + \lambda w_2(x) & \text{if } |x| > R', \\ z(x) + z_1(x) + \lambda z_2(x) & \text{if } |x| < R'' \end{cases}$$

is then an entire solution of  $L\hat{u} = f$ . In view of (3.12), the function  $u(x) = \hat{u}(x) + K_0$  is a solution of (1.2) in  $R^2$ , satisfying (3.4), provided  $K_0$  is a sufficiently large positive constant.

REMARK. If  $L = \Delta$  then for any locally Hölder continuous function  $f(x)$  with compact support  $K$  for which

$$\Phi \equiv \int_{\mathbb{R}^n} f(x) dx \neq 0$$

there does not exist a bounded entire solution of  $\Delta v = f$  in  $\mathbb{R}^2$ . Indeed, suppose  $\Phi > 0$  and let

$$w(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log |x - y| dy .$$

Then  $\Delta w = f$  in  $\mathbb{R}^2$  and

$$w(x) = \frac{\Phi}{2\pi} \log |x| + O(1) \quad \text{if } x \rightarrow \infty .$$

If there is a bounded entire solution  $v(x)$  of  $\Delta v = f$  in  $\mathbb{R}^2$  then the function  $u = w - v$  is harmonic in  $\mathbb{R}^2$  and  $u(x) \rightarrow \infty$  if  $x \rightarrow \infty$ . Consequently  $u$  must attain its minimum (in  $\mathbb{R}^2$ ) at a finite point. By the maximum principle,  $u(x) \equiv \text{const.}$ , which is impossible.

4. An application. Consider the Cauchy problem

$$(4.1) \quad \frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i} \quad \text{if } 0 < t < \infty, x \in \mathbb{R}^n$$

$$(4.2) \quad u(0, x) = f(x) \text{ if } x \in \mathbb{R}^n .$$

We shall assume:  $a_i(x)$  are locally Hölder continuous and

$$(4.3) \quad |a_i(x)| \leq \frac{A}{(1 + |x|)^{2+\nu}} \quad (\nu > 0, A > 0) ,$$

$f(x)$  is continuous and

$$(4.4) \quad |f(x) - f(y)| \leq N|x - y| \quad (N > 0) .$$

It is then well known [1] that the problem (4.1), (4.2) has a unique solution in the class of functions  $v(t, x)$  satisfying, for each  $T > 0$ ,

$$|v(t, x)| \leq Ce^{c|x|^2} \quad (0 \leq t \leq T, x \in \mathbb{R}^n)$$

for some positive constants  $C, c$  depending on  $v, T$ .

**THEOREM 3.** *Let (4.3), (4.4) hold, and let  $n \geq 3$ . Then the solution  $u(t, x)$  of (4.1), (4.2) satisfies*

$$(4.5) \quad \left| u(t, x) - \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{-\frac{|x - \xi|^2}{4t}\right\} f(\xi) d\xi \right| \leq M$$

for all  $t \geq 0, x \in \mathbb{R}^n$  where  $M$  is a constant.

*Proof.* We can write  $u(t, x)$  in the form (see [3])

$$(4.6) \quad u(t, x) = Ef(\xi_x(t))$$

where  $E$  is the expectation and  $\xi_x(t)$  is a solution of the stochastic integral equation

$$(4.7) \quad \xi_x(t) = x + \int_0^t a(\xi_x(s))ds + 2 \int_0^t dw(s);$$

here  $w(t)$  is  $n$ -dimensional Brownian motion. Similarly (for  $a_i \equiv 0$ )

$$(4.8) \quad \frac{1}{(4\pi t)^{n/2}} \int_{R^n} \exp\left\{-\frac{|x - \xi|^2}{4t}\right\} f(\xi) d\xi = Ef(x + 2w(t)).$$

By Theorem 1 there exists a bounded solution  $v_j(x)$  of

$$\Delta v_j + \sum_{i=1}^n a_i(x) \frac{\partial v_j}{\partial x_i} = |a_j(x)| \text{ in } R^n.$$

By Ito's formula [3],

$$E \int_0^t |a_j(\xi_x(s))| ds = Ev_j(\xi_x(t)) - v_j(x).$$

Hence,

$$E \left| \int_0^t a_j(\xi_x(s)) ds \right| \leq C$$

where  $C$  is a constant independent of  $(t, x)$ . Recalling (4.7), we conclude that

$$(4.9) \quad E |\xi_x(t) - x - 2w(t)| \leq C.$$

Combining (4.6), (4.8) with (4.4), (4.9), the assertion of the theorem follows.

For  $n = 2$  one can employ Theorem 2 and establish the inequality

$$\left| u(t, x) - \frac{1}{(4\pi t)^{1/2}} \int_{R^2} \exp\left\{-\frac{|x - \xi|^2}{4t}\right\} f(\xi) d\xi \right| \leq M \log(2 + t + |x|).$$

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