$G_\delta$-DIAGONALS AND METRIZATION THEOREMS

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The topological space $X$ is said to have a $G_\delta$-diagonal if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a $G_\delta$-set in $X \times X$. It is easy to see that if $X$ has a coarser metrizable topology, then $X$ has a $G_\delta$-diagonal. The main result is that a completely regular pseudocompact space with a regular $G_\delta$-diagonal is metrizable.

A considerable amount of research has been done on the question of what topological properties imply metrizability in the presence of a $G_\delta$-diagonal. For example, it is well-known that the existence of a $G_\delta$-diagonal is sufficient for metrizability in any of the following classes of spaces:

- compact Hausdorff spaces
- linearly ordered spaces
- paracompact $p$-spaces.

A question still open is whether a countably compact regular space with a $G_\delta$-diagonal must be metrizable. A space $X$ is said to have a regular $G_\delta$-diagonal if the diagonal $\Delta$ is the intersection of countably many closures of open subsets of $X \times X$ (see [5]). It is known that a countably compact space with a regular $G_\delta$-diagonal is metrizable [1].

2. The main result.

**Definition 2.1.** A space $X$ is pseudocompact if every real-valued continuous function on $X$ is bounded.

Pseudocompact spaces were first defined and investigated by Hewitt in [3]. The following characterization of completely regular pseudocompact spaces may be found in [2], page 134.

**Lemma 2.2.** Let $X$ be a completely regular space. $X$ is pseudocompact if and only if for every sequence $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ of nonvoid open subsets of $X$, $\bigcap_{n=1}^{\infty} \overline{cl}_X(G_n) \neq \emptyset$.

**Lemma 2.3.** Let $X$ be a completely regular pseudocompact space. Suppose $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ is a sequence of open sets such that

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \overline{cl}_X(G_n) = \{x\}$$

for a point $x$ of $X$. Then the sets $G_n$ form a local neighborhood
base at x.

Proof. Let $G$ be an open set containing $x$. Suppose

$$G_n \cap (X - G) \neq \emptyset$$

for every $n$. Choose $H$ open such that $x \in H \subset \text{cl}_x(H) \subset G$. Then, $(G_n \cap (X - \text{cl}_x(H)))_{n=1}^\infty$ is a decreasing sequence of nonvoid open subsets of $X$. Thus, by Lemma 2.2, there is a point $p$ of $X$ such that $p \in \bigcap_{n=1}^\infty \text{cl}_x(G_n \cap (X - \text{cl}_x(H)))$. But, $p$ belongs to $\bigcap_{n=1}^\infty \text{cl}_x G_n$, a contradiction! Therefore, there must be an integer $n$ such that $G_n \subset G$.

**DEFINITION 2.4.** Let $\mathcal{G}$ be an open cover of $X$, $x \in X$, and $H \subset X$. Then,

$$\text{st}(x, \mathcal{G}) = \bigcup \{G \in \mathcal{G} : x \in G\}$$

$$\text{st}(R, \mathcal{G}) = \bigcup \{G \in \mathcal{G} : G \cap H \neq \emptyset\}.$$ 

The following result was announced by Moore in [4].

**LEMMA 2.5.** (Moore's metrization theorem) A topological space is metrizable if

1. $X$ is Hausdorff, and
2. There is a decreasing sequence $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_n \supset \cdots$ of open covers of $X$ such that for every $x$ in $X$, the sets $\text{st}(\text{st}(x, \mathcal{G}_n), \mathcal{G}_n)$ for $n = 1, 2, 3, \cdots$ form a local neighborhood base at $x$.

Our main result appears below.

**THEOREM 2.6.** Let $X$ be a completely regular pseudocompact space. If $X$ has a regular $G_\delta$-diagonal, then $X$ is metrizable.

Proof. $\Delta = \{(x, x) : x \in X\}$. Then, there is a decreasing sequence $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ of open subsets of $X \times X$ such that

$$\Delta = \bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty \text{cl}_{X \times X}(G_n).$$ 

For each $x$ in $X$, choose a sequence $(g_n(x))$ of open subsets of $X$ such that $(x, x) \in g_n(x) \times g_n(x) \subset G_n$ for each $n$. Then, for each $n$ let

$$\mathcal{G}_n = \bigcup_{k \geq n} \{g_k(x) : x \in X\}.$$ 

Then, $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_n \supset \cdots$ is a decreasing sequence of open covers of $X$.

