

# Pacific Journal of Mathematics

**MULTIPLICITY AND THE AREA OF AN  $(n - 1)$  CONTINUOUS  
MAPPING**

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## MULTIPLICITY AND THE AREA OF AN $(n - 1)$ CONTINUOUS MAPPING

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**For a class of mappings considered by Goffman and Ziemer [Annals of Math. 92 (1970)] it is shown that the area is given by the integral of a multiplicity function and a convergence theorem is proved.**

1. **Introduction.** A theory of surface area for mappings beyond the class of continuous mappings was initiated in [2]. This theory includes certain essentially discontinuous mappings for which it seems natural that the area be given by the classical integral formula.

Let  $Q = R^n \cap \{x: 0 < x_i < 1 \text{ for } 1 \leq i \leq n\}$ . For each  $i \in \{1, \dots, n\}$  and  $r \in I = \{t: 0 < t < 1\}$  let  $P_i(r) = Q \cap \{x: x_i = r\}$ . A mapping  $f: Q \rightarrow R^m$ ,  $n \leq m$ , is said to be  $n - 1$  continuous if, for each  $i$ ,  $f|P_i(r)$  is continuous for almost every (in the sense of 1-dimensional Lebesgue measure)  $r \in I$ . A sequence  $\{f_j\}$  of mappings from  $Q$  into  $R^m$  is said to converge  $n - 1$  to  $f$  if, for each  $i$ ,  $f_j|P_i(r)$  converges uniformly to  $f|P_i(r)$  for almost every  $r \in I$ .

The area of an  $n - 1$  continuous mapping  $f: Q \rightarrow R^m$  is defined as

$$A(f) = \inf \lim_{j \rightarrow \infty} a(f_j)$$

where the infimum is taken over all sequences  $\{f_j\}$  of quasilinear mappings converging  $n - 1$  to  $f$  and  $a(f_j)$  denotes the elementary area of  $f_j$ . In [2] it was shown that  $A(f)$  coincides with Lebesgue area if  $f$  is continuous.

For real  $p \geq 1$ , let  $W_p^1(Q)$  denote those functions in  $L^p(Q)$  whose distribution first partial derivatives are functions in  $L^p(Q)$ . Suppose  $f: Q \rightarrow R^m$  with  $f = (f^1, \dots, f^m)$  and  $f^i \in W_{p_i}^1(Q)$ ,  $p_i > n - 1$  for  $1 \leq i \leq m$  and  $\sum_{j=1}^n 1/p_{i_j} \leq 1$  whenever  $1 \leq i_1 < \dots < i_n \leq m$ . It was shown in [3] that  $f$  is  $n - 1$  continuous and

$$A(f) = \int_Q |Jf(x)| dx .$$

In this paper we prove the following

**THEOREM.** *If  $f: Q \rightarrow R^n$  with  $f^i \in W_{p_i}^1(Q)$ ,  $p_i > n - 1$  and  $\sum_{i=1}^n 1/p_i \leq 1$ , then there is a nonnegative integer valued lower semicontinuous function  $N(f, y)$  on  $R^n$  such that*

$$(1) \quad A(f) = \int_{R^n} N(f, y) dy$$

and, if  $\{f_j\}$  is any sequence of quasi-linear mappings converging  $n - 1$  to  $f$  with  $A(f) = \lim_{j \rightarrow \infty} a(f_j)$ , then

$$(2) \quad \lim_{j \rightarrow \infty} \int_{R^n} |N(f, y) - N(f_j, y)| dy = 0$$

and

$$(3) \quad \int_Q \phi(f(x)) Jf(x) dx = \lim_{j \rightarrow \infty} \int_Q \phi(f_j(x)) Jf_j(x) dx$$

whenever  $\phi$  is a continuous real valued function on  $R^n$  with compact support.

2. *Proof of (1) and (2).* Suppose  $f$  satisfies the hypothesis of the theorem. By a full set of hyperplanes we will mean a subset  $P$  of  $\{P_i(r): 1 \leq i \leq n \text{ and } 0 < r < 1\}$  such that, for each  $i$ ,  $P_i(r) \in P$  for almost every  $r \in I$ .

