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**INTRINSIC TOPOLOGIES IN TOPOLOGICAL LATTICES AND
SEMILATTICES**

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This paper demonstrates that the topology of a compact topological lattice or semilattice can be defined intrinsically, i.e., in terms of the algebraic structure. Properties of various intrinsic topologies are explored.

A variety of ways have been suggested for defining topologies from the algebraic structure of a lattice (see e.g. [4] or [12]). If one is given a topological lattice, a natural question is whether the given topology agrees with one or more of these intrinsic topologies. Some results of this nature may be found in [5] or [13]. In this paper we show that the topology of a compact topological lattice or semilattice can always be defined intrinsically; these results extend to a large class of locally compact lattices.

A *topological lattice* is a lattice L equipped with a Hausdorff topology for which the operations of join and meet are continuous as mappings from $L \times L$ into L . A *topological semilattice* is a (meet) semilattice together with a Hausdorff topology for which the meet operation is continuous.

If A is a subset of a lattice or semilattice, we define

$$L(A) = \{y: y \leq x \text{ for some } x \in A\}$$

and

$$M(A) = \{z: x \leq z \text{ for some } x \in A\}.$$

A subset B of a semilattice is an *ideal* if $L(B) = B$. A set A is *convex* if $x, z \in A$ and $x \leq y \leq z$ imply $y \in A$. A lattice L is *locally convex* if it has an open base of convex sets. A *closed interval* is a set of the form

$$[a, b] = \{x: a \leq x \leq b\}.$$

For the definition of undefined lattice properties employed in this paper, the reader is referred to [4].

The topological closure of a set A will be denoted by A^* .

1. **Intrinsic topologies.** The following intrinsic topologies on a lattice L are considered in this paper.

(1) The interval topology (I). If L has a 0 and 1, the interval topology is defined by taking as a subbase for the closed sets all sets

$\{L(x): x \in L\}$ and all sets $\{M(x): x \in L\}$. If L does not have universal bounds, then a set $K \subset L$ is closed if $K \cap [a, b]$ is closed in the interval topology of the sublattice $[a, b]$ for all a, b with $a \leq b$.

(2) The order topology (0). A net $\{x_\alpha\}$ in L is said to *order-converge* to x if there exist a monotonic ascending net $\{t_\alpha\}$ with $x = \sup t_\alpha (t_\alpha \uparrow x)$ and a monotonic descending net $\{u_\alpha\}$ with $x = \inf u_\alpha (u_\alpha \downarrow x)$ such that for all α , $t_\alpha \leq x_\alpha \leq u_\alpha$. A subset A of L is *closed* in the order topology if $\{x_\alpha\} \subset A$ and x_α order converges to x imply that $x \in A$. Note that if x_α order-converges to x , then for any cofinal subset of the domain directed set it remains true that x_α order-converges to x . Hence the order topology may be defined equivalently by declaring a set U of L open if $x \in U$ and x_α order-converges to x imply x_α is residually in U .

(3) The convex-order topology (CO). A subset U of L is a basic open set for the convex-order topology if (i) U is convex and (ii) if x_α order-converges to x , $x \in U$, then x_α is residually in U . Again, the second condition is equivalent to U being open in the order topology.

We now list some easily derived properties of these intrinsic topologies.

PROPOSITION 1. (1) *The CO topology is locally convex.*

(2) *The 0 topology is finer than the CO topology.*

(3) *Any homomorphism from L to a locally convex lattice that is continuous in the 0 topology is continuous in the CO topology.*

(4) *If the 0 topology is locally convex, then it agrees with the CO topology.*

PROPOSITION 2. *The 0 topology is finer than the I topology.*

Proof. [4, p. 251].

We shall call a topology on a lattice *agreeable* if (i) the topology is locally convex and (ii) if $t_\alpha \uparrow x$ or $t_\alpha \downarrow x$ then t_α converges to x in the topology.

PROPOSITION 3. *If τ is an agreeable topology on a lattice L , then the CO topology is finer than τ .*

Proof. Since τ is locally convex, it suffices to show that if a convex set U is in τ , then it is open in the CO topology. Suppose that x_α is a net that order-converges to $x \in U$. Then there exist $t_\alpha \uparrow x$, $u_\alpha \downarrow x$ such that for all α , $t_\alpha \leq x_\alpha \leq u_\alpha$. Since τ is agreeable, t_α and u_α are residually in U , and since U is convex x_α is residually in U .

