

# Pacific Journal of Mathematics

**ACTIONS OF TORUS  $T^n$  ON  $(n + 1)$ -MANIFOLDS  $M^{n+1}$**

JINGYAL PAK

# ACTIONS OF TORUS $T^n$ ON $(n + 1)$ -MANIFOLDS $M^{n+1}$

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Let  $\xi$  be a principal  $T^l$ -bundle over a lens space  $L(p, q)$ . It is shown here that the total space of  $\xi$  can be identified with  $L(k, q) \times S_1^1 \times \cdots \times S_l^1$ , for some  $k \leq p$ . Let  $(T^n, M^{n+1})$  be an effective torus action on an orientable  $(n+1)$ -dimensional manifold. An elementary examination of the parity of dimensions of the slice  $S_x$  at  $x \in M$  and of the orbit  $T^n(x)$ , shows that the circle subgroups are the only possible stability groups on  $M^{n+1}$ . From these two results and the cross-sectioning theorem we can conclude that  $T^{n+1}$  and  $L(k, q) \times T^{n-2}$  are the only possible types of compact closed orientable  $(n + 1)$ -manifolds which allow  $T^n$  actions.

It is shown in [3] that  $T^4$  and  $L(p, q) \times S^1$  are the only compact closed orientable 4-manifolds which allow effective  $T^3$  actions. The purpose of this note is to show, using an argument similar to that of [3], that  $T^{n+1}$  and  $L(m, q) \times T^{n-2}$  are the only possible compact closed orientable  $(n + 1)$ -manifolds which allow effective  $T^n$  actions for  $n \geq 3$ . Here  $L(m, q)$  includes the case of  $S^2 \times S^1$  and  $S^3$ . The key lemma used in the proof of this theorem is that every principal  $T^l$ -bundle over the lens space  $L(p, q)$  can be identified with  $L(k, q) \times T^l$  for suitable  $k \leq p$ . In later papers we intend to work on  $T^n$  actions on compact closed non-orientable  $(n + 1)$ -manifolds  $M^{n+1}$  and  $(n + 2)$ -manifolds  $M^{n+2}$ .

Let  $G$ , a compact Lie group, act on a space  $X$ . If  $x \in X$ ,  $G_x = \{g \in G \mid g(x) = x\}$  will denote the stability group, or isotropy group of  $G$  at  $x \in X$ .  $G(x) = \{g(x) \mid g \in G\}$  will be called the orbit of  $x \in X$ . The orbit space, the set of all orbits, will be denoted by  $X/G = X^*$  or  $\bar{X}$  with the quotient topology, and the orbit map by  $\Pi: X \rightarrow X^*$ . For each  $x \in X$ , one can find a certain subset  $S_x$  called the slice at  $x$  [1, Chapter VIII], with the following properties:

- (i)  $S_x$  is invariant under  $G_x$ .
- (ii) If  $g \in G$ ,  $y, y' \in S_x$ , and  $g(y) = y'$ , then  $g \in G_x$ .
- (iii) There exists a "cell neighborhood"  $C$  of  $G/G_x$  such that

$C \times S_x$  is homeomorphic to a neighborhood of  $x$ . That is, if  $f: C \rightarrow G/G_x$  is a local cross-section in  $G/G_x$  then the map  $F: C \times S_x \rightarrow X$  defined by  $F(c, s) = f(c)s$  is a homeomorphism of  $C \times S_x$  onto an open set containing  $S_x$  in  $X$ . The principal orbits are those for which the stability groups are identity. An action is effective if  $g(x) = x$  for every  $x \in X$  implies  $g = e$ . We shall assume that  $G$  is acting smoothly and effectively on a smooth orientable manifold. By the slice theorem, given in [1, Chapter VIII], it follows that if  $T^n$  acts effectively on a

compact closed  $(n + 1)$ -manifold  $M^{n+1}$ , then there exist principal  $T^n$  orbits and the orbit space  $M/T^n = M^*$  is a compact 1-manifold which we denote by  $S^1$  or  $[0, 1]$ .

LEMMA 1. *Let  $(T^n, M^{n+1})$  be a transformation group. Then the circle subgroups of  $T^n$  are the only possible nontrivial stability groups on  $M^{n+1}$ .*

*Proof.* Let  $T^i \times F$ ,  $i = 1, \dots, n$ , be a subgroup of  $T^n$ , where  $T^i$  is  $i$ -dimensional torus subgroup of  $T^n$  and  $F$  is any finite subgroup of  $T^n$  complementary to  $T^i$ . We assume that if  $i = 1$ , then  $F$  is nontrivial.

First we show that no nontrivial finite subgroup  $F$  of  $T^n$  can be a stability group. If  $M^* = S^1$  then every point in  $M^*$  corresponds to a principal orbit, so that we don't have a finite group as a stability group. In any case, if we have a finite stability group  $F$  at  $x$ , then  $x$  is isolated. The orbit is  $n$ -dimensional and the slice is a 1-dimensional interval. Thus  $F$  must be  $Z_2$  which reverses the orientation (a contradiction, since  $M$  is orientable and  $T^n$  is connected).