(i) For $x$ in $X$, $\bigcap_{n=1}^\infty \text{cl}_x(\text{st}(x, \mathcal{G}_n)) = \{x\}$. Let $y \neq x$. Then,
there is an integer \( n \) such that \((x, y) \in \text{cl}_{XX} (G_m)\) for \( m \geq n \). Then, there are neighborhoods \( U \) and \( V \) of \( x \) and \( y \) respectively such that \((U \times V) \cap G_m = \emptyset\) for \( m \geq n \). Suppose that \( V \cap \text{st}(x, \mathcal{E}_n) \neq \emptyset \). Then, there is an integer \( k \geq n \) and a point \( z \) of \( X \) such that \( x \) is in \( g_k(z) \) and \( V \cap g_k(z) \neq \emptyset \). Then,
\[
\emptyset = (U \times V) \cap g_k \supset (U \times V) \cap (g_k(z) \times g_k(z)) \neq \emptyset .
\]
Contradiction! Thus, it must be that
\[
V \cap \text{st}(x, \mathcal{E}_n) = \emptyset \quad \text{and} \quad y \in \text{cl}_x (\text{st}(x, \mathcal{E}_n)) .
\]
(ii) We conclude by Lemma 2.3 that \((\text{st}(x, \mathcal{E}_n)) \) forms a local base at \( x \), for each \( x \) in \( X \).

(iii) For \( x \) in \( X \), \( \bigcap_{n=1}^{\infty} \text{cl}_x (\text{st}(x, \mathcal{E}_n), \mathcal{E}_n)) = \{x\} \). Let \( y \neq x \).
Then, there is an integer \( n \) such that \( m \geq n \) implies that
\[
(x, y) \in \text{cl}_{XX} (G_m) .
\]
Then, there are neighborhoods \( U \) and \( V \) of \( x \) and \( y \) respectively such that \((U \times V) \cap G_m = \emptyset\) for \( m \geq n \). There are integers \( k \) and \( j \) such that \( \text{st}(x, \mathcal{E}_k) \subset U \) and \( \text{st}(y, \mathcal{E}_j) \subset V \). Let \( m = \max \{n, k, j\} \). Then, \((\text{st}(x, \mathcal{E}_m) \times \text{st}(y, \mathcal{E}_m)) \cap G_m \subset (U \times V) \cap G_m = \emptyset\). Suppose
\[
\text{st}(y, \mathcal{E}_m) \cap \text{st}(x, \mathcal{E}_m, \mathcal{E}_m) \neq \emptyset .
\]
Then, there is an integer \( k \geq m \) and a point \( z \) of \( X \) such that
\[
g_k(z) \cap \text{st}(x, \mathcal{E}_m) \neq \emptyset
\]
and \( \text{st}(y, \mathcal{E}_m) \cap g_k(z) \neq \emptyset \). Then,
\[
(\text{st}(x, \mathcal{E}_m) \times \text{st}(y, \mathcal{E}_m)) \cap (g_k(z) \times g_k(z)) \neq \emptyset .
\]
Contradiction! Thus, it must be that \( \text{st}(y, \mathcal{E}_m) \cap \text{st}(x, \mathcal{E}_m, \mathcal{E}_m) = \emptyset \) and hence \( y \in \text{cl}_x (\text{st}(x, \mathcal{E}_m), \mathcal{E}_m)) \).

(iv) We conclude by Lemma 2.3 that \((\text{st}(x, \mathcal{E}_m) \cap \mathcal{E}_m)) \) forms a local base at \( x \), for each \( x \) in \( X \).

(v) By Moore's Metrization Theorem (Lemma 2.5), \( X \) is metrizable.!!

**Corollary 2.7.** If \( X \) is a completely regular pseudocompact space with a coarser metric topology, then \( X \) is metrizable.

