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In this paper, using the Bergman kernel function $K_D(z, \bar{z})$, we give necessary and sufficient conditions that a pseudoconformal mapping f(z) be starlike or convex in some bounded schlicht domain D for which the kernel function $K_D(z, \bar{z})$ becomes infinitely large when the point $z \in D$ approaches the boundary of D in any way. We also consider starlike and convex mappings from the polydisk or unit hypersphere into C^n .

Generalizing the results obtained by M. S. Robertson [10] using the principle of subordination, T. J. Suffridge has established necessary and sufficient conditions that a function be univalent and map the polydisk or

$$D_p = \left\{ z ext{:} \left[\sum\limits_{j=1}^n |z_j|^p
ight]^{1/p} < 1, \ p \ge 1
ight\}$$

onto a starlike or convex domain [11].

Similar problems have been considered by T. Matsuno [8] for one hypershere. In this paper we deal with the same problems in terms of the Bergman kernel function $K_D(z, \bar{z})$, and show the results are equivalent to theorems of Suffridge in case of polydisk or hypersphere.

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1. Preliminaries. We consider bounded schlicht domains D in C^{n} for which the kernel function becomes infinite everywhere on the boundary ∂D , i.e., it is the union of an increasing sequence of strictly pseudo-convex domains

(1.1)
$$D_t = [z: \varphi_t(z) \equiv K_D(z, \overline{z}) - t < 0, z \in D]$$

for some number t > 0, where $z = (z_1, \dots, z_n)'$. (See [3]). First we have

LEMMA 1.1. If D is a bounded domain, the Bergman kernel function $K_D(z, \bar{z})$ is strictly plurisubharmonic and

(1.2)
$$1/\omega(D) \leq K_D(z, \overline{z}) \leq 1/\pi^n (l(z))^{2n}$$

where $l(z) = \min_{\tau \in \partial D} \rho(\tau, z)$, $\rho(\tau, z) = \max_{j} \{ |\tau_j - z_j|, j = 1, \dots, n \}$ and $\omega(D)$ signifies the euclidean volume of D.

Proof. The minimum value of the integral $||f||_D^2 = \int_D |f(\zeta)|^2 dv_{\zeta}$ for functions $f(\zeta) \in \mathscr{L}^2(D)$ satisfying the condition $df(z)/d\zeta \cdot u = 1$, where $u = (u_1, \dots, u_n)'$ is an arbitrary nonzero column vector, is

(1.3)
$$1/u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \zeta} u = \int_D \left| \frac{u^* \frac{\partial K_D(\zeta, \bar{z})}{\partial \zeta^*}}{u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \zeta} u} \right|^2 dv_{\zeta} . \quad (See [1], [2].)$$

Here we define partial derivatives of a function $g(\zeta, \overline{\tau})$ as

(1.4)
$$\begin{array}{l} \partial^2 g(\zeta,\,\overline{\tau})/\partial\tau^*\partial\zeta &= (\partial/\partial\overline{\tau}_1,\,\cdots,\,\partial/\partial\overline{\tau}_n)'\times(\partial/\partial\zeta_1,\,\cdots,\,\partial/\partial\zeta_n)\times g(\zeta,\,\overline{\tau}) \\ &= \begin{pmatrix} \partial^2/\partial\overline{\tau}_1\partial\zeta_1,\,\cdots,\,\partial^2/\partial\overline{\tau}_1\partial\zeta_n \\ \\ \\ \partial^2/\partial\overline{\tau}_n\partial\zeta_1,\,\cdots,\,\partial^2/\partial\overline{\tau}_n\partial\zeta_n \end{pmatrix} \times g(\zeta,\,\overline{\tau}) , \end{array}$$

and if $g(\zeta)$ is a function of only ζ , we denote $dg(\zeta)/d\zeta = (\partial/\partial\zeta_1, \cdots, \partial/\partial\zeta_n) \times g(\zeta)$, where the sign \times designates the Kronecker product and the sign * denotes the transposed conjugate matrix. (Cf. [7].)

