

# Pacific Journal of Mathematics

**ON THE FITTING LENGTH OF A SOLUBLE LINEAR GROUP**

TREVOR ONGLEY HAWKES

## ON THE FITTING LENGTH OF A SOLUBLE LINEAR GROUP

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**Let  $G$  be a finite soluble completely reducible linear group of degree  $n$  over a perfect field. It is shown that the Fitting length  $l(G)$  of  $G$  satisfies the inequality**

$$l(G) \leq 3 + 2 \log_3(n/2),$$

**and that this bound is best possible for infinitely many values of  $n$ .**

Let  $G$  be a soluble completely reducible linear group of degree  $n > 1$  over a perfect field  $k$ . Huppert shows in [3], Satz 10, that the derived length of  $G$  is at most  $6 \log_2 n$ . This is therefore an upper bound for the Fitting length  $l(G)$  of  $G$  as well. In this note we assume in addition that  $G$  is finite and prove that

$$l(G) \leq 3 + 2 \log_3(n/2).$$

We show further that this bound is actually achieved for infinitely many values of  $n$ .

**LEMMA 1.** *Let  $K$  be a normal subgroup and  $M$  a maximal subgroup of a finite soluble group  $G$ . Then  $l(M \cap K) \geq l(K) - 2$ .*

*Proof.* Let  $l(K) = l$ . The result is clearly true when  $l \leq 2$ ; therefore assume  $l > 2$  and proceed by induction on  $|G|$ . Set  $F = F(K)$ , the Fitting subgroup of  $K$ . Since  $F$  is the direct product of its Sylow subgroups, each of which is normal in  $G$ , we may assume by the  $R_0$ -closure of the class  $L(l)$  of groups of Fitting length at most  $l$  that  $F$  has a Sylow  $p$ -complement  $S$  such that  $l(K/S) = l$ . Suppose  $S \neq 1$ . If  $S \leq M$ , then  $M/S$  is maximal subgroup of  $G/S$ , and so by induction  $l(M \cap K/S) \geq l(K/S) - 2 = l - 2$ . But then  $l(M \cap K) \geq l - 2$ , as required. On the other hand, if  $S \not\leq M$ , then  $M \cap K/M \cap S \cong S(M \cap K)/S = K/S$  has Fitting length  $l$ , whence  $l(M \cap K) = l$ . Hence we may assume that  $S = 1$  and that  $F$  is a  $p$ -group. Set  $L/F = F(K/F)$ . Then  $l(K/L) = l - 2$  and  $L/F$  is a  $p'$ -group. There are two possibilities to consider:

- (a)  $L \not\leq M$ . In this case  $L(M \cap K) = K$  and therefore  $M \cap K/M \cap L \cong K/L$  has Fitting length  $l - 2$ . Hence  $l(M \cap K) \geq l - 2$ .
- (b)  $L \leq M$ . In this case, denoting the Fitting subgroup of  $M \cap K$  by  $\bar{F}$ , since  $F \leq \bar{F}$  and  $C_K(F) \leq F$ , we see that  $\bar{F}$  is a  $p$ -group. But then  $\bar{F}/F$  is a normal  $p$ -subgroup of  $M \cap K/F$ , and so  $\bar{F}/F \leq$

$C_{K/F}(L/F) \leq L/F$ , a  $p'$ -group. Thus  $F = \bar{F}$  and  $l(M \cap K) = 1 + l(M \cap K/F)$ . By induction,  $l(M \cap K/F) \leq l(K/F) - 2 = l - 3$ , and so in this case too the conclusion of the lemma holds.

**LEMMA 2.** *Let  $T$  be an extra-special group of order  $2^l$ , and let  $A$  be a soluble subgroup of  $\text{Aut}(T)$  acting irreducibly on  $T/\Phi(T)$ . Then  $l(A) \leq 4$ .*

*Proof.* By Huppert, [4], III.13.9 (b),  $A$  is a subgroup of an orthogonal group of dimension 6 over the field of 2 elements; hence by Dieudonné [2], p. 68,  $|A|$  divides  $2^7 \cdot 3^2 \cdot 5 \cdot 7$  or  $2^7 \cdot 3^4 \cdot 5$ . Since  $T/\Phi(T)$  is an irreducible  $Z_2[A]$ -module,  $2 \nmid |F(A)|$  and hence  $|F(A)|$  is a divisor of  $3^2 \cdot 5 \cdot 7$  or  $3^4 \cdot 5$ . Let  $F(A)/K$  be a chief factor of  $A$  and set  $\bar{A} = A/C_A(F(A)/K)$ . If  $|F(A)/K| = 3, 3^2, 5$  or  $7$ , examination of the corresponding linear groups shows that  $l(\bar{A}) \leq 3$ . If  $|F(A)/K| = 3^3$ ,  $\bar{A}$  is isomorphic with a subgroup of  $GL(3, 3)$ . Its order therefore divides  $|GL(3, 3)| = 2^5 \cdot 3^3 \cdot 13$ . But its order also divides  $2^7 \cdot 3^4 \cdot 5/3^3$  and therefore divides  $2^5 \cdot 3$ . But then  $O_{2,3,2}(\bar{A}) = \bar{A}$  and so we have  $l(\bar{A}) \leq 3$ . Since a Sylow 3-subgroup of  $GL(6, 2)$  has order  $3^4$  and is non-Abelian, there are no other possibilities for the order of  $F(A)/K$ . Hence  $A/F(A)$ , which is a subdirect product of the groups  $\bar{A}$ , has Fitting length at most 3. Thus  $l(A) \leq 4$ , as claimed.

