

# Pacific Journal of Mathematics

**$G_\delta$ -DIAGONALS AND METRIZATION THEOREMS**

WILLIAM GEORGE MCARTHUR

## $G_\delta$ -DIAGONALS AND METRIZATION THEOREMS

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**The topological space  $X$  is said to have a  $G_\delta$ -diagonal if the diagonal  $\Delta = \{(x, x) : x \in X\}$  is a  $G_\delta$ -set in  $X \times X$ . It is easy to see that if  $X$  has a coarser metrizable topology, then  $X$  has a  $G_\delta$ -diagonal. The main result is that a completely regular pseudocompact space with a regular  $G_\delta$ -diagonal is metrizable.**

A considerable amount of research has been done on the question of what topological properties imply metrizability in the presence of a  $G_\delta$ -diagonal. For example, it is well-known that the existence of a  $G_\delta$ -diagonal is sufficient for metrizability in any of the following classes of spaces:

compact Hausdorff spaces  
linearly ordered spaces  
paracompact  $p$ -spaces.

A question still open is whether a countably compact regular space with a  $G_\delta$ -diagonal must be metrizable. A space  $X$  is said to have a *regular  $G_\delta$ -diagonal* if the diagonal  $\Delta$  is the intersection of countably many closures of open subsets of  $X \times X$  (see [5]). It is known that a countably compact space with a regular  $G_\delta$ -diagonal is metrizable [1].

### 2. The main result.

**DEFINITION 2.1.** A space  $X$  is *pseudocompact* if every real-valued continuous function on  $X$  is bounded.

Pseudocompact spaces were first defined and investigated by Hewitt in [3]. The following characterization of completely regular pseudocompact spaces may be found in [2], page 134.

**LEMMA 2.2.** *Let  $X$  be a completely regular space.  $X$  is pseudocompact if and only if for every sequence  $G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$  of nonvoid open subsets of  $X$ ,  $\bigcap_{n=1}^{\infty} \text{cl}_X(G_n) \neq \emptyset$ .*

**LEMMA 2.3.** *Let  $X$  be a completely regular pseudocompact space. Suppose  $G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$  is a sequence of open sets such that*

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \text{cl}_X(G_n) = \{x\}$$

*for a point  $x$  of  $X$ . Then the sets  $G_n$  form a local neighborhood*

base at  $x$ .

*Proof.* Let  $G$  be an open set containing  $x$ . Suppose

$$G_n \cap (X - G) \neq \emptyset$$

for every  $n$ . Choose  $H$  open such that  $x \in H \subset \text{cl}_X(H) \subset G$ . Then,  $(G_n \cap (X - \text{cl}_X(H)))_{n=1}^\infty$  is a decreasing sequence of nonvoid open subsets of  $X$ . Thus, by Lemma 2.2, there is a point  $p$  of  $X$  such that  $p \in \bigcap_{n=1}^\infty \text{cl}_X(G_n \cap (X - \text{cl}_X(H)))$ . But,  $p$  belongs to  $\bigcap_{n=1}^\infty \text{cl}_X G_n$ , a contradiction! Therefore, there must be an integer  $n$  such that  $G_n \subset G$ !!

DEFINITION 2.4. Let  $\mathcal{S}$  be an open cover of  $X$ ,  $x \in X$ , and  $H \subset X$ . Then,

$$\text{st}(x, \mathcal{S}) = \bigcup \{G \in \mathcal{S} : x \in G\}$$

$$\text{st}(R, \mathcal{S}) = \bigcup \{G \in \mathcal{S} : G \cap H \neq \emptyset\}.$$

The following result was announced by Moore in [4].

LEMMA 2.5. (*Moore's metrization theorem*) A topological space is metrizable if

- (1)  $X$  is Hausdorff, and
- (2) There is a decreasing sequence  $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \subset \mathcal{S}_n \supset \dots$  of open covers of  $X$  such that for every  $x$  in  $X$ , the sets  $\text{st}(\text{st}(x, \mathcal{S}_n), \mathcal{S}_n)$  for  $n = 1, 2, 3, \dots$  form a local neighborhood base at  $x$ .

Our main result appears below.

THEOREM 2.6. Let  $X$  be a completely regular pseudocompact space. If  $X$  has a regular  $G_\delta$ -diagonal, then  $X$  is metrizable.