On the other hand, if we put $f(\zeta) = u^*(\zeta - z)/|u|^2$, then

$$rac{df(z)}{d\zeta} u = u^* u / |\, u\,|^2 = 1$$
 ,

therefore

(1.5)
$$1/u^* \frac{\partial^2 K_D(z, \overline{z})}{\partial \zeta^* \partial \zeta} u \leq \int_D \left| \frac{u^*(\zeta - z)}{|u|^2} \right|^2 dv_\zeta \leq \frac{1}{|u|^2} \int_D |\zeta - z|^2 dv_\zeta \leq \frac{L^2 \omega(D)}{|u|^2} ,$$

where $L = \max_{\tau \in \partial D} |\tau - z|$ and $|u| = (\sum_{j=1}^{n} |u_j|^2)^{1/2}$. Thus

$$u^* rac{\partial^2 K_{\scriptscriptstyle D}(\pmb{z},\,ar{\pmb{z}})}{\partial \zeta^* \partial \zeta} u > 0$$

for all $z \in D$, that is, $K_D(z, \overline{z})$ is strictly plurisubharmonic (see [3]). Next it is well known that the minimum value of the integral $||f||_D^2$ under the condition $f(z) = 1, z \in D$, becomes $1/K_D(z, \overline{z})$. Then, for the function $f(\zeta) \equiv 1$, we have

(1.6)
$$1/K_D(z, \overline{z}) = \int_D |K_D(\zeta, \overline{z})/K_D(z, \overline{z})|^2 dv_{\zeta} \leq \int_D dv_{\zeta} = \omega(D)$$
.

Also, using the Cauchy integral formula, we obtain

(1.7)
$$\begin{aligned} &\left| \left(\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})} \right)_{\zeta=z} \right| \\ &\leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{|K_D(\zeta, \bar{z})/K_D(z, \bar{z})|}{r_1 \cdots r_n} r_1 d\theta_1 \cdots r_n d\theta_n , \end{aligned}$$

where $\zeta_j - z_j = r_j e^{i\theta_j}$, $0 < r_j < l(z)$, $(j = 1, \dots, n)$. We get therefore by the Schwarz integral inequality

$$(1.8) l^{2n}/2^n \leq \frac{1}{(2\pi)^n} \int_{\rho(\zeta,z) < l} \int \left| \frac{K_D(\zeta, \overline{z})}{K_D(z, \overline{z})} \right| dv_{\zeta} \\ \leq \frac{1}{(2\pi)^n} \left[(\pi l^2)^n \int_{\rho(\zeta,z) < l} \int \left| \frac{K_D(\zeta, \overline{z})}{K_D(z, \overline{z})} \right|^2 dv_{\zeta} \right]^{1/2}.$$

Then

(1.9)
$$\pi^{n/2} l^n \leq \left[\int_D \left| \frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})} \right|^2 dv_{\zeta} \right]^{1/2} = (1/K_D(z, \bar{z}))^{1/2} \, ,$$

hence we have (1.2) from (1.6) and (1.9).

2. Convex mappings. We consider the above mentioned domains D and D_t , and suppose that $\partial K_D(z, \overline{z})/\partial z \approx 0$, $z \approx 0$, in D, and $K_D(0, 0) = \min_{z \in D} K_D(z, \overline{z})$ at only z = 0. For a holomorphic univalent function w = f(z) of D, let

(2.1)
$$\varphi_t(z) = \varphi_t(f^{-1}(w)) \equiv \Phi_t(w), t > K_D(0, 0),$$

and let $\Delta = f(D)$, $\Delta_t = f(D_t)$. Then we have

corresponding to (1.1). On the boundary $\partial D_i: \varphi_i(z) = 0$, the total differential of $\varphi_i(z)$ becomes

(2.3)
$$d\varphi_t = \frac{\partial \varphi_t}{\partial z} dz + dz^* \frac{\partial \varphi_t}{\partial z^*} = 2 \mathscr{R} \left[\frac{\partial \varphi_t}{\partial z} dz \right] = 0 ,$$

where $dz = (dz_1, \dots, dz_n)'$. Consequently, since $\partial \varphi_t / \partial z^* = \partial K_D(z, \overline{z}) / \partial z^*$ is perpendicular to all tangential vectors dz of the boundary ∂D_t at $z, \partial \varphi_t / \partial z^*$ is a normal vector of ∂D_t at z. And we can derive

(2.4)
$$\mathscr{R}\left[\frac{\partial \Phi_t}{\partial w}dw\right] = \mathscr{R}\left[\frac{\partial \Phi_t}{\partial z}\left(\frac{dz}{dw}\right)\left(\frac{dw}{dz}\right)dz\right] = \mathscr{R}\left[\frac{\partial \varphi_t}{\partial z}dz\right] = 0$$
,

hence $\partial \Phi_t / \partial w^*$ is also a normal vector of the boundary $\partial \Delta_t : \Phi_t(w) = 0$ at w = f(z). (See [5], [6].)