2. **The interval topology in complete lattices.** The interval topology has received rather thorough investigation. In this section we summarize results concerning its relationship to compact topological lattices.

PROPOSITION 4. *Let L be a complete lattice.*

(1) *L is compact in the interval topology.*

(2) *If (L, τ) is a topological lattice, then τ is finer than the interval topology.*

(3) *If L is Hausdorff in the interval topology, then the order and interval topology coincide.*

Proof. (1) This is a result of O. Frink. A proof may be found [4, p. 250].

(2) Since in a topological lattice $M(x)$ and $L(x)$ are closed for each $x \in L$, and these sets are a subbasis for the closed sets of the interval topology, the result follows.

(3) See [3] or [15].

The next theorem contains the central results on compact topological lattices with the interval topology.

THEOREM 5. *The following are equivalent in a compact topological lattice (L, τ) :*

(1) *(L, I) is Hausdorff.*

(2) *$\tau = 0 = I = \text{CO}$.*

(3) *(L, τ) has a basis of open convex sublattices.*

(4) *(L, τ) has a base of neighborhoods at each point of closed intervals.*

(5) *If $y \not\leq x$ then there exists z such that x is in the interior of $L(z)$ and $y \not\leq z$, and dually.*

(6) *Every net has an order-convergent subnet.*

Proof. The equivalence of 3, 4, 5 has been shown by E. B. Davies [6, Theorem 5]. K. Atsumi has shown the equivalence of 1 and 6 [3, Theorem 3]. D. Strauss has shown the equivalence of 1 and 3 [13, Theorem 5]. Conditions 3 and 1 together with part 2 of Proposition 4 imply $\tau = I$. Part 3 of Proposition 4 further implies $I = 0$. Since CO is trapped between I (since I is locally convex) and 0, it also agrees with them. Hence Conditions 3 and 1 imply 2. Condition 2 easily implies Condition 1 since τ is Hausdorff. Hence the six conditions are equivalent.

We remark that if (L, τ) is compact topological lattice of finite breadth, then $\tau = I$ [5]. Hence all the equivalences of Theorem 5 apply to (L, τ) . It is known that a finite-dimensional compact con-

nected topological lattice has finite breadth [9].

For complete distributive lattices one obtains a purely algebraic description of lattices which are topological lattices in the interval topology.

THEOREM 6. *Let L be a distributive lattice. The following are equivalent:*

- (1) L is complete and completely distributive.
- (2) L is complete and (L, I) is Hausdorff.
- (3) L is complete and L can be embedded in a product of unit intervals (under coordinatewise order) by a lattice isomorphism which preserves all joins and all meets.
- (4) L admits a topology τ for which (L, τ) is a compact topological lattice with enough continuous lattice homomorphisms into the unit interval (with usual order) to separate points.
- (5) L admits a topology τ for which (L, τ) is a compact topological lattice with a basis of open convex sublattices.

Proof. Theorems 4 and 5 of [6] imply the equivalence of Conditions 4 and 5. Strauss has shown the equivalence of Conditions 1 and 2 [13, Theorem 7] and the implication of Condition 3 by Condition 2 [13, Theorem 6]. It is readily seen that Condition 3 implies that L is a closed subset in the product topology of unit intervals (where the unit interval carries its normal topology); hence L is a compact topological lattice in its relative topology. Since a product of intervals has a basis of open convex sublattices, the intersection of this basis with L endows L with such a basis. Hence Condition 3 implies Condition 5. That Condition 5 implies Condition 2 follows from Theorem 5 above.

THEOREM 7. *Let B be a Boolean lattice. The following are equivalent:*

- (1) B is complete and completely distributive.
- (2) B admits a topology τ for which (B, τ) is a compact topological lattice.
- (3) B is isomorphic with the Boolean lattice of subsets of some set.
- (4) B is isomorphic to a product of $\{0, 1\}$ with $0 < 1$.
- (5) B is complete and (B, I) is Hausdorff.

Proof. By Theorem 6, Conditions 1 and 5 are equivalent and imply Condition 2. Strauss has shown Condition 2 implies Condition 1 [13, Theorem 1].