If  $\pi \subset Q$  is an  $n$ -cube such that  $f|_{\partial\pi}$  is continuous and  $y \in R^n - f(\partial\pi)$ , let  $0(f, \pi, y)$  denote the topological index of  $y$  with respect to the mapping  $f|_{\partial\pi}$  [4, p. 123]. If  $y \in f(\partial\pi)$  let  $0(f, \pi, y) = 0$ .

Let  $P$  be a full set of hyperplanes such that  $f|_{P_i(r)}$  is continuous whenever  $P_i(r) \in P$ . In harmony with [1, page 173] let, for  $y \in R^n$ ,

$$N(f, y) = \sup \sum |0(f, \pi, y)|$$

where the supremum is taken over all finite collections  $G$  of non overlapping  $n$ -cubes  $\pi \subset Q$  whose  $n - 1$  faces all lie in elements of  $P$ . From the properties of the topological index, it is easily seen that  $N(f, y)$  is a lower semicontinuous function of  $y$ .

If  $g: Q \rightarrow R^n$  is quasi-linear, then  $N(g, y)$  is independent of the choice of  $P$  and

$$a(g) = \int_{R^n} N(g, y) dy.$$

By [3, 3.5] we know that  $f$  possesses a regular approximate differential almost everywhere in  $Q$ . Using the arguments of [1, page 424] one verifies that

$$\int_Q |Jf(x)| dx \leq \int_{R^n} N(f, y) dy$$

whenever  $N(f, y)$  is computed relative to a full set  $P$  of hyperplanes such that the restriction of  $f$  to each element of  $P$  is continuous.

Suppose  $\{f_j\}$  is a sequence of quasi-linear mappings converging  $n - 1$  to  $f$  with  $A(f) = \lim_{j \rightarrow \infty} a(f_j)$ . Let  $P$  be a full set of hyperplanes on each of which the sequence converges uniformly to  $f$  and define  $N(f, y)$  relative to  $P$ . For each  $y \in R^n$  we have

$$N(f, y) \leq \lim_{j \rightarrow \infty} N(f_j, y)$$

and hence

$$\int_{R^n} N(f, y) dy \leq \lim_{j \rightarrow \infty} \int_{R^n} N(f_j, y) dy = A(f).$$

If  $\bar{P} \subset P$  is a full set of hyperplanes and  $\bar{N}(f, y)$  is defined relative to  $\bar{P}$ , then, clearly  $\bar{N}(f, y) \leq N(f, y)$  for all  $y \in R^n$ . Since  $A(f) = \int |Jf(x)| dx$ , it follows that  $N(f, y)$  is determined as an element of  $L^1(R^n)$  independent of the choice of the sequence  $\{f_j\}$ . Thus (1) is proved and (2) follows because  $N(f, y)$  is integer valued and

$$N(f, y) \leq \lim_{j \rightarrow \infty} N(f_j, y)$$

for almost every  $y \in R^n$  whenever  $\{f_j\}$  is a sequence of quasilinear mappings converging  $n - 1$  to  $f$  with  $A(f) = \lim_{j \rightarrow \infty} a(f_j)$ .

*Proof of (3).* Suppose  $f$  and  $\{f_j\}$  satisfy the conditions of the theorem and let  $P$  be a full set of hyperplanes on each of which  $\{f_j\}$  converges uniformly to  $f$ .

For  $y \in R^n$  let

$$N^\pm(f, y) = \sup_{\pi \in G} \sum_{\pi \in G} \frac{1}{2} [|0(f, \pi, y)| \pm 0(f, \pi, y)]$$

where the supremum is taken over all finite collections  $G$  of non overlapping  $n$ -cubes  $\pi \subset Q$  whose  $n - 1$  faces all lie in elements of  $P$ . Clearly

$$N^\pm(f, y) \leq N(f, y) \leq N^+(f, y) + N^-(f, y).$$

It is readily seen that

$$N^\pm(f, y) \leq \lim_{j \rightarrow \infty} N^\pm(f_j, y)$$

and that the  $N^\pm(f, y)$  are lower semicontinuous functions of  $y$ .