We can expand $\Phi_t(w + dw)$ into a Taylor series:

$$(2.5) \qquad \begin{split} \varPhi_t(w+dw) &= \varPhi_t(w) + 2\mathscr{R}\left[\frac{\partial \varPhi_t}{\partial w}dw\right] \\ &+ 2\mathscr{R}\left[\frac{\partial^2 \varPhi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2 \varPhi_t}{\partial w^*\partial w}dw\right] + 0(|dw|^2) , \end{split}$$

where $dw^2 = (dw_1, \dots, dw_n)' \times (dw_1, \dots, dw_n)'$. (See [3], Chap. IX.) Since

$$\mathscr{R}\left[rac{\partial \varPhi_{t}}{\partial w}dw
ight]=0$$

at $w \in \partial \mathcal{A}_t$, it follows that

$$(2.6) \quad \varPhi_i(w+dw) = 2\mathscr{R}\left[\frac{\partial^2 \varPhi_i}{\partial w^2}dw^2 + dw^* \frac{\partial^2 \varPhi_i}{\partial w^* \partial w}dw\right] + 0(|dw|^2) \ .$$

If the point (w + dw) lie always the outside of Δ_i for all $w \in \partial \Delta_i$ and tangential vectors dw at w, i.e., $\Phi_i(w + dw) > 0$, then Δ_i is convex. From (2.6), we must have the following condition in order to consist always $\Phi_i(w + dw) > 0$:

(2.7)
$$\mathscr{R}\left[\frac{\partial^2 \Phi_t}{\partial w^2} dw^2 + dw^* \frac{\partial^2 \Phi_t}{\partial w^* \partial w} dw\right] > 0 .$$

Now we can calculate as follows by formulas of matrix derivatives described in [7]:

$$rac{\partial^2 arPsi_t}{\partial w^2} = rac{\partial}{\partial w} \Big(rac{\partial arphi_t}{\partial z} \Big(rac{dw}{dz} \Big)^{-1} \Big) = rac{\partial}{\partial z} \Big(rac{\partial arPsi_t}{\partial z} \Big(rac{dw}{dz} \Big)^{-1} \Big) \Big(\Big(rac{dw}{dz} \Big)^{-1} imes E \Big)
onumber \ = rac{\partial^2 arPsi_t}{\partial z^2} \Big(\Big(rac{dw}{dz} \Big)^{-1} imes \Big(rac{dw}{dz} \Big) \Big)^{-1} - rac{\partial arPsi_t}{\partial z} \Big(rac{dw}{dz} \Big)^{-1} rac{d^2 w}{dz^2} \Big(\Big(rac{dw}{dz} \Big)^{-1} imes \Big(rac{dw}{dz} \Big)^{-1} \Big) \,,$$

(2.9)
$$\frac{\partial^2 \Phi_t}{\partial w^2} dw^2 = \left\{ \frac{\partial^2 \varphi_t}{\partial z^2} - \frac{\partial \varphi_t}{\partial z} \left(\frac{dw}{dz} \right)^{-1} \frac{d^2 w}{dz^2} \right\} dz^2 ,$$

$$(2.10) \quad dw^* \frac{\partial^2 \varPhi_t}{\partial w^* \partial w} dw = dw^* \Big\{ \left(\frac{dw}{dz} \right)^{-1} * \frac{\partial^2 \varphi_t}{\partial z^* \partial z} \left(\frac{dw}{dz} \right)^{-1} \Big\} dw = dz^* \frac{\partial^2 \varphi_t}{\partial z^* \partial z} dz .$$

Then, substituting (2.9) and (2.10) into (2.7), we obtain

$$(2.11) \qquad \mathscr{R}\left[\left\{\frac{\partial^2 \varphi_t}{\partial z^2} - \frac{\partial \varphi_t}{\partial z} \left(\frac{dw}{dz}\right)^{-1} \frac{d^2 w}{dz^2}\right\} dz^2 + dz^* \frac{\partial^2 \varphi_t}{\partial z^* \partial z} dz\right] > 0.$$

Thus we have the following Lemma.