*Proof.*  $\Delta = \{(x, x) : x \in X\}$ . Then, there is a decreasing sequence  $G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$  of open subsets of  $X \times X$  such that

$$\Delta = \bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty \text{cl}_{X \times X}(G_n).$$

For each  $x$  in  $X$ , choose a sequence  $(g_n(x))$  of open subsets of  $X$  such that  $(x, x) \in g_n(x) \times g_n(x) \subset G_n$  for each  $n$ . Then, for each  $n$  let

$$\mathcal{S}_n = \bigcup_{k \geq n} \{g_k(x) : x \in X\}.$$

Then,  $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \supset \mathcal{S}_n \supset \dots$  is a decreasing sequence of open covers of  $X$ .

- (i) For  $x$  in  $X$ ,  $\bigcap_{n=1}^\infty \text{cl}_X(\text{st}(x, \mathcal{S}_n)) = \{x\}$ . Let  $y \neq x$ . Then,

there is an integer  $n$  such that  $(x, y) \notin \text{cl}_{X \times X}(G_m)$  for  $m \geq n$ . Then, there are neighborhoods  $U$  and  $V$  of  $x$  and  $y$  respectively such that  $(U \times V) \cap G_m = \emptyset$  for  $m \geq n$ . Suppose that  $V \cap \text{st}(x, \mathcal{E}_n) \neq \emptyset$ . Then, there is an integer  $k \geq n$  and a point  $z$  of  $X$  such that  $x$  is in  $g_k(z)$  and  $V \cap g_k(z) \neq \emptyset$ . Then,

$$\emptyset = (U \times V) \cap G_k \supset (U \times V) \cap (g_k(z) \times g_k(z)) \neq \emptyset .$$

Contradiction! Thus, it must be that

$$V \cap \text{st}(x, \mathcal{E}_n) = \emptyset \quad \text{and} \quad y \notin \text{cl}_X(\text{st}(x, \mathcal{E}_n)) .$$

(ii) We conclude by Lemma 2.3 that  $(\text{st}(x, \mathcal{E}_n))$  forms a local base at  $x$ , for each  $x$  in  $X$ .

(iii) For  $x$  in  $X$ ,  $\bigcap_{n=1}^\infty \text{cl}_X(\text{st}(\text{st}(x, \mathcal{E}_n), \mathcal{E}_n)) = \{x\}$ . Let  $y \neq x$ . Then, there is an integer  $n$  such that  $m \geq n$  implies that

$$(x, y) \notin \text{cl}_{X \times X}(G_m) .$$

Then, there are neighborhoods  $U$  and  $V$  of  $x$  and  $y$  respectively such that  $(U \times V) \cap G_m = \emptyset$  for  $m \geq n$ . There are integers  $k$  and  $j$  such that  $\text{st}(x, \mathcal{E}_k) \subset U$  and  $\text{st}(y, \mathcal{E}_j) \subset V$ . Let  $m = \max\{n, k, j\}$ . Then,  $(\text{st}(x, \mathcal{E}_m) \times \text{st}(y, \mathcal{E}_m)) \cap G_m \subset (U \times V) \cap G_m = \emptyset$ . Suppose

$$\text{st}(y, \mathcal{E}_m) \cap \text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m) \neq \emptyset .$$

Then, there is an integer  $k \geq m$  and a point  $z$  of  $X$  such that

$$g_k(z) \cap \text{st}(x, \mathcal{E}_m) \neq \emptyset$$

and  $\text{st}(y, \mathcal{E}_m) \cap g_k(z) \neq \emptyset$ . Then,

$$(\text{st}(x, \mathcal{E}_m) \times \text{st}(y, \mathcal{E}_m)) \cap (g_k(z) \times g_k(z)) \neq \emptyset .$$

Contradiction! Thus, it must be that  $\text{st}(y, \mathcal{E}_m) \cap \text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m) = \emptyset$  and hence  $y \notin \text{cl}_X(\text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m))$ .

(iv) We conclude by Lemma 2.3 that  $(\text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m))$  forms a local base at  $x$ , for each  $x$  in  $X$ .

(v) By Moore's Metrization Theorem (Lemma 2.5),  $X$  is metrizable!!

**COROLLARY 2.7.** *If  $X$  is a completely regular pseudocompact space with a coarser metric topology, then  $X$  is metrizable.*

*Proof.* If  $X$  has a coarser metric topology, so does  $X \times X$ !!

