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**EXTENSIONS OF INEQUALITIES OF THE LAGUERRE AND
TURÁN TYPE**

MERRELL LEE PATRICK

EXTENSIONS OF INEQUALITIES OF THE LAGUERRE AND TURÁN TYPE

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It is shown that

$$\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} F^{(n+j)}(z) F^{(n+2k-j)}(z) \geq 0,$$

for $-\infty < z < \infty$, $n \geq 1$ and $k \geq 0$, where $F(z)$ is an entire function of a special type. For $k = 1$ this simply is the well known Laguerre inequality

$$(F^{(n+1)}(z))^2 - F^{(n)}(z)F^{(n+2)}(z) \geq 0$$

$-\infty < z < \infty$, $n \geq 0$. From these inequalities we obtain the inequalities

$$\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} u_{n+j}(x) u_{n+2k-j}(x) \geq 0$$

which hold for such values of x , for which the functions $u_n = u_n(x)$ have a generating function of the type

$$\sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z).$$

1. Introduction. Consider the entire function of the form

$$(1.1) \quad F(z) = C e^{-\alpha z^2 + \beta z} \prod_m (1 - z/z_m) e^{z/z_m}$$

where $\alpha \geq 0$, C , β and all z_m are real, and $\sum_m z_m^{-2}$ is convergent. These functions have been studied by Laguerre (see Borel [1; pp. 32-47]), Pólya and Schur [9], and others. Apart from constant factors, these functions are the only ones which are limits of polynomials with real zeros only.

In § 2 we show, for integers $n \geq 0$, that

$$(1.2) \quad |F^{(n)}(x + iy)|^2 = \sum_{k=0}^{\infty} L_k(F^{(n)}; x) y^{2k}$$

where

$$(1.3) \quad L_k(F^{(n)}; x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} F^{(n+j)}(x) F^{(n+2k-j)}(x)$$

and $F^{(n)}(z)$ denotes the n th derivative of $F(z) = F^{(0)}(z)$. Furthermore, we show that

$$(1.4) \quad L_k(F^{(n)}; x) \geq 0$$

for each $n \geq 0$, $k = 0, 1, 2, \dots$, and all real x . Patrick [8] obtained inequality (1.4) for polynomials with real zeros only. For $k = 1$, (1.4) is the well known and useful Laguerre inequality.

$$(1.5) \quad (F^{(n+1)}(z))^2 - F^{(n)}(z)F^{(n+2)}(z) \geq 0, \quad -\infty < z < \infty, n \geq 0.$$

Finally, in § 3, we use (1.4) to obtain the inequalities

$$(1.6) \quad \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} u_{n+j}(x) u_{n+2k-j}(x) \geq 0$$

which holds for such values of x , for which the functions $u_n = u_n(x)$ have a generating function of the type

$$(1.7) \quad \sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z)$$

where $F(z)$ is of the form (1.1).

For $k = 1$, (1.6) is the interesting and much studied Turán inequality

$$(1.8) \quad (u_{n+1}(x))^2 - u_n(x)u_{n+2}(x) \geq 0, n \geq 0.$$

In 1948 Szegő [13] called attention to (1.8) for Legendre polynomials. Since that time the inequality has been studied for other special functions by Mukherjee and Nanjundiah [6]; Nanjundiah [7]; Szász [11], [12]; Thiruvengkatachar and Nanjundiah [14]; Venkatachaliengar and Rao [15], and others. These authors have shown that the functions which satisfy (1.8) include the Legendre, Tchebychef, ultraspherical, Hermite and Laguerre polynomials and the Bessel and modified Bessel functions. A summary of these results are contained in Skovgaard [10]. Also in [10], Skovgaard uses the Laguerre inequality (1.5) to obtain the Turán inequality for sequences $u_n = u_n(x)$, $n = 0, 1, 2, \dots$ satisfying (1.7). He shows that for such sequences the Turán inequality is a special case of the Laguerre inequality. Then he is able to easily prove the inequality (1.8) for most of the above mentioned functions. These results served as motivation for the work which resulted in this paper.

Some other papers containing results related to the Turán type expressions or inequalities are Carlitz [2]; A. E. Danese [3], [4]; Karlin and Szegő [5]; and Webster [16].

2. "Extended" Laguerre inequalities. We now prove the following theorem concerning entire functions of the form (1.1).