LEMMA 2.1. For a fixed value t, a holomorphic univalent function w = f(z) of D have convex image Δ_t of D_t defined by (1.1) if and only if at every point z on the boundary ∂D_t

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$$(2.12) \quad \mathscr{R} \bigg[\alpha^* \frac{\partial^2 K_D(z, \bar{z})}{\partial z^* \partial z} \alpha + \Big\{ \frac{\partial^2 K_D(z, \bar{z})}{\partial z^2} - \frac{\partial K_D'(z, \bar{z})}{\partial z} \Big(\frac{df}{dz} \Big)^{-1} \frac{d^2 f}{dz^2} \Big\} \alpha^2 \bigg] > 0$$

for all unit vectors α satisfying

$$\mathscr{R}\left[rac{\partial K_{\scriptscriptstyle D}(z,\,ar{z})}{\partial z}lpha
ight]=0$$
 .

DEFINITION. We define the class \mathscr{D} of bounded schlicht domains D for which the kernel function $K_D(z, \bar{z})$ becomes infinite everywhere on the boundary ∂D , $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ only at $z = 0, \partial K_D(z, \bar{z})/\partial z \approx 0, z \approx 0$, in D, and there is the holomorphic mapping g(z) of D into D satisfying g(0) = 0, for some one $z^{(1)}$ of two arbitrary points $z^{(1)}, z^{(2)}(\approx 0)$ in $D g(z^{(1)}) = z^{(2)}$, and $K_D(z, \bar{z}) \geq K_D(g(z), \overline{g(z)})$.

For example, let D be a minimal domain or representative domain with center at the origin which is the image domain of $E = \{\zeta : |\zeta| = (\sum_{j=1}^{n} |\zeta_j|^2)^{1/2} < 1\}$ under the biholomorphic mapping $z = \varphi(\zeta)$ satisfying $0 = \varphi(0)$. Then det $(d\varphi(\zeta)/d\zeta) \equiv \text{const.}$ when D is a minimal, domain and $d\varphi(\zeta)/d\zeta \equiv \text{const.}$ when D is a representative domain (see [4], Theorem 3.1). Hence, for any holomorphic mapping g(z) of D into Dsatisfying g(0) = 0, we have $K_D(z, \overline{z}) \geq K_D(g(z), \overline{g(z)})$ because $K_E(\zeta, \overline{\zeta}) \geq K_E(\Phi(\zeta), \overline{\Phi(\zeta)})$ under the holomorphic mapping $\Phi(\zeta) \equiv \varphi^{-1}[g(\varphi(\zeta))], \Phi(0) = 0$, of E into E. Also we have $K_D(0, 0) = \min_{z \in D} K_D(z, \overline{z})$ at only the origin. Moreover, for arbitrary points $z^{(1)}, z^{(2)} \in D$, if $|\varphi^{-1}(z^{(2)})| \leq |\varphi^{-1}(z^{(1)})|$, then

$$g(\pmb{z}) \,\equiv\, arphi \Bigl(rac{|arphi^{-1}(\pmb{z}^{(2)})|}{|arphi^{-1}(\pmb{z}^{(1)})|} \,\, U_2 \, U_1^{\,*} arphi^{-1}(\pmb{z}) \Bigr)$$

is a holomorphic mapping of D into D satisfying g(0) = 0 and $g(z^{(1)}) = z^{(2)}$ where

and U_1 , U_2 are unitary matrices. And we observe

$$\partial K_{\scriptscriptstyle D}(z,\,ar z)/\partial z = \partial K_{\scriptscriptstyle E}(\zeta,\,ar \zeta)/\partial \zeta \cdot (darphi(\zeta)/d\zeta)^{-1} \rightleftharpoons 0,\, z \rightleftharpoons 0$$
 ,

because

$$\partial K_{\scriptscriptstyle E}(\zeta,\,ar\zeta)/\partial\zeta = (n\,+\,1)\zeta^*K_{\scriptscriptstyle E}(\zeta,\,ar\zeta)/(1\,-\,|\zeta|^2) \rightleftharpoons 0,\,\zeta \rightleftharpoons 0$$
 .

