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**MULTIPLICITY AND THE AREA OF AN $(n - 1)$ CONTINUOUS
MAPPING**

RONALD FRANCIS GARIEPY

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For a class of mappings considered by Goffman and Ziemer [Annals of Math. 92 (1970)] it is shown that the area is given by the integral of a multiplicity function and a convergence theorem is proved.

1. Introduction. A theory of surface area for mappings beyond the class of continuous mappings was initiated in [2]. This theory includes certain essentially discontinuous mappings for which it seems natural that the area be given by the classical integral formula.

Let $Q = R^n \cap \{x: 0 < x_i < 1 \text{ for } 1 \leq i \leq n\}$. For each $i \in \{1, \dots, n\}$ and $r \in I = \{t: 0 < t < 1\}$ let $P_i(r) = Q \cap \{x: x_i = r\}$. A mapping $f: Q \rightarrow R^m$, $n \leq m$, is said to be $n - 1$ continuous if, for each i , $f|P_i(r)$ is continuous for almost every (in the sense of 1-dimensional Lebesgue measure) $r \in I$. A sequence $\{f_j\}$ of mappings from Q into R^m is said to converge $n - 1$ to f if, for each i , $f_j|P_i(r)$ converges uniformly to $f|P_i(r)$ for almost every $r \in I$.

The area of an $n - 1$ continuous mapping $f: Q \rightarrow R^m$ is defined as

$$A(f) = \inf \lim_{j \rightarrow \infty} a(f_j)$$

where the infimum is taken over all sequences $\{f_j\}$ of quasilinear mappings converging $n - 1$ to f and $a(f_j)$ denotes the elementary area of f_j . In [2] it was shown that $A(f)$ coincides with Lebesgue area if f is continuous.

For real $p \geq 1$, let $W_p^1(Q)$ denote those functions in $L^p(Q)$ whose distribution first partial derivatives are functions in $L^p(Q)$. Suppose $f: Q \rightarrow R^m$ with $f = (f^1, \dots, f^m)$ and $f^i \in W_{p_i}^1(Q)$, $p_i > n - 1$ for $1 \leq i \leq m$ and $\sum_{j=1}^n 1/p_{i_j} \leq 1$ whenever $1 \leq i_1 < \dots < i_n \leq m$. It was shown in [3] that f is $n - 1$ continuous and

$$A(f) = \int_Q |Jf(x)| dx.$$

In this paper we prove the following

THEOREM. *If $f: Q \rightarrow R^n$ with $f^i \in W_{p_i}^1(Q)$, $p_i > n - 1$ and $\sum_{i=1}^n 1/p_i \leq 1$, then there is a nonnegative integer valued lower semicontinuous function $N(f, y)$ on R^n such that*

$$(1) \quad A(f) = \int_{R^n} N(f, y) dy$$

and, if $\{f_j\}$ is any sequence of quasi-linear mappings converging $n - 1$ to f with $A(f) = \lim_{j \rightarrow \infty} a(f_j)$, then

$$(2) \quad \lim_{j \rightarrow \infty} \int_{R^n} |N(f, y) - N(f_j, y)| dy = 0$$

and

$$(3) \quad \int_Q \phi(f(x)) Jf(x) dx = \lim_{j \rightarrow \infty} \int_Q \phi(f_j(x)) Jf_j(x) dx$$

whenever ϕ is a continuous real valued function on R^n with compact support.

2. *Proof of (1) and (2).* Suppose f satisfies the hypothesis of the theorem. By a full set of hyperplanes we will mean a subset P of $\{P_i(r): 1 \leq i \leq n \text{ and } 0 < r < 1\}$ such that, for each i , $P_i(r) \in P$ for almost every $r \in I$.

If $\pi \subset Q$ is an n -cube such that $f|_{\partial\pi}$ is continuous and $y \in R^n - f(\partial\pi)$, let $0(f, \pi, y)$ denote the topological index of y with respect to the mapping $f|_{\partial\pi}$ [4, p. 123]. If $y \in f(\partial\pi)$ let $0(f, \pi, y) = 0$.