In case  $g: Q \rightarrow R^n$  is quasi-linear,  $N^\pm(g, y)$  are independent of the choice of  $P$  and

$$N(g, y) = N^+(g, y) + N^-(g, y)$$

for almost every  $y \in R^n$ .

For each positive integer  $j$ , let

$$E_j^\pm = \{y: N^\pm(f_k, y) < N^\pm(f, y) \text{ for some } k \geq j\}.$$

and let  $E_j = E_j^+ \cup E_j^-$ .

Since the functions  $N^\pm$  are integer valued we have

$$\lim_{j \rightarrow \infty} \mathcal{L}_n(E_j) = 0$$

where  $\mathcal{L}_n$  denotes  $n$  dimensional Lebesgue measure. Now

$$\begin{aligned} & \int_{R^n} |N^+(f_j, y) - N^+(f, y)| dy \\ & \leq \int_{R^n} N^+(f_j, y) dy - \int_{R^n - E_j^+} N^+(f, y) dy + \int_{E_j^+} (f, y) dy \\ & \leq \int_{R^n} (N^+(f_j, y) + N^-(f_j, y)) dy \\ & \quad - \int_{R^n - E_j} (N^+(f, y) + N^-(f, y)) dy + \int_{E_j} N^+(f, y) dy \\ & \leq \int_{R^n} N(f_j, y) dy - \int_{R^n - E_j} N(f, y) dy + \int_{E_j} N(f, y) dy \\ & = a(f_j) - A(f) + 2 \int_{E_j} N(f, y) dy . \end{aligned}$$

Thus

$$\lim_{j \rightarrow \infty} \int_{R^n} |N^\pm(f_j, y) - N^\pm(f, y)| dy = 0 .$$

Now

$$\begin{aligned} 0 & \leq \int_{R^n} [N^+(f, y) + N^-(f, y) - N(f, y)] dy \\ & \leq \int_{R^n} |N^+(f, y) - N^+(f_j, y)| dy + \int_{R^n} |N^-(f, y) - N^-(f_j, y)| dy \\ & \quad + \int_{R^n} |N(f, y) - N(f_j, y)| dy . \end{aligned}$$

Thus,  $N(f, y) = N^+(f, y) + N^-(f, y)$  for almost every  $y \in R^n$ .

Let  $n(f, y) = N^+(f, y) - N^-(f, y)$ . Then

$$\lim_{j \rightarrow \infty} \int_{R^n} |n(f, y) - n(f_j, y)| dy = 0 .$$

Suppose  $\phi$  is a real valued continuous function on  $R^n$  with compact support. If  $g: Q \rightarrow R^n$  is quasi-linear (or of class  $C^1$ ) then

$$\int_Q \phi(g(x)) Jg(x) dx = \int_{R^n} \phi(y) n(g, y) dy .$$

Suppose  $\{\bar{f}_j\}$  is a sequence of modifiers of  $f$ .

Then, from [3, 3.2], the sequence  $\{\bar{f}_j\}$  converges  $n - 1$  to  $f$  and

$$\lim_{j \rightarrow \infty} \int_Q |Jf(x) - J\bar{f}_j(x)| dx = 0 .$$

Hence

$$\begin{aligned} \int_Q \phi(f(x))Jf(x)dx &= \lim_{j \rightarrow \infty} \int_Q \phi(\bar{f}_j(x))J\bar{f}_j(x)dx \\ &= \lim_{j \rightarrow \infty} \int_{R^n} \phi(y)n(\bar{f}_j, y)dy = \int_{R^n} \phi(y)n(f, y)dy . \end{aligned}$$

Thus

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_Q \phi(f_j(x))Jf_j(x)dx &= \lim_{j \rightarrow \infty} \int_{R^n} \phi(y)n(f_j, y)dy \\ &= \int_{R^n} \phi(y)n(f, y)dy = \int_Q \phi(f(x))Jf(x)dx \end{aligned}$$

and (3) is proved.

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Received October 26, 1971.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.



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