THEOREM 2.1. Let D be a bounded schlicht domain of the class \mathscr{D} . Suppose $f: D \to C^n$ is holomorphic, f(0) = 0, and $\det (df/dz) \rightleftharpoons 0$ for all $z \in D$. Then f is a univalent map of D onto a convex domain if and only if

$$(2.13) \quad \mathscr{R}\left[\alpha^* \frac{\partial^2 K_D(z, \bar{z})}{\partial z^* \partial z} \alpha + \left\{ \frac{\partial^2 K_D(z, \bar{z})}{\partial z^2} - \frac{\partial K_D(z, \bar{z})}{\partial z} \left(\frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} \right\} \alpha^2 \right] > 0$$

for all unit vectors α satisfying

$$\mathscr{R}\left[rac{\partial K_{\scriptscriptstyle D}(z,\,ar{z})}{\partial z}lpha
ight]=0$$
 .

Proof. The Bergman kernel function $K_D(z, \overline{z})$ of this domain D becomes infinite on ∂D . Then we define D_t and Δ_t by (1.1) and (2.2) respectively. If $\Delta = f(D)$ is schlicht and convex, then all Δ_t also become convex, i.e., for any $w^{(1)}, w^{(2)} \in \partial \Delta_t$,

$$(2.14) w^{\scriptscriptstyle (0)} = \tau w^{\scriptscriptstyle (2)} + (1-\tau) w^{\scriptscriptstyle (1)} \in \varDelta_t, \quad 0 < \tau < 1 \, .$$

In fact, if we put $z^{(1)} = f^{-1}(w^{(1)}), z^{(2)} = f^{-1}(w^{(2)})$, then $K_D(z^{(1)}, \overline{z^{(1)}}) = K_D(z^{(2)}, \overline{z^{(2)}}) = t$. Setting

(2.15)
$$F(z) \equiv \tau f(g(z)) + (1 - \tau) f(z)$$

where g(z) is a holomorphic mapping of D into D satisfying g(0) = 0and $g(z^{(1)}) = z^{(2)}$, we observe that F(0) = 0 and $F(z) \prec f(z)$ because the mapping $f: D \to C^n$ is convex. Hence

$$\psi(z) \equiv f^{-1}(F(z))$$

is a holomorphic mapping of D into D, so we have

$$K_{\scriptscriptstyle D}(z^{\scriptscriptstyle (1)}, \overline{z^{\scriptscriptstyle (1)}}) \geqq K_{\scriptscriptstyle D}(\psi(z^{\scriptscriptstyle (1)}), \overline{\psi(z^{\scriptscriptstyle (1)}})) = K_{\scriptscriptstyle D}(f^{-1}(w^{\scriptscriptstyle (0)}), \overline{f^{-1}(w^{\scriptscriptstyle (0)})}) \; .$$

Consequently $f^{-1}(w^{(0)}) \in D_i$, so $w^{(0)} \in \mathcal{A}_i$. Thus, by Lemma 2.1, (2.13) holds for all $z \in D$. Contrary, if (2.13) is realized for all $z \in D$, every \mathcal{A}_i is convex. Therefore we can conclude that the mapped domain \mathcal{A} is convex.

Particularly if D is a unit hypersphere, then

$$K_{_D} \,\, (\pmb{z},\, \overline{\pmb{z}}) \,= \, rac{n!}{\pi^n (1 \,-\, |\, \pmb{z}\,|^2)^{n+1}} \,\,.$$

Thus we have the following result by Theorem 2.1.

THEOREM 2.2. Let D be the unit hypersphere and let $f: D \rightarrow C^n$ be holomorphic, f(0) = 0 and $det(df/dz) \neq 0$ for all $z \in D$. Then f(D) is convex if and only if

$$(2.17) \qquad \qquad \mathscr{R}\bigg[|Az|^2 + z^* \Big(\frac{df}{dz}\Big)^{-1} \frac{d^2f}{dz^2} (Az \times Az)\bigg] \ge 0 ,$$

where

$$A = egin{pmatrix} A_1 & 0 \ & \ddots \ 0 & A_n \end{pmatrix}, A_j \geqq 0, j = 1, \cdots, n \;,$$

and the equality holds only if Az = 0.

Proof. We can compute as follows setting $K = K_D(z, \bar{z})$:

$$\partial K/\partial z = (n+1)\frac{z^*}{1-|z|^2}K,$$

(2.19)
$$\partial^2 K / \partial z^2 = (n+1)(n+2) \frac{(z \times z)^*}{(1-|z|^2)^2} K$$
,

(2.20)
$$\partial^2 K / \partial z^* \partial z = (n+1) \frac{(1-|z|^2)E + (n+2)zz^*}{(1-|z|^2)^2} K$$
.