Tarski has shown that Condition 1 implies Condition 3 (see [14] or [4, p. 119]). If B is isomorphic to all subsets of a set X , then it

can be identified with $\{0, 1\}^x$ by a lattice isomorphism. Hence Condition 3 implies Condition 4. Since any product of complete chains is completely distributive [4, p. 120], Condition 4 implies Condition 1.

3. The convex-order topology. In the preceding section we gave conditions under which a topological lattice had the interval topology and for which all the intrinsic topologies collapsed to this topology. The conditions for a topological lattice to have the order or convex-order topologies are much more general.

THEOREM 8. *Let (L, τ) be a topological lattice with τ a regular, agreeable topology. If each $x \in L$ has a complete neighborhood, then $\tau = \text{CO}$. (A subset is complete if every increasing net in the subset has a sup in the subset, and dually).*

Proof. By Proposition 3, the CO topology is finer than τ .

Conversely, let U be a basic open convex set in the CO topology. If $U \notin \tau$, then there exists x in U and a net $\{x_\alpha\}$ converging to x in (L, τ) such that $x_\alpha \notin U$ for all α .

Let N be a complete neighborhood of x in τ . Let D be the set of all sequences $\{W_n: n = 1, 2, \dots\}$ satisfying for all n ,

- (i) $x \in W_n^\circ, W_n = W_n^* \subset N$
- (ii) $(W_n \vee W_{n+1}) \cup (W_n \wedge W_{n+1}) \subset W_{n+1}^\circ$.

If $\{W_n\}, \{V_n\} \in D$, we define $\{W_n\} \geq \{V_n\}$ if $W_n \subset V_n$ for all n . It is straightforward to verify that (D, \leq) is a directed set. If $\{W_n\} \in D$, let $W = \bigcap W_n$. Condition (i) implies $x \in W \subset N$ and W is closed. Condition (ii) implies W is a sublattice. Since τ is agreeable, N is complete, and W is closed, W has a largest element w^+ and a smallest element w^- .

If V is an closed neighborhood of x contained in N , then employing the regularity of τ and the continuity of \vee and \wedge , one can construct $\{V_n\} \in D$ such that $V = V_1$. Hence $v^+ \in \bigcap V_n \subset V$. Thus the net $\{w^+: \{W_n\} \in D\}$ is a monotonic decreasing net which converges to x in the τ -topology. It follows from the continuity of the lattice operations that $\{w^+\} \downarrow x$. Dually $\{w^-\} \uparrow x$. Hence residually many of the $\{w^+\}$ and $\{w^-\}$ are in U . Fix $\{W_n\} \in D$ such that $w^+, w^- \in U$.

For each n , pick $x_n \in \{x_\alpha\} \cap W_n$. If $m > n$, then

$$\begin{aligned} \bigvee_{k=n}^m x_k &\in \bigvee_{k=n}^m W_k \subset \left(\bigvee_{k=n}^{m-2} W_k\right) \vee W_{m-1} \vee W_{m-1} \\ &\subset \left(\bigvee_{k=n}^{m-3} W_k\right) \vee W_{m-2} \vee W_{m-2} \subset \dots \subset W_{n-1}. \end{aligned}$$

Thus for all $m > n, y_m = \bigvee_{k=n}^m x_k \in W_{n-1}$. Since $W_{n-1} \subset N, W_{n-1}$ is closed, N is complete, and the sequence y_m is monotonic increasing,

there exists $a_n \in W_{n-1}$ such that $a_n = \sup\{x_k: k \geq n\}$. The sequence a_n is a decreasing sequence contained in N , and hence converges to $a = \inf\{a_n\}$. Since the sequence $\{a_n\}$ is eventually in each W_n and each W_n is closed, we conclude $a \in W = \bigcap W_n$. Hence $a \leq w^+$.

Dually let $b_n = \inf\{x_k: k \geq n\}$ and $b = \sup\{b_n\}$. Then $w^- \leq b$. Since $b_n \leq a_n$ for all n , $w^- \leq b \leq a \leq w^+$. Since U is convex, $a, b \in U$. Since $a_n \downarrow a$ and $b_n \uparrow b$ and $a, b \in U$, there exists m such that $a_m, b_m \in U$. Since $b_m \leq x_m \leq a_m$, we have $x_m \in U$. However, this is in contradiction to $x_m \in \{x_\alpha\}$ and $x_\alpha \notin U$ for all α .