Now consider the case of  $T^i \times F$ ,  $i = 1, \dots, n$ . The orbit will be  $(n - i)$ -dimensional, and there is an  $(n + 1) - (n - i) = (i + 1)$  dimensional disk slice on which  $T^i \times F$  must act as a rotation. But  $T^i \times F \not\subset SO(i + 1)$  for  $i = 1, \dots, n$ . Thus there is no point  $x \in M$  such that  $T_x^n = T^i \times F$  for  $i = 1, \dots, n$ . This also implies that the fixed point set  $F(T^n, M^{n+1}) = \emptyset$  for  $n > 1$ .

LEMMA 2. *Let  $(T^n, M^{n+1})$  be a transformation group. Then the orbit map  $\Pi: M^{n+1} \rightarrow M^*$  has a cross-section.*

*Proof.* If  $M^* = S^1$ , then the  $T^n$ -bundle is trivial. If  $M^* = [0, 1]$ , then the action corresponding over  $(0, 1)$  is the trivial  $T^n$ -bundle, so that we have a cross-section over  $(0, 1)$ . Now we can extend this cross-section trivially to both ends.

LEMMA 3. *If  $M^{n+1}$  is a principal  $T^{n-2}$ -bundle over  $L(p, q)$ ,  $n \geq 3$ , then  $M^{n+1}$  can be written as  $L(k, q) \times T^{n-2}$  for some integer  $k \leq p$ .*

*Proof.* By taking a circle subgroup  $T_1^1$  of  $T^{n-2}$  and the complementary subgroup  $T^{n-3}$  to  $T_1^1$  in  $T^{n-2}$ , we can consider  $M/T^{n-3}$  as a principal  $T_1^1$ -bundle over  $L(p, q)$ . Without loss of generality we can take  $T_1^1$  be the first factor of  $T^{n-2} = T^1 \times \dots \times T^1$ . But, this bundle is classified by  $[L(p, q), K(z, 2)] \cong Z_p$ , and (see [5]) for any element  $f_i \in [L(p, q), K(z, 2)]$ ,  $i \in Z_p$ , the total space of the principal  $T_1^1$ -bundle determined by  $f_i$  is  $L(m, q) \times S^1$ , where  $m = \text{gcd}(i, p)$ . Take a circle

subgroup  $T_2^1$  in  $T^{n-3}$  as in the first case and denote the complementary subgroup by  $T^{n-4}$ . Then  $M/T^{n-4}$  is principal  $T_2^1$ -bundle over  $L(m, q) \times S^1$ . This bundle is also classified by

$$[L(m, q) \times S^1, K(Z, 2)] \cong H^2(L(m, q) \times S^1, Z) .$$

Let  $\xi \in [L(m, q) \times S^1, K(Z, 2)]$  and denote its total space by  $E'$ . Consider the following diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\Pi'} & L(m, q) \times S^1 \\ \downarrow & & \downarrow \Pi \\ E'' & \xrightarrow{\Pi''} & L(m, q) . \end{array}$$

Here  $E''$  is the total space of  $\xi$  restricted to  $L(m, q) \times t$ , where  $t$  is any chosen point of  $S^1$ . Here  $\Pi'$  and  $\Pi''$  are bundle maps and  $\Pi$  is the projection map onto the first coordinate  $L(m, q)$ . Now  $E'$  is the pull-back of  $E''$  relative to the projection map  $\Pi$ , so that we have  $E' = E'' \times S^1$ . Since  $\xi$  restricted to  $L(m, q) \times t$  is an element of  $[L(m, q), K(Z, 2)] \cong Z_m$  we can consider  $f_j \in [L(m, q), K(Z, 2)]$ , for some  $j \in Z_m$  as representing this bundle element whose total space is  $E''$ . But  $E'' \cong L(d, q) \times S^1$  as before, where  $d = \text{gcd}(j, m)$ . Hence  $E' \cong L(d, q) \times S^1 \times S^1 \cong L(d, q) \times T^2$ . Repeating this process a finite number of times we eventually get  $M \cong L(k, q) \times T^{n-2}$  for some  $k \leq p$ .

**THEOREM.** *If  $T^n$  acts effectively on a compact closed orientable  $(n + 1)$ -manifold  $M^{n+1}$ , then  $M^{n+1}$  must be either  $T^{n+1}$  or  $L(k, q) \times T^{n-2}$  for  $n \geq 3$ .*

*Proof.* If  $M^* = S^1$ , then every point on  $S^1$  corresponds to a principal orbit, and the total space is a  $T^n$ -bundle over  $S^1$ . But these bundles are classified by

$$[S^1, K(Z, 2) \times \dots \times K(Z, 2)] = H^2(S^1, Z + \dots + Z) = 0 ,$$

so that the bundle is trivial and  $M = S^1 \times T^n = T^{n+1}$ .