Then, from (2.13), we have

(2.21)
$$\mathscr{R}\left[(n+2)\{|z^*\alpha|^2 + (z^*\alpha)^2\} + (1-|z|^2)\left\{1 - z^*\left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}\alpha^2\right\}\right] > 0.$$

Since

$$|z^*lpha|^2+\mathscr{R}(z^*lpha)^2=0$$

from

$$\mathscr{R}\left[rac{\partial K}{\partial z}lpha
ight]=0, ext{ i.e., } \mathscr{R}[z^*lpha]=0$$
 ,

we conclude

(2.22)
$$\mathscr{R}\left[1-z^*\left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}\alpha^2\right]>0.$$

Moreover, under the condition $\mathscr{R}[z^*\alpha] = 0$ it becomes that $z^*\alpha = ip(p \ge 0, i = \sqrt{-1})$, because both α and $-\alpha$ are satisfy (2.22). Therefore we can put $\alpha = i(Az/|Az|)$ when $Az \ge 0$, where

$$A = egin{pmatrix} A_1 & 0 \ & \ddots & \ 0 & A_n \end{pmatrix}, \ A_j \ge 0, \ (j = 1, \ \cdots, \ n) \ ,$$

are chosen arbitrarily. Thus we obtain (2.17) from (2.22).

REMARK 1. Suffridge's Theorem 5 [11] shows that

$$F=rac{df}{dz}\Big[A^{2}z+\Big(rac{df}{dz}\Big)^{\!-\!1}rac{d^{2}f}{dz^{2}}(Az imes Az)\Big]\!ig/2,\ w\,=\Big(rac{df}{dz}\Big)^{\!-\!1}F\!\in\!\mathscr{P}_{2}$$
 ,

i.e.,

$$egin{aligned} \mathscr{R} &\sum_{j=1}^n w_j |z_j|^2 / z_j = \mathscr{R} z^* iggl[A^2 z + \Big(rac{df}{dz}\Big)^{-1} rac{d^2 f}{dz^2} (Az imes Az) iggr] iggr/ 2 \ &= \mathscr{R} iggl[|Az|^2 + z^* \Big(rac{df}{dz}\Big)^{-1} rac{d^2 f}{dz^2} (Az imes Az) iggr] iggr/ 2 \geqq 0 \;, \end{aligned}$$

is the necessary and sufficient condition for convexity.

Next, if D is the polydisk $\{z \in C^n \colon |z_j| < 1, j = 1, \dots, n\}$, the kernel function $K_D(z, \overline{z})$ becomes $1/\pi^n (1 - |z_1|^2)^2 \cdots (1 - |z_n|^2)^2$. Hence

$$\partial K/\partial z = 2K \cdot z^* Z,$$



$$(2.25) \qquad \qquad \partial^2 K/\partial z^*\partial z = 4K\boldsymbol{\cdot} Zzz^*Z + 2K\boldsymbol{\cdot} Z^2 \;,$$

where

$$Z = egin{pmatrix} 1/(1 - |z_1|^2) & 0 \ & \ddots & \ 0 & 1/(1 - |z_n|^2) \end{pmatrix}.$$

Substituting formally (2.23), (2.24), and (2.25) into (2.13) and setting

$$\mathscr{R}(z^*Zlpha)^{\scriptscriptstyle 2}+|z^*Zlpha|^{\scriptscriptstyle 2}=0 \,\, ext{and} \,\, lpha=irac{Z^{-1/2}Az}{|Z^{-1/2}Az|}$$

where

$$Z^{_{-1/2}}=egin{pmatrix} \sqrt[]{1-|z_1|^2}&0\ &\ddots\ &\ &\ddots\ &\ &0&\sqrt{1-|z_n|^2} \end{pmatrix}$$
 ,

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in place of the condition

$$\mathscr{R}\!\left[rac{\partial K_{\scriptscriptstyle D}(z,\,ar{z})}{\partial z}\!lpha
ight]=2K\!\cdot \mathscr{R}[z^*Z\!lpha]=0\;,$$

we arrive at

$$(2.26) \qquad \mathscr{R}igg[|Az|^2 + z^*Z \Bigl(rac{df}{dz} \Bigr)^{-1} rac{d^2f}{dz^2} (Z imes Z)^{-1/2} (Az imes Az) \Bigr] \geqq 0 \; ,$$

where the equality holds only if Az = 0.

