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G_δ -DIAGONALS AND METRIZATION THEOREMS

WILLIAM GEORGE MCARTHUR

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The topological space X is said to have a G_δ -diagonal if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a G_δ -set in $X \times X$. It is easy to see that if X has a coarser metrizable topology, then X has a G_δ -diagonal. The main result is that a completely regular pseudocompact space with a regular G_δ -diagonal is metrizable.

A considerable amount of research has been done on the question of what topological properties imply metrizability in the presence of a G_δ -diagonal. For example, it is well-known that the existence of a G_δ -diagonal is sufficient for metrizability in any of the following classes of spaces:

- compact Hausdorff spaces
- linearly ordered spaces
- paracompact p -spaces.

A question still open is whether a countably compact regular space with a G_δ -diagonal must be metrizable. A space X is said to have a *regular G_δ -diagonal* if the diagonal Δ is the intersection of countably many closures of open subsets of $X \times X$ (see [5]). It is known that a countably compact space with a regular G_δ -diagonal is metrizable [1].

2. The main result.

DEFINITION 2.1. A space X is *pseudocompact* if every real-valued continuous function on X is bounded.

Pseudocompact spaces were first defined and investigated by Hewitt in [3]. The following characterization of completely regular pseudocompact spaces may be found in [2], page 134.

LEMMA 2.2. *Let X be a completely regular space. X is pseudocompact if and only if for every sequence $G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$ of nonvoid open subsets of X , $\bigcap_{n=1}^{\infty} \text{cl}_X(G_n) \neq \emptyset$.*

LEMMA 2.3. *Let X be a completely regular pseudocompact space. Suppose $G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$ is a sequence of open sets such that*

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \text{cl}_X(G_n) = \{x\}$$

for a point x of X . Then the sets G_n form a local neighborhood

base at x .

Proof. Let G be an open set containing x . Suppose

$$G_n \cap (X - G) \neq \emptyset$$

for every n . Choose H open such that $x \in H \subset \text{cl}_X(H) \subset G$. Then, $(G_n \cap (X - \text{cl}_X(H)))_{n=1}^\infty$ is a decreasing sequence of nonvoid open subsets of X . Thus, by Lemma 2.2, there is a point p of X such that $p \in \bigcap_{n=1}^\infty \text{cl}_X(G_n \cap (X - \text{cl}_X(H)))$. But, p belongs to $\bigcap_{n=1}^\infty \text{cl}_X G_n$, a contradiction! Therefore, there must be an integer n such that $G_n \subset G$.!!

DEFINITION 2.4. Let \mathcal{S} be an open cover of X , $x \in X$, and $H \subset X$. Then,

$$\text{st}(x, \mathcal{S}) = \bigcup \{G \in \mathcal{S} : x \in G\}$$

$$\text{st}(R, \mathcal{S}) = \bigcup \{G \in \mathcal{S} : G \cap H \neq \emptyset\}.$$

The following result was announced by Moore in [4].

LEMMA 2.5. (*Moore's metrization theorem*) A topological space is metrizable if

- (1) X is Hausdorff, and
- (2) There is a decreasing sequence $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \subset \mathcal{S}_n \supset \dots$ of open covers of X such that for every x in X , the sets $\text{st}(\text{st}(x, \mathcal{S}_n), \mathcal{S}_n)$ for $n = 1, 2, 3, \dots$ form a local neighborhood base at x .

Our main result appears below.

THEOREM 2.6. Let X be a completely regular pseudocompact space. If X has a regular G_δ -diagonal, then X is metrizable.

Proof. $\Delta = \{(x, x) : x \in X\}$. Then, there is a decreasing sequence $G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$ of open subsets of $X \times X$ such that

$$\Delta = \bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty \text{cl}_{X \times X}(G_n).$$

For each x in X , choose a sequence $(g_n(x))$ of open subsets of X such that $(x, x) \in g_n(x) \times g_n(x) \subset G_n$ for each n . Then, for each n let

$$\mathcal{S}_n = \bigcup_{k \geq n} \{g_k(x) : x \in X\}.$$

Then, $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \supset \mathcal{S}_n \supset \dots$ is a decreasing sequence of open covers of X .

- (i) For x in X , $\bigcap_{n=1}^\infty \text{cl}_X(\text{st}(x, \mathcal{S}_n)) = \{x\}$. Let $y \neq x$. Then,

there is an integer n such that $(x, y) \notin \text{cl}_{X \times X}(G_m)$ for $m \geq n$. Then, there are neighborhoods U and V of x and y respectively such that $(U \times V) \cap G_m = \emptyset$ for $m \geq n$. Suppose that $V \cap \text{st}(x, \mathcal{E}_n) \neq \emptyset$. Then, there is an integer $k \geq n$ and a point z of X such that x is in $g_k(z)$ and $V \cap g_k(z) \neq \emptyset$. Then,

$$\emptyset = (U \times V) \cap G_k \supset (U \times V) \cap (g_k(z) \times g_k(z)) \neq \emptyset .$$

Contradiction! Thus, it must be that

$$V \cap \text{st}(x, \mathcal{E}_n) = \emptyset \quad \text{and} \quad y \notin \text{cl}_X(\text{st}(x, \mathcal{E}_n)) .$$

(ii) We conclude by Lemma 2.3 that $(\text{st}(x, \mathcal{E}_n))$ forms a local base at x , for each x in X .