The next lemma is a standard and easily proved result about topological lattices (see [7] or [13]).

LEMMA 9. *Let K be a compact subset of a topological lattice. If $\{x_\alpha\}$ is a monotonically increasing (decreasing) net in K , then the net converges to its sup (inf).*

THEOREM 10. *Let L be a topological lattice which is (i) compact or (ii) locally compact and connected. Then L has the convex order topology.*

Proof. If L is compact, it is well known via the work of Nachbin [10] that L is locally convex. This fact together with Lemma 9 implies the topology on L is agreeable and L is complete. The conclusion then follows from Theorem 8.

If L is locally compact and connected, Anderson has shown L is locally convex [1]. Suppose $u_\alpha \downarrow x$. Let U be a compact neighborhood of x . Since $[x, u_\alpha] = (L \wedge u_\alpha) \vee x$ is connected, if u_α is not residually in U , then cofinally there exists y_α in the boundary of U such that $x \leq y_\alpha \leq u_\alpha$. By compactness of U , we can assume by picking subnets if necessary that $\{y_\alpha\}$ converges to some y in the boundary of U .

Fix some α . If $\beta > \alpha$, then $y_\beta \leq u_\beta \leq u_\alpha$. Thus $y_\beta \wedge u_\alpha = y_\beta$ for all $\beta > \alpha$ for which y_β is defined. Since $y_\beta \wedge u_\alpha$ converges to $y \wedge u_\alpha$, we have $y \wedge u_\alpha = y$, i.e., $y \leq u_\alpha$ for all u_α not in U . Since $x = \inf\{u_\alpha\}$, $y \leq x$. Similarly, since each $y_\alpha \geq x$, by continuity of \wedge , $y \geq x$. Hence $y = x$. But this is impossible since x is not in the boundary of U . Thus we conclude the topology of L is agreeable. Since L is locally compact, Lemma 9 implies each point has a complete neighborhood. Hence by Theorem 8, L has the convex order topology.

It is a consequence of the preceding theorem that a lattice admits at most one topology for which it is a compact (or locally compact connected) topological lattice, namely the convex order topology. This theorem also allows a nice algebraic condition for continuity of homomorphisms between compact (or locally compact connected) topological lattices. It follows that any isomorphism between such lattices is a

homeomorphism.

PROPOSITION 11. *Let L and K be lattices, f a homomorphism from L into K . If $u_\alpha \downarrow x (t_\alpha \uparrow x)$ implies $f(u_\alpha) \downarrow f(x) (f(t_\alpha) \uparrow f(x))$, then f is continuous if L and K are given the convex order topologies.*

Proof. Let U be a basic convex, open set in K . Then $f^{-1}(U)$ is convex in L . Suppose $x \in f^{-1}(U)$ and $\{x_\alpha\}$ order converges to x . Then there exists $u_\alpha \downarrow x, t_\alpha \uparrow x$ such that for all $\alpha, u_\alpha \geq x_\alpha \geq t_\alpha$. Then $f(u_\alpha) \geq f(x_\alpha) \geq f(t_\alpha)$ and by hypothesis $f(u_\alpha) \downarrow f(x)$ and $f(t_\alpha) \uparrow f(x)$. Hence since U is open $f(x_\alpha)$ is eventually in U . Thus x_α is eventually in $f^{-1}(U)$. Hence $f^{-1}(U)$ is open and f is continuous.

It is shown in [13] that if (L, τ) is a topological lattice for which τ is a first countable regular topology for which every point has a σ -complete neighborhood, then τ is finer than the order topology. If further, τ is agreeable, Propositions 2 and 3 show τ is the order topology. Since in the proof of Theorem 10, it was shown that the topology of a locally compact connected or a compact topological lattice is agreeable, it follows that

THEOREM 12. *Let L be a compact or locally compact connected topological lattice which is metrizable. Then L has the order topology.*

The theorem for the compact case appears in [7] and [13]. It is not known whether the theorem remains true without metrizability.