THEOREM 2.3. Let D be the polydisk and let $f: D \to C^n$ be holomorphic, f(0) = 0 and det $(df/dz) \approx 0$ for all $z \in D$. Then f is a univalent map of D onto a convex domain if and only if the condition (2.26) is fulfilled.

Proof. If f is a convex mapping, then by Suffridge's Theorem 3 [11] $f = T(\varphi_1(z_1), \dots, \varphi_n(z_n))'$ where T is a nonsingular linear transformation and each $\varphi_j(z_j)$ is a univalent mapping from the unit disk in the plane onto convex domain in the plane. Then we have

Substituting this into the left side of (2.26), we get

(2.28)
$$\mathscr{R}\left[\sum_{j=1}^n A_j^2 |z_j|^2 \{1 + z_j \varphi_j''(z_j)/\varphi_j'(z_j)\}\right].$$

Hence from the hypothesis $\mathscr{R}[1 + z_j \mathcal{P}''_j(z_j)/\mathcal{P}'_j(z_j)] > 0, j = 1, \dots, n$, we get the inequality (2.26).

We will prove the converse. Fix $k, 1 \leq k \leq n$ and choose $A_k = 1, A_h = 0, h \approx k, 1 \leq h \leq n$. From (2.26)

(2.29)
$$\mathscr{R}\left[|z_k|^2 + \frac{z_k^2(1-|z_k|^2)}{\det J}\sum_{j=1}^n \frac{\overline{z}_j}{1-|z_j|^2}C_j^{k^2}\right] \ge 0$$
,

where J = df/dz and $G_j^{k^2}$ is obtained from det J by replacing the *j*th column by the column $\partial^2 f/\partial z_k^2 = (\partial^2 f_1/\partial z_k^1, \dots, \partial^2 f_n/\partial z_k^2)'$. For $l, 1 \leq l \leq n, l \approx k$, setting $|z_j| < 1/2, j \approx l, 1 \leq j \leq n, (1 - |z_k|^2)/(1 - |z_l|^2)$ tends to infinity when $|z_l| \to 1$. Then we must have always

(2.30)
$$\mathscr{R}\left[\frac{1}{\det J}\frac{z_k^2}{z_l}\ G_l^{k^2}\right] \ge 0$$

from the condition (2.29). Here, since it becomes 0 at $z_k = 0$, we see that $G_l^{k^2} \equiv 0$ for each $l, l \rightleftharpoons k, 1 \leq l \leq n$. Next, if we set $A_k = A_l = 1, A_m = 0, m \neq k, l$, then (2.26) becomes as follows from the above results:

$$(2.31) \qquad \mathscr{R} \bigg[|z_k|^2 + |z_l|^2 + \frac{|z_k|^2 z_k G_k^{k^2}}{\det J} + \frac{|z_l|^2 z_l G_l^{l^2}}{\det J} \\ + \frac{2^{z_k z_l} \sqrt{(1 - |z_k|^2)(1 - |z_l|^2)}}{\det J} \sum_{j=1}^n \frac{\overline{z}_j G_j^{kl}}{(1 - |z_j|^2)} \bigg] \ge 0 \ .$$

For $s, 1 \leq s \leq n$, setting

$$|z_h| < 1/2, \, h
ightarrow s, 1 \leq h \leq n, \; rac{\sqrt{(1 - |z_k|^2)(1 - |z_l|^2)}}{1 - |z_s|^2}$$

tends to infinity when $|z_s| \rightarrow 1$. Then we must have always

(2.32)
$$\mathscr{R}\left[\frac{1}{\det J}\frac{z_k z_l}{z_s}G_s^{kl}\right] \ge 0.$$

Since it attains to the minimum value 0 at $z_k z_l = 0$, we must have $G_s^{kl} \equiv 0$ for each s. Thus we arrive at the conditions of the Theorem 3 of Suffridge following his methods. So we can conclude that f is a convex mapping.

3. Starlike mappings. We now consider univalent functions of D which map D onto a starlike domain with respect to 0. First we set up the definition of starlikeness following Suffridge:

DEFINITION. A holomorphic mapping $f: D \to C^n$ is starlike if f is univalent, f(0) = 0 and $(1 - \tau)f \prec f$ for all $\tau \in I = [0, 1]$.