(iii) For x in X , $\bigcap_{n=1}^\infty \text{cl}_X(\text{st}(\text{st}(x, \mathcal{E}_n), \mathcal{E}_n)) = \{x\}$. Let $y \neq x$. Then, there is an integer n such that $m \geq n$ implies that

$$(x, y) \notin \text{cl}_{X \times X}(G_m) .$$

Then, there are neighborhoods U and V of x and y respectively such that $(U \times V) \cap G_m = \emptyset$ for $m \geq n$. There are integers k and j such that $\text{st}(x, \mathcal{E}_k) \subset U$ and $\text{st}(y, \mathcal{E}_j) \subset V$. Let $m = \max\{n, k, j\}$. Then, $(\text{st}(x, \mathcal{E}_m) \times \text{st}(y, \mathcal{E}_m)) \cap G_m \subset (U \times V) \cap G_m = \emptyset$. Suppose

$$\text{st}(y, \mathcal{E}_m) \cap \text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m) \neq \emptyset .$$

Then, there is an integer $k \geq m$ and a point z of X such that

$$g_k(z) \cap \text{st}(x, \mathcal{E}_m) \neq \emptyset$$

and $\text{st}(y, \mathcal{E}_m) \cap g_k(z) \neq \emptyset$. Then,

$$(\text{st}(x, \mathcal{E}_m) \times \text{st}(y, \mathcal{E}_m)) \cap (g_k(z) \times g_k(z)) \neq \emptyset .$$

Contradiction! Thus, it must be that $\text{st}(y, \mathcal{E}_m) \cap \text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m) = \emptyset$ and hence $y \notin \text{cl}_X(\text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m))$.

(iv) We conclude by Lemma 2.3 that $(\text{st}(\text{st}(x, \mathcal{E}_m), \mathcal{E}_m))$ forms a local base at x , for each x in X .

(v) By Moore's Metrization Theorem (Lemma 2.5), X is metrizable!!

COROLLARY 2.7. *If X is a completely regular pseudocompact space with a coarser metric topology, then X is metrizable.*

Proof. If X has a coarser metric topology, so does $X \times X$!!

EXAMPLE 2.8. The space $E \cap [0, 1]$ of [2], problem 3J is pseudocompact, Hausdorff, and has a coarser metric topology. Since the space is not completely regular, it is not metrizable.

EXAMPLE 2.9. The space \mathcal{P} of [2], Problem 5I is pseudocompact, completely regular, and the diagonal in $\mathcal{P} \times \mathcal{P}$ is a G_δ -set. But, \mathcal{P} is not metrizable.

3. Some remarks on the countably compact case.

DEFINITION 3.1. A space X is countably compact if every countable family of closed sets with the finite intersection property has nonempty intersection.

PROPOSITION 3.2. *If X is countably compact, regular, with a G_δ -diagonal, then X is first countable.*

Proof. Suppose $\Delta = \bigcap_n G_n$ where the sets $G_1, G_2, \dots, G_n, \dots$ are open subsets of $X \times X$. For x in X , choose a sequence $(g_n(x))$ of open subsets of X which contain x such that for each n ,

$$\text{cl}_X(g_{n+1}(x)) \subset g_n(x) \quad \text{and} \quad g_n(x) \times g_n(x) \subset G_n.$$

Note that $\bigcap_{n=1}^{\infty} \text{cl}_X(g_n(x)) = \{x\}$. Now, suppose G is an open subset of X which contains x . If it is true that no set $g_n(x)$ is contained in G , then $(\text{cl}_X(g_n(x)) \cap (X-G))_n$ is a countable collection of closed sets with the finite intersection property. Thus, since X is countably compact, $(\bigcap_{n=1}^{\infty} \text{cl}_X(g_n(x))) \cap (X-G) \neq \emptyset$. Contradiction! Hence, there must exist an integer n such that $g_n(x) \subset G$. This shows that $(g_n(x))_n$ forms a neighborhood base at x and hence X is first countable.!!

PROPOSITION 3.3. *If X is countably compact, regular, with a G_δ -diagonal, then $X \times X$ is countably compact, regular, and has a G_δ -diagonal.*

Proof. It is well-known that regularity is productive and that countable compactness is countably productive in the presence of first countability. Now, suppose that $\Delta = \bigcap_n G_n$ with the sets G_n open in $X \times X$. Let $\Delta' = \{(x, y), (x, y) : x, y \in X\}$. For each n , let

$$g_n(x, y) = g_n(x) \times g_n(y)$$

where the sets $g_n(x)$ are as in Proposition 3.2. Let

$$H_n = \bigcup_{(x, y) \in X \times X} (g_n(x, y) \times g_n(x, y)).$$

Claim: $\Delta' = \bigcap_{n=1}^{\infty} H_n$. Clearly, $\Delta' \subset \bigcap_{n=1}^{\infty} H_n$. Suppose $(x_1, y_1) \neq (x_2, y_2)$.

Case I. $x_1 \neq x_2$. Then, there is an integer n such that

$(x_1, x_2) \notin G_n$. Suppose $((x_1, y_1), (x_2, y_2)) \in H_n$. Then, there is a pair (x, y) in $X \times X$ such that $(x_1, y_1) \in g_n(x, y)$ and $(x_2, y_2) \in g_n(x, y)$. Then, $x_1 \in g_n(x)$ and $x_2 \in g_n(x)$ which implies that $(x_1, x_2) \in G_n$. Contradiction!

Case II. $y_1 \neq y_2$. Similar argument to that of Case I. Thus, $X \times X$ has a G_δ -diagonal!!

PROPOSITION 3.4. *Every countably compact, regular, space with a G_δ -diagonal is metrizable if and only if every countably compact, regular, space with a G_δ -diagonal is normal.*

Proof. If X has a G_δ -diagonal and $X \times X$ is normal, then X has a regular G_δ -diagonal!!

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