We state without proof the following elementary arithmetical facts.

**LEMMA 3.** (a) *If  $d \geq 3, 3^d \geq d\sqrt{12}$ ;*  
 (b) *If  $d \geq 4, 2^d \geq d\sqrt{12}$ ;*

We now come to our main result.

**THEOREM.** *Let  $G$  be a finite soluble completely reducible linear group of degree  $n$  over a perfect field  $k$ . Let  $l(G) = l > 1$ . Then*

$$n \geq 2 \cdot 3^{\eta(l-3)/2},$$

where  $\eta = 0$  for  $l = 2, 3$  and  $\eta = 1$  for  $l \geq 4$ .

*Proof.* Since a linear group of degree one is Abelian, the theorem is clearly true for  $l = 2, 3$ . Therefore assume  $l \geq 4$ . We may suppose there is an  $n$ -dimensional  $k$ -space  $V$  on which  $G$  acts (faithfully and completely reducibly). We proceed by induction on the integer  $m = |G| + \dim_k(V)$ , assuming the theorem has already been proved for all groups  $G$  and all fields  $k$  giving smaller values of  $m$ . Let  $V = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_r$  be a decomposition of  $V$  into irreducible components  $\mathcal{U}_i$ . Set  $K_i = \ker(G \text{ on } \mathcal{U}_i)$ . If  $G/K_i \in L(l-1)$  for every  $i$ ,

we have,  $G \in R_0 L(l - 1) = L(l - 1)$  since  $\bigcap_{i=1}^r K_i = 1$ . Since this is not the case, we have  $l(G/K_i) = l$  for some  $i$ , and therefore when  $r > 1$  we may apply induction to the triple  $(G/K_i, \mathcal{Z}_i, k)$  to give the result. Therefore assume  $V$  is irreducible as a  $k[G]$ -module. Since  $G$  is finite and  $k$  is perfect, we can find a finite extension  $\bar{k}$  of  $k$  which is a splitting field for  $G$  and its subgroups such that  $\bar{V} = \bar{k} \otimes_k V$  is completely reducible; in fact  $\bar{V} = V_1 \oplus \dots \oplus V_s$  is the direct sum of algebraically conjugate irreducible  $\bar{k}[G]$ -modules. If  $s > 1$ ,  $\dim_{\bar{k}}(V_i) < \dim_{\bar{k}}(\bar{V}) = \dim_k(V)$ . Since  $\ker(G \text{ on } V_i) = \ker(G \text{ on } V) = 1$ , we can apply induction to the triple  $(G, V_i, \bar{k})$  to give the result. Therefore we may assume that  $s = 1$  and without loss of generality that  $k = \bar{k}$  is a splitting field for  $G$  and its subgroups.

Let  $H$  be a subgroup of  $G$  critical for the class  $L(l - 1)$ ; thus  $H \in L(l) \setminus L(l - 1)$  and all proper subgroups of  $H$  belong to  $L(l - 1)$ . By Lemma 5.2 and Theorem 5.3 of [1] there is a prime  $q$  dividing  $|F(G)|$  such that  $H$  has a special normal  $q$ -subgroup  $Q$  such that  $Q/\Phi(Q)$  is a chief factor of  $H$  on which  $H$  induces a group of automorphisms of Fitting length exactly  $l - 1$ . If  $k$  has finite characteristic  $p$ , by the irreducibility of  $V$  we have  $O_p(G) = 1$ ; thus  $q \neq \text{char } k$ . Hence there exists a composition factor  $V^*$  of  $V|_H$  not centralized by  $Q$ . The subgroup  $Q^* = C_Q(V^*)$  is proper and normal in  $H$ , and therefore  $Q^*\Phi(Q) = Q$  or  $\Phi(Q)$ . But  $Q^*\Phi(Q) = Q$  implies  $Q^*$  is not proper. Therefore  $Q^* \leq \Phi(Q)$ . But then  $l(H/Q^*) = l$ . If  $H < G$ , induction applied to the triple  $(H/Q^*, V^*, k)$  gives the result. Therefore we suppose  $H = G$  is critical for  $L(l)$ .