**Proof.** If \( X \) has a coarser metric topology, so does \( X \times X \).!!

**Example 2.8.** The space \( E \cap [0, 1] \) of \([2]\), problem 3J is pseudocompact, Hausdorff, and has a coarser metric topology. Since the space is not completely regular, it is not metrizable.
EXAMPLE 2.9. The space $\Psi$ of [2], Problem 51 is pseudocompact, completely regular, and the diagonal in $\Psi \times \Psi$ is a $G_\delta$-set. But, $\Psi$ is not metrizable.

3. Some remarks on the countably compact case.

DEFINITION 3.1. A space $X$ is countably compact if every countable family of closed sets with the finite intersection property has nonempty intersection.

PROPOSITION 3.2. If $X$ is countably compact, regular, with a $G_\delta$-diagonal, then $X$ is first countable.

Proof. Suppose $A = \bigcap_n G_n$ where the sets $G_1, G_2, \ldots, G_n, \ldots$ are open subsets of $X \times X$. For $x$ in $X$, choose a sequence $(g_n(x))$ of open subsets of $X$ which contain $x$ such that for each $n$,

$$\text{cl}_X (g_{n+1}(x)) \subset g_n(x)$$

and

$$g_n(x) \times g_n(x) \subset G_n.$$  

Note that $\bigcap_{n=1}^\infty \text{cl}_X (g_n(x)) = \{x\}$. Now, suppose $G$ is an open subset of $X$ which contains $x$. If it is true that no set $g_n(x)$ is contained in $G$, then $(\text{cl}_X (g_n(x)) \cap (X-G))_n$ is a countable collection of closed sets with the finite intersection property. Thus, since $X$ is countably compact, $(\bigcap_{n=1}^\infty \text{cl}_X (g_n(x))) \cap (X-G) \neq \emptyset$. Contradiction! Hence, there must exist an integer $n$ such that $g_n(x) \subset G$. This shows that $(g_n(x))_n$ forms a neighborhood base at $x$ and hence $X$ is first countable.!!

PROPOSITION 3.3. If $X$ is countably compact, regular, with a $G_\delta$-diagonal, then $X \times X$ is countably compact, regular, and has a $G_\delta$-diagonal.

Proof. It is well-known that regularity is productive and that countable compactness is countably productive in the presence of first countability. Now, suppose that $A = \bigcap_n G_n$ with the sets $G_n$ open in $X \times X$. Let $A' = \{(x, y), (x, y) : x, y \in X\}$. For each $n$, let

$$g_n(x, y) = g_n(x) \times g_n(y)$$

where the sets $g_n(x)$ are as in Proposition 3.2. Let

$$H_n = \bigcup_{(x, y) \in X \times X} (g_n(x, y) \times g_n(x, y)).$$

Claim: $A' = \bigcap_{n=1}^\infty H_n$. Clearly, $A' \subset \bigcap_{n=1}^\infty H_n$. Suppose $(x_1, y_1) \neq (x_2, y_2)$.

Case I. $x_1 \neq x_2$. Then, there is an integer $n$ such that

$$\text{cl}_X (g_{n+1}(x_1)) \subset g_n(x_1)$$

and

$$g_n(x_1) \times g_n(x_1) \subset H_n.$$
Suppose \((x_1, y_1), (x_2, y_2) \in H_n\). Then, there is a pair \((x, y)\) in \(X \times X\) such that \((x_1, y_1) \in g_n(x, y)\) and \((x_2, y_2) \in g_n(x, y)\). Then, \(x_1 \in g_n(x)\) and \(x_2 \in g_n(x)\) which implies that \((x_1, x_2) \in G_n\). Contradiction!

Case II. \(y_1 \neq y_2\). Similar argument to that of Case I. Thus, \(X \times X\) has a \(G_\delta\)-diagonal.

PROPOSITION 3.4. Every countably compact, regular, space with a \(G_\delta\)-diagonal is metrizable if and only if every countably compact, regular, space with a \(G_\delta\)-diagonal is normal.

Proof. If \(X\) has a \(G_\delta\)-diagonal and \(X \times X\) is normal, then \(X\) has a regular \(G_\delta\)-diagonal.

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