Let P be a full set of hyperplanes such that $f|_{P_i(r)}$ is continuous whenever $P_i(r) \in P$. In harmony with [1, page 173] let, for $y \in R^n$,

$$N(f, y) = \sup \sum |0(f, \pi, y)|$$

where the supremum is taken over all finite collections G of non overlapping n -cubes $\pi \subset Q$ whose $n - 1$ faces all lie in elements of P . From the properties of the topological index, it is easily seen that $N(f, y)$ is a lower semicontinuous function of y .

If $g: Q \rightarrow R^n$ is quasi-linear, then $N(g, y)$ is independent of the choice of P and

$$a(g) = \int_{R^n} N(g, y) dy.$$

By [3, 3.5] we know that f possesses a regular approximate differential almost everywhere in Q . Using the arguments of [1, page 424] one verifies that

$$\int_Q |Jf(x)| dx \leq \int_{R^n} N(f, y) dy$$

whenever $N(f, y)$ is computed relative to a full set P of hyperplanes such that the restriction of f to each element of P is continuous.

Suppose $\{f_j\}$ is a sequence of quasi-linear mappings converging $n - 1$ to f with $A(f) = \lim_{j \rightarrow \infty} a(f_j)$. Let P be a full set of hyperplanes on each of which the sequence converges uniformly to f and define $N(f, y)$ relative to P . For each $y \in R^n$ we have

$$N(f, y) \leq \lim_{j \rightarrow \infty} N(f_j, y)$$

and hence

$$\int_{R^n} N(f, y) dy \leq \lim_{j \rightarrow \infty} \int_{R^n} N(f_j, y) dy = A(f).$$

If $\bar{P} \subset P$ is a full set of hyperplanes and $\bar{N}(f, y)$ is defined relative to \bar{P} , then, clearly $\bar{N}(f, y) \leq N(f, y)$ for all $y \in R^n$. Since $A(f) = \int |Jf(x)| dx$, it follows that $N(f, y)$ is determined as an element of $L^1(R^n)$ independent of the choice of the sequence $\{f_j\}$. Thus (1) is proved and (2) follows because $N(f, y)$ is integer valued and

$$N(f, y) \leq \lim_{j \rightarrow \infty} N(f_j, y)$$

for almost every $y \in R^n$ whenever $\{f_j\}$ is a sequence of quasilinear mappings converging $n - 1$ to f with $A(f) = \lim_{j \rightarrow \infty} a(f_j)$.

Proof of (3). Suppose f and $\{f_j\}$ satisfy the conditions of the theorem and let P be a full set of hyperplanes on each of which $\{f_j\}$ converges uniformly to f .

For $y \in R^n$ let

$$N^\pm(f, y) = \sup_{\pi \in G} \frac{1}{2} [|0(f, \pi, y)| \pm 0(f, \pi, y)]$$

where the supremum is taken over all finite collections G of non overlapping n -cubes $\pi \subset Q$ whose $n - 1$ faces all lie in elements of P . Clearly

$$N^\pm(f, y) \leq N(f, y) \leq N^+(f, y) + N^-(f, y).$$

It is readily seen that

$$N^\pm(f, y) \leq \lim_{j \rightarrow \infty} N^\pm(f_j, y)$$

and that the $N^\pm(f, y)$ are lower semicontinuous functions of y .

In case $g: Q \rightarrow R^n$ is quasi-linear, $N^\pm(g, y)$ are independent of the choice of P and

$$N(g, y) = N^+(g, y) + N^-(g, y)$$

for almost every $y \in R^n$.

For each positive integer j , let

$$E_j^\pm = \{y: N^\pm(f_k, y) < N^\pm(f, y) \text{ for some } k \geq j\}.$$

and let $E_j = E_j^+ \cup E_j^-$.