4. Compact semilattices. In this section we give an internal characterization of the topology of a compact semilattice. If S is a semilattice we say I is an *ideal* of S if $L(I) = I$. If A is an ideal in S , define A^+ by $x \in A^+$ if there exists a net x_α in A such that $x_\alpha \uparrow x$.

THEOREM 13. *Let S be a compact topological semilattice. An ideal A of S is closed if and only if $A = A^+$.*

Proof. Suppose A is closed. If $x \in A$, then the constant net x is a monotonic increasing net increasing to x . Hence $A \subset A^+$. If x_α is a net in A and $x_\alpha \uparrow x$, then x_α converges to x in the topology of S (a monotonically increasing net converges to its sup in a compact topological semilattice). Hence $x \in A$. Thus $A = A^+$.

Conversely let $A = A^+$. Let $y \in A^*$. Let D be the set of all sequences $\{W_n: n = 1, 2, \dots, \}$ satisfying for all n ,

- (i) $x \in W_n^\circ, W_n = W_n^*$
- (ii) $W_n \wedge W_n \subset W_{n-1}^\circ$.

If $\{W_n\}, \{V_n\} \in D$, we define $\{W_n\} \geq \{V_n\}$ if $W_n \subset V_n$ for all n . Then (D, \leq) is a directed set. If $\{W_n\} \in D$, let $W = \bigcap W_n$. Then W is closed and is a subsemilattice. Hence W has a minimal element w^- . As in the proof of Theorem 8, $\{w^-: \{W_n\} \in D\}$ is a monotonically increasing net and $w^- \uparrow y$.

Fix a specific w^- associated with a $\{W_n\}$. Since $y \in A^*$, for each n there exists $b_n \in W_n \cap A$. Let $\partial_n = \bigwedge_{m > n} b_m$. Then ∂_n is an increasing sequence, each $\partial_n \in A$ since A is an ideal, and as in the proof of Theorem 8, $\partial_n \uparrow \partial \in W$. Since $A = A^+$, $\partial \in A$. Since $w^- \leq \partial$ and A is an ideal, $w^- \in A$. But since the net $\{w^-\} \uparrow y$, we conclude $y \in A$. Hence A is closed.

Theorem 13 makes possible an algebraic description of the closure of an ideal in a compact topological semilattice.

COROLLARY 14. *Let I be an ideal of a compact topological semilattice S . Then $I^* = I^{++}$.*

Proof. Since $I \subset I^*$, we have $I^+ \subset (I^*)^+$. By Theorem 13, $(I^*)^+ = I^*$. Hence $I^+ \subset I^*$. A repetition of the argument with I^+ replacing I shows $I^{++} \subset I^*$.

Let $y \in I^+$ and $x \leq y$. Then there exists a net $\{y_\alpha\}$ in I such that $y_\alpha \uparrow y$. Then $x \wedge y_\alpha \uparrow x$ and $x \wedge y_\alpha \in I$ for all α . Thus $x \in I^+$; hence we have shown I^+ is an ideal. It is essentially shown in the proof of Theorem 13 that if $y \in I^*$, then $y \in (L(I^+))^+$. Since I^+ is an ideal $L(I^+) = I^+$. Thus $y \in I^{++}$. Hence $I^{++} = I^*$.

A principal application of Theorem 13 is an algebraic or intrinsic method of defining the topology of a compact topological semilattice. It is known that if S is a compact topological semilattice, then the space of all closed ideals S' of S ordered by inclusion and considered as a subspaces of 2^S is a compact distributive topological lattice; furthermore the mapping sending s into $L(s)$ is a topological isomorphism from S into S' (see e.g. [8, Theorem 1.2]). Since the closed ideals of S can be identified algebraically as those ideals for which $I = I^+$ and since the topology of S' can be defined algebraically as the convex-order topology (Theorem 10), the topology of S is determined by its algebraic structure.

THEOREM 15. *Let f be a homomorphism from a compact topological semilattice S onto a compact topological semilattice T . If f has the property that for $x_\alpha \uparrow x, f(x_\alpha) \uparrow f(x)$ and for $y_\alpha \downarrow y, f(y_\alpha) \downarrow f(y)$, then f is continuous.*

The proof of this theorem breaks down conveniently into several steps.