If  $M^* = [0, 1]$ , then by Lemma 1 there are only two circle subgroups of  $T^n$  corresponding to the stability groups at 0 and 1. Let  $T_0$  be a subgroup generated by these two circle subgroups. Then any  $(n - 2)$ -dimensional subgroup  $T^{n-2}$  of  $T^n$  which is complementary to  $T_0$  acts freely on  $M$ . Then  $M/T^{n-2}$  is a 3-dimensional orientable manifold  $\bar{M}$  and  $T_0$  acts on it so that  $\bar{M} \setminus T_0 \cong [0, 1]$ . But  $T_0$  actions on 3-manifolds whose orbit spaces are isomorphic to  $[0, 1]$  are classified as lens spaces  $L(p, q)$  in [2]. Now, since  $T^{n-2}$  acts freely on  $M$ ,  $M$  is a principal  $T^{n-2}$ -bundle over  $L(p, q)$ . But these bundles can be written as  $L(k, q) \times T^{n-2}$  by the Lemma 3.

REMARK. Since the maximal torus subgroup of  $SO(m)$  is  $T^n$  where  $m = 2n$  or  $m = 2n + 1$ , we see that  $(T^n, M^m)$  can have no fixed points unless  $m > 2n$  or  $m > 2n + 1$ . Also we can see from the theorem that a compact simply-connected  $(n + 1)$ -manifold does not allow effective  $T^n$  actions for  $n \geq 3$ . Thus extending a result of R. Richardson, Jr. [4] which says that  $T^3$  cannot act effectively on the 4-dimensional sphere  $S^4$ .

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Tsuyoshi Andô, <i>Closed range theorems for convex sets and linear liftings</i> .....	393
Richard David Bourgin, <i>Conically bounded sets in Banach spaces</i> .....	411
Robert Jay Buck, <i>Hausdorff dimensions for compact sets in <math>R^n</math></i> .....	421
Henry Cheng, <i>A constructive Riemann mapping theorem</i> .....	435
David Fleming Dawson, <i>Summability of subsequences and stretchings of sequences</i> .....	455
William Thomas Eaton, <i>A two sided approximation theorem for 2-spheres</i> .....	461
Jay Paul Fillmore and John Herman Scheuneman, <i>Fundamental groups of compact complete locally affine complex surfaces</i> .....	487
Avner Friedman, <i>Bounded entire solutions of elliptic equations</i> .....	497
Ronald Francis Gariepy, <i>Multiplicity and the area of an <math>(n - 1)</math> continuous mapping</i> .....	509
Andrew M. W. Glass, <i>Archimedean extensions of directed interpolation groups</i> .....	515
Morisuke Hasumi, <i>Extreme points and unicity of extremum problems in <math>H^1</math> on polydiscs</i> .....	523
Trevor Ongley Hawkes, <i>On the Fitting length of a soluble linear group</i> .....	537
Garry Arthur Helzer, <i>Semi-primary split rings</i> .....	541
Melvin Hochster, <i>Expanded radical ideals and semiregular ideals</i> .....	553
Keizō Kikuchi, <i>Starlike and convex mappings in several complex variables</i> .....	569
Charles Philip Lanski, <i>On the relationship of a ring and the subring generated by its symmetric elements</i> .....	581
Jimmie Don Lawson, <i>Intrinsic topologies in topological lattices and semilattices</i> .....	593
Roy Bruce Levow, <i>Counterexamples to conjectures of Ryser and de Oliveira</i> .....	603
Arthur Larry Lieberman, <i>Some representations of the automorphism group of an infinite continuous homogeneous measure algebra</i> .....	607
William George McArthur, <i><math>G_\delta</math>-diagonals and metrization theorems</i> .....	613
James Murdoch McPherson, <i>Wild arcs in three-space. II. An invariant of non-oriented local type</i> .....	619
H. Millington and Maurice Sion, <i>Inverse systems of group-valued measures</i> .....	637
William James Rae Mitchell, <i>Simple periodic rings</i> .....	651
C. Edward Moore, <i>Concrete semispaces and lexicographic separation of convex sets</i> .....	659
Jingyal Pak, <i>Actions of torus <math>T^n</math> on <math>(n + 1)</math>-manifolds <math>M^{n+1}</math></i> .....	671
Merrell Lee Patrick, <i>Extensions of inequalities of the Laguerre and Turán type</i> .....	675
Harold L. Peterson, Jr., <i>Discontinuous characters and subgroups of finite index</i> .....	683
S. P. Philipp, <i>Abel summability of conjugate integrals</i> .....	693
R. B. Quintana and Charles R. B. Wright, <i>On groups of exponent four satisfying an Engel condition</i> .....	701
Marlon C. Rayburn, <i>On Hausdorff compactifications</i> .....	707
Martin G. Ribe, <i>Necessary convexity conditions for the Hahn-Banach theorem in metrizable spaces</i> .....	715
Ryōtarō Satō, <i>On decomposition of transformations in infinite measure spaces</i> .....	733
Peter Drummond Taylor, <i>Subgradients of a convex function obtained from a directional derivative</i> .....	739
James William Thomas, <i>A bifurcation theorem for <math>k</math>-set contractions</i> .....	749
Clifford Edward Weil, <i>A topological lemma and applications to real functions</i> .....	757
Stephen Andrew Williams, <i>A nonlinear elliptic boundary value problem</i> .....	767
Pak-Ken Wong, <i>*-actions in <math>A^*</math>-algebras</i> .....	775