Since the functions N^\pm are integer valued we have

$$\lim_{j \rightarrow \infty} \mathcal{L}_n(E_j) = 0$$

where \mathcal{L}_n denotes n dimensional Lebesgue measure. Now

$$\begin{aligned} & \int_{R^n} |N^+(f_j, y) - N^+(f, y)| dy \\ & \leq \int_{R^n} N^+(f_j, y) dy - \int_{R^n - E_j^+} N^+(f, y) dy + \int_{E_j^+} (f, y) dy \\ & \leq \int_{R^n} (N^+(f_j, y) + N^-(f_j, y)) dy \\ & \quad - \int_{R^n - E_j} (N^+(f, y) + N^-(f, y)) dy + \int_{E_j} N^+(f, y) dy \\ & \leq \int_{R^n} N(f_j, y) dy - \int_{R^n - E_j} N(f, y) dy + \int_{E_j} N(f, y) dy \\ & = \alpha(f_j) - A(f) + 2 \int_{E_j} N(f, y) dy . \end{aligned}$$

Thus

$$\lim_{j \rightarrow \infty} \int_{R^n} |N^\pm(f_j, y) - N^\pm(f, y)| dy = 0 .$$

Now

$$\begin{aligned} 0 & \leq \int_{R^n} [N^+(f, y) + N^-(f, y) - N(f, y)] dy \\ & \leq \int_{R^n} |N^+(f, y) - N^+(f_j, y)| dy + \int_{R^n} |N^-(f, y) - N^-(f_j, y)| dy \\ & \quad + \int_{R^n} |N(f, y) - N(f_j, y)| dy . \end{aligned}$$

Thus, $N(f, y) = N^+(f, y) + N^-(f, y)$ for almost every $y \in R^n$.

Let $n(f, y) = N^+(f, y) - N^-(f, y)$. Then

$$\lim_{j \rightarrow \infty} \int_{R^n} |n(f, y) - n(f_j, y)| dy = 0 .$$

Suppose ϕ is a real valued continuous function on R^n with compact support. If $g: Q \rightarrow R^n$ is quasi-linear (or of class C^1) then

$$\int_Q \phi(g(x)) Jg(x) dx = \int_{R^n} \phi(y) n(g, y) dy .$$

Suppose $\{\bar{f}_j\}$ is a sequence of modifiers of f .

Then, from [3, 3.2], the sequence $\{\bar{f}_j\}$ converges $n - 1$ to f and

$$\lim_{j \rightarrow \infty} \int_Q |Jf(x) - J\bar{f}_j(x)| dx = 0 .$$

Hence

$$\begin{aligned} \int_Q \phi(f(x))Jf(x)dx &= \lim_{j \rightarrow \infty} \int_Q \phi(\bar{f}_j(x))J\bar{f}_j(x)dx \\ &= \lim_{j \rightarrow \infty} \int_{R^n} \phi(y)n(\bar{f}_j, y)dy = \int_{R^n} \phi(y)n(f, y)dy . \end{aligned}$$

Thus

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_Q \phi(f_j(x))Jf_j(x)dx &= \lim_{j \rightarrow \infty} \int_{R^n} \phi(y)n(f_j, y)dy \\ &= \int_{R^n} \phi(y)n(f, y)dy = \int_Q \phi(f(x))Jf(x)dx \end{aligned}$$

and (3) is proved.

REFERENCES

1. L. Cesari, *Surface Area*, Annals of Mathematics Studies No. 35, Princeton University Press, Princeton, N. J. 1956.
2. C. Goffman and F. C. Liu, *Discontinuous mappings and surface area*, Proc. London Math. Soc., **20** (1970), 237-248.
3. C. Goffman and W. Ziemer, *Higher dimensional mappings for which the area formula holds*, Annals of Math., **92** (1970), 482-488.
4. T. Rado and P. V. Reichelderfer, *Continuous Transformations in Analysis*, Springer-Verlag, Berlin, 1955.

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