(i) If $t \in T$, $f^{-1}(t)$ has a least element. Since f is a homomorphism $f^{-1}(t)$ is a semilattice. Hence it is a monotonically decreasing net indexed by itself. Since S is compact, the net monotonically decreases to some s . Hence by hypothesis $f(s) = t$. Thus s is a least element for $f^{-1}(t)$.

(ii) If A is an ideal, $f(A)^+ = f(A^+)$. Suppose $y \in f(A)^+$. Then there exists a net $y_\alpha \uparrow y$ where $y_\alpha \in f(A)$ for all α . There exists $w_\alpha \in A$ such that $f(w_\alpha) = y_\alpha$ for each α . There exists x_α , the least element of $f^{-1}(y_\alpha)$; hence $x_\alpha \leq w_\alpha$. Since A is an ideal, $x_\alpha \in A$. If $\alpha \leq \beta$, then $f(x_\alpha \wedge x_\beta) = f(x_\alpha) \wedge f(x_\beta) = y_\alpha \wedge y_\beta = y_\alpha$; hence $x_\alpha \wedge x_\beta \in f^{-1}(y_\alpha)$. Since x_α is the least element of $f^{-1}(y_\alpha)$, $x_\alpha = x_\alpha \wedge x_\beta$. Hence the net x_α is increasing. Since S is compact, $x_\alpha \uparrow x$ for some $x \in A^+$. By hypothesis $f(x_\alpha) \uparrow f(x)$, i.e., $y_\alpha \uparrow f(x)$. Thus $p = f(x) \in f(A^+)$. Conversely, let $t = f(s) \in f(A^+)$. Then there exists a net $s_\alpha \uparrow s$, $s_\alpha \in A$ for each α . By hypothesis $f(s_\alpha) \uparrow f(s)$. Hence $t \in f(A)^+$. Thus $f(A)^+ = f(A^+)$.

(iii) f induces a homomorphism $f': S' \rightarrow T'$, the lattices of closed ideals of S and T resp. If A is a closed ideal of S , define $f'(A)$ to be $f(A)$. Since f is onto, $f(A)$ is an ideal. Also $f(A)^+ = f(A^+) = f(A)$; hence $f(A)$ is closed, i.e., $f'(A) \in T'$. Always $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) \subset f(A) \cap f(B)$. Suppose $t \in f(A) \cap f(B)$; then there exists $a \in A, b \in B$ such that $f(a) = t = f(b)$. Let x be the least element of $f^{-1}(t)$; then $x \leq a, x \leq b$. If A and B are ideals, then $x \in A \cap B$. Hence $t = f(x) \in f(A \cap B)$. Thus $f(A \cap B) = f(A) \cap f(B)$.

(iv) f' preserves limits of increasing and decreasing nets.

In S' and T' the limit of a decreasing net is just the intersection. An argument similar to the one just given to show f' preserves finite intersections will show f' also preserves arbitrary intersections. If $\{A_\alpha\}$ is an increasing net in S' , then the limit is $(\cup A_\alpha)^*$ and the limit of $f(A_\alpha)$ is $(\cup f(A_\alpha))^*$. Now $f((\cup A_\alpha)^*) = f((\cup A_\alpha)^{++}) = (f \cup A_\alpha)^{++}$ (by two applications of (ii)) $= (\cup f(A_\alpha))^{++} = (\cup f(A_\alpha))^*$. Hence f' preserves limits.

(v) The homomorphism f is continuous. Theorems 10 and 11 imply that f' is continuous. Since S and T are embedded in S' and T' , f' restricted to their images is continuous. But this restriction of f' is just f .

COROLLARY 16. *Let h be an isomorphism from a compact topological semilattice S onto a compact topological semilattice T . Then h is a homeomorphism. Hence a fixed semilattice admits at most one topology for which it is a compact topological semilattice.*

Proof. Clearly h and h^{-1} preserve limits of increasing and decreasing nets. Hence the conclusion follows from Theorem 15.

For any two compact topologies, the identity mapping must be a

homeomorphism. Hence the two agree.

Anderson and Hunter [2] have studied some classes of groups and semigroups in which each automorphism is continuous; this property they call van der Waerden property. Corollary 16 shows that compact semilattices are such semigroups.

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