**EXAMPLE 2.8.** The space  $E \cap [0, 1]$  of [2], problem 3J is pseudocompact, Hausdorff, and has a coarser metric topology. Since the space is not completely regular, it is not metrizable.

**EXAMPLE 2.9.** The space  $\mathcal{P}$  of [2], Problem 5I is pseudocompact, completely regular, and the diagonal in  $\mathcal{P} \times \mathcal{P}$  is a  $G_\delta$ -set. But,  $\mathcal{P}$  is not metrizable.

### 3. Some remarks on the countably compact case.

**DEFINITION 3.1.** A space  $X$  is countably compact if every countable family of closed sets with the finite intersection property has nonempty intersection.

**PROPOSITION 3.2.** *If  $X$  is countably compact, regular, with a  $G_\delta$ -diagonal, then  $X$  is first countable.*

*Proof.* Suppose  $\Delta = \bigcap_n G_n$  where the sets  $G_1, G_2, \dots, G_n, \dots$  are open subsets of  $X \times X$ . For  $x$  in  $X$ , choose a sequence  $(g_n(x))$  of open subsets of  $X$  which contain  $x$  such that for each  $n$ ,

$$\text{cl}_X(g_{n+1}(x)) \subset g_n(x) \quad \text{and} \quad g_n(x) \times g_n(x) \subset G_n.$$

Note that  $\bigcap_{n=1}^\infty \text{cl}_X(g_n(x)) = \{x\}$ . Now, suppose  $G$  is an open subset of  $X$  which contains  $x$ . If it is true that no set  $g_n(x)$  is contained in  $G$ , then  $(\text{cl}_X(g_n(x)) \cap (X-G))_n$  is a countable collection of closed sets with the finite intersection property. Thus, since  $X$  is countably compact,  $(\bigcap_{n=1}^\infty \text{cl}_X(g_n(x))) \cap (X-G) \neq \emptyset$ . Contradiction! Hence, there must exist an integer  $n$  such that  $g_n(x) \subset G$ . This shows that  $(g_n(x))_n$  forms a neighborhood base at  $x$  and hence  $X$  is first countable.!!

**PROPOSITION 3.3.** *If  $X$  is countably compact, regular, with a  $G_\delta$ -diagonal, then  $X \times X$  is countably compact, regular, and has a  $G_\delta$ -diagonal.*

*Proof.* It is well-known that regularity is productive and that countable compactness is countably productive in the presence of first countability. Now, suppose that  $\Delta = \bigcap_n G_n$  with the sets  $G_n$  open in  $X \times X$ . Let  $\Delta' = \{((x, y), (x, y)): x, y \in X\}$ . For each  $n$ , let

$$g_n(x, y) = g_n(x) \times g_n(y)$$

where the sets  $g_n(x)$  are as in Proposition 3.2. Let

$$H_n = \bigcup_{(x, y) \in X \times X} (g_n(x, y) \times g_n(x, y)).$$

**Claim:**  $\Delta' = \bigcap_{n=1}^\infty H_n$ . Clearly,  $\Delta' \subset \bigcap_{n=1}^\infty H_n$ . Suppose  $(x_1, y_1) \neq (x_2, y_2)$ .

*Case I.*  $x_1 \neq x_2$ . Then, there is an integer  $n$  such that

$(x_1, x_2) \notin G_n$ . Suppose  $((x_1, y_1), (x_2, y_2)) \in H_n$ . Then, there is a pair  $(x, y)$  in  $X \times X$  such that  $(x_1, y_1) \in g_n(x, y)$  and  $(x_2, y_2) \in g_n(x, y)$ . Then,  $x_1 \in g_n(x)$  and  $x_2 \in g_n(x)$  which implies that  $(x_1, x_2) \in G_n$ . Contradiction!

*Case II.*  $y_1 \neq y_2$ . Similar argument to that of Case I. Thus,  $X \times X$  has a  $G_\delta$ -diagonal.!!

**PROPOSITION 3.4.** *Every countably compact, regular, space with a  $G_\delta$ -diagonal is metrizable if and only if every countably compact, regular, space with a  $G_\delta$ -diagonal is normal.*

*Proof.* If  $X$  has a  $G_\delta$ -diagonal and  $X \times X$  is normal, then  $X$  has a regular  $G_\delta$ -diagonal.!!

The author would like to thank Professor Robert Heath of the University of Pittsburgh for his many enlightening remarks on this subject matter.

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Received October 29, 1971.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.



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