Let  $A$  be an Abelian normal subgroup of  $G$ . Let  $V|_A = W_1 \oplus \dots \oplus W_t$  be the decomposition into homogeneous components  $W_i$ . Suppose  $t > 1$ , and let  $M$  be a maximal subgroup of  $G$  containing the stabilizer  $S$  of  $W_1$ . By Clifford theory  $W_1$  is an irreducible  $S$ -module and  $V = W_1^G = (W_1^M)^G$ . Furthermore,  $Y = W_1^M$  is an irreducible  $k[M]$ -module. Applying induction to the triple  $(M/\ker(M \text{ on } Y), Y, k)$  gives  $\dim_k(Y) \geq 2 \cdot 3^{\gamma'(l-3)/2}$ , where  $l' = l(M)$ . If  $|G:M| = 2$ , then  $M \triangleleft G$  and clearly  $l' = l - 1$ . But then  $n = 2 \dim_k(Y) \geq 2 \cdot 2 \cdot 3^{\gamma'(l-4)/2} > 2 \cdot 3^{\gamma(l-3)/2}$ . Therefore suppose  $|G:M| \geq 3$ . By Lemma 1  $l(M) \geq l - 2$ , and so again by induction we have  $n \geq 3 \dim_k(Y) \geq 3 \cdot 2 \cdot 3^{\gamma'(l-5)/2} \geq 2 \cdot 3^{\gamma(l-3)/2}$ . Therefore we may assume that  $t = 1$ , and, since  $k$  is a splitting field for the subgroups of  $G$ , that every Abelian normal subgroup of  $G$  is cyclic and contained in  $Z(G)$ .

Thus  $Q$  is an extra-special group, say of order  $q^{2d+1}$ . By Huppert [4], V.16.14, the faithful irreducible  $k[Q]$ -modules have dimension  $q^d$ . Since  $V$  is faithful for  $Q$ , we have  $n = \dim_k(V) \geq q^d$ .  $G$  induces on  $U = Q/\Phi(Q)$  a soluble irreducible group  $S$  of symplectic linear transformations over  $Z_q$ , and  $l(S) = l - 1$ . If  $l = 4$  or  $5$ ,  $q^d \geq 6$ ; for the ir-

reducible soluble subgroups of  $\text{Sp}(2, 2)$ ,  $\text{Sp}(2, 3)$ ,  $\text{Sp}(2, 5)$  and  $\text{Sp}(4, 2) \cong S_6$  all have Fitting length at most 2. In these cases we have  $n \geq q^d \geq 6 \geq 2 \cdot 3^{(l-3)/2}$ . If  $l \geq 6$ , induction applied to  $(G/\ker(G \text{ on } U), U, \mathbf{Z}_q)$  shows that  $d \geq 3^{(l-4)/2} \geq 3$ . Thus, if  $q \neq 2$ , by induction and Lemma 3(a), we have

$$n \geq q^d \geq 3^d \geq d\sqrt{12} \geq 2 \cdot \sqrt{3} \cdot 3^{(l-4)/2} = 2 \cdot 3^{(l-3)/2}.$$

And if  $l \geq 6$  and  $q = 2$ , by Lemma 2 and induction we have  $d \geq \max\{4, 3^{(l-4)/2}\}$ . Hence using Lemma 3(b) we have

$$n \geq 2^d \geq d\sqrt{12} \geq 2 \cdot 3^{(l-3)/2}.$$

This completes the proof.

The bound for this theorem can actually be achieved whenever  $l$  is odd and  $k = \mathbf{Z}_3$ . For let  $l = 2l' + 1$  and let  $H$  be the holomorph of an elementary Abelian group  $A$  of order 9.  $H/A \cong GL(2, 3)$  has Fitting length 3. Let  $W = (\dots(H \wr S_3) \wr \dots \wr S_3)$ , the successive wreath product of  $H$  with  $l' - 1$  copies of the symmetric group of degree 3 according to its natural representation. It is easy to check that  $W$  has a self-centralizing elementary Abelian normal 3-subgroup  $N$  such that  $l(W/N) = 2(l' - 1) + 3 = l$ .  $N$  is a faithful irreducible  $\mathbf{Z}_3[W/N]$ -module of  $\mathbf{Z}_3$ -dimension  $2 \cdot 3^{l'-1} = 2 \cdot 3^{(l-3)/2}$ .

We conclude by remarking that the above methods give better bounds for  $l(G)$  in terms of  $n$  if the smallest prime divisor of  $|G|$  is greater than 2 or, more generally, if the 2-length of  $G$  is restricted.

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