THEOREM 3.1. Let D be a bounded schlicht domain for which the kernel function $K_D(z, \overline{z})$ becomes infinite everywhere on the boundary, $\frac{K_D(0, 0) = \min_{z \in D} K_D(z, \overline{z})}{g(z)}$ at only the origin, and $K_D(z, \overline{z}) \geq K_D(g(z), \overline{g(z)})$ for any holomorphic mapping g(z) of D into D satisfying g(0) = 0. Suppose $f: D \to C^n$ is holomorphic, f(0) = 0 and det $(df/dz) \approx 0$ for all $z \in D$. Then f is starlike if and only if

(3.1)
$$\mathscr{R}\left[\frac{\partial K_{D}(z,\bar{z})}{\partial z}\left(\frac{df}{dz}\right)^{-1}f\right] > 0$$

for all $z \in D$, $z \rightleftharpoons 0$.

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REMARK 2. Domains which belong to the above mentioned class \mathscr{D} satisfy the conditions of this Theorem.

Proof. If f is starlike, then all image Δ_t are starlike, that is, for all $w^{(1)} \in \partial \Delta_t$ we have $w^{(0)} = (1 - \tau)w^{(1)} \in \Delta_t$, $\tau \in I$. In fact, if we set $z^{(1)} = f^{-1}(w^{(1)})$, $K_D(z^{(1)}, \overline{z^{(1)}}) = t$ and $\psi(z) \equiv f^{-1}((1 - \tau)f(z))$, then we obtain

$$(3.2) K_D(z^{(1)}, \overline{z^{(1)}}) \ge K_D(\psi(z^{(1)}), \overline{\psi(z^{(1)})}) = K_D(f^{-1}(w^{(0)}), \overline{f^{-1}(w^{(0)})}) ,$$

because $\psi(z)$ is a mapping of D into D and $\psi(0) = 0$. Then it holds that $f^{-1}(w^{(0)}) \in D_t$ which yields $w^{(0)} \in \mathcal{A}_t$. Now, since

$$arPsi_t \! \left(w + arepsilon \! rac{\partial arPsilon_t}{\partial w^*}
ight) = 2 arepsilon \left| rac{\partial arPsilon_t}{\partial w^*}
ight|^{\! 2} + 0 (arepsilon^2) > 0$$

when $\varepsilon > 0$ is sufficiently small and $w \in \partial \varDelta_t$, $N_w \equiv \partial \Phi_t / \partial w^*$ is the outward normal vector at the boundary point $w \in \partial \varDelta_t$. Hence $(1 - \tau)w \in \varDelta_t (w \in \partial \varDelta_t, 0 < \tau \leq 1)$ implies

(3.3)
$$\cos\left(-N_{w}, -w\right) = \mathscr{R}\left[\frac{\partial \Phi_{t}}{\partial w}w\right] / \left|\frac{\partial \Phi_{t}}{\partial w^{*}}\right| |w| > 0$$

which yields (3.1) by virtue of

$$\frac{\partial \Phi_t}{\partial w} w = \frac{\partial K}{\partial z} \left(\frac{df}{dz} \right)^{-1} f(z) \; .$$

Conversely, if (3.1) holds, then we conclude $(1-\tau)w \in \Delta_t$, $w \in \partial \Delta_t$, $0 < \tau < \varepsilon (< 1)$ for some $\varepsilon > 0$ by (3.3). Moreover, we can conclude $(1-\tau)w \in \Delta_t$, $w \in \partial \Delta_t$, $0 < \tau \leq 1$, because, if $(1-\tau_1)w \equiv w^{(1)} \in \partial \Delta_t$ and $(1-\tau)w \in \Delta_t$, $0 < \tau < \tau_1$ for some $\tau_1 < 1$, then $(1-\tau)w^{(1)} \notin \Delta_t$, $w^{(1)} \in \partial \Delta_t$ which is a contradiction. Then the image domain Δ of D becomes starlike.

COROLLARY 3.1. Let D be the unit hypersphere, and let $f: D \rightarrow C^n$ be holomorphic, f(0) = 0 and det $(df/dz) \approx 0$ for all $z \in D$. Then f(z) is starlike if and only if

(3.4)
$$\mathscr{R}\left[z^*\left(\frac{df}{dz}\right)^{-1}f\right] > 0$$

for all $z \in D$, $z \rightleftharpoons 0$.

Proof. Substituting (2.18) into (3.1), we obtain the required result.

REMARK 3. The conditions of Suffridge's Theorem 4 [11]: $f = Jw, w \in \mathscr{P}_2$ are the same as (3.4).

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