THEOREM 1. *Let $F(z) = F(x + iy)$ be an entire function of the*

form (1.1) and let $F^{(n)}(z)$ denote the n th derivative of $F(z)$. Then for any integer $n \geq 0$ and any $z = x + iy \neq \infty$

$$(2.1) \quad |F^{(n)}(x + iy)|^2 = \sum_{k=0}^{\infty} L_k(F^{(n)}; x)y^{2k}$$

where

$$(2.2) \quad L_k(F^{(n)}; x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} F^{(n+j)}(x) F^{(n+2k-j)}(x).$$

Furthermore

$$(2.3) \quad L_k(F^{(n)}; x) \geq 0 \quad \text{for } k = 0, 1, 2, \dots$$

Proof. We first prove (2.1)-(2.2) for $n = 0$. Since $F^{(0)}(z) = F(z)$ is an entire function we can write for any z

$$(2.4) \quad F(z) = F(x + iy) = \sum_{k=0}^{\infty} A_k(F; x)y^k$$

and

$$(2.5) \quad F(\bar{z}) = F(x - iy) = \sum_{k=0}^{\infty} (-1)^k A_k(F; x)y^k$$

where

$$(2.6) \quad A_k(F; x) = \frac{i^k}{k!} F^{(k)}(x), \quad k = 0, 1, 2, \dots$$

Since C, α, β , and all z_m are real, $\overline{F(z)} = F(\bar{z})$ and so from (2.4) and (2.5) we have

$$(2.7) \quad |F(x + iy)|^2 = \sum_{m=0}^{\infty} C_m(F; x)y^m$$

where

$$(2.8) \quad C_m(F; x) = (-1)^m A_0(F; x)A_m(F; x) + (-1)^{m-1} A_1(F; x)A_{m-1}(F; x) \\ + \dots + (-1)A_{m-1}(F; x)A_1(F; x) + A_m(F; x)A_0(F; x).$$

When m is odd, it follows from (2.6) and (2.8) that

$$(2.9) \quad C_m(F; x) \\ = \sum_{j=0}^M \left(\frac{i^j}{j!} \cdot \frac{(-1)^{m-j} i^{m-j}}{(m-j)!} + \frac{i^{m-j}}{(m-j)!} \cdot \frac{(-1)^j i^j}{j!} \right) F^{(j)}(x) F^{(m-j)}(x) \\ = \sum_{j=0}^M \frac{(-1)^j i^m}{j(m-j)!} [(-1)^m + 1] F^{(j)}(x) F^{(m-j)}(x) = 0$$

where $M = [(m + 1)/2]$. When m is even, say $m = 2k$, we have from (2.6) and (2.8)

$$(2.10) \quad C_{2k}(F; x) = \frac{(F^{(k)}(x))^2}{(k!)^2} + \sum_{j=0}^{k-1} 2 \frac{(-1)^{j+k}}{j!(2k-j)!} F^{(j)}(x) F^{(2k-j)}(x).$$

It follows easily that $L_k(F^{(n)}; x)$ in (2.2) with $n = 0$ is equal to $C_{2k}(F; x)$ in (2.10). From (2.7), (2.9), and (2.10) we have the desired result (2.1)–(2.2) for $n = 0$.

We now prove (2.1)–(2.2) for $n \geq 1$. Skovgaard [10] shows that the derivative of a function of the form (1.1) is also of the form (1.1). Therefore, all derivatives $F^{(n)}(z)$ of $F(z)$ are of the form (1.1) and (2.1)–(2.2) for any $n \geq 1$ follows from the proof of the $n = 0$ case by replacing $F(z)$ by $F^{(n)}(z)$.

Next we prove (2.3) for $n = 0$. Using (1.1) we have

$$(2.11) \quad |F(z)|^2 = C^2 e^{-\alpha(z^2 + \bar{z}^2) + \beta(z + \bar{z})} (z\bar{z})^r \prod_m (1 - z/z_m)(1 - \bar{z}/z_m) e^{z + \bar{z}/z_m}$$

or

$$(2.12) \quad |F(x + iy)|^2 = C^2 e^{-2\alpha x^2 + 2\beta x} e^{2\alpha y^2} (x^2 + y^2)^r \prod_m [(1 - x/z_m)^2 + y^2/z_m] e^{2x/z_m}.$$

Since e^{2x/z_m} is real and positive for all x we can write

$$\prod_m [(1 - x/z_m)^2 + y^2/z_m^2] e^{2x/z_m} = \sum_{k=0}^{\infty} C_k(x) y^{2k}$$

where $C_k(x) \geq 0$ for $k = 0, 1, \dots$, and all x . Also $\alpha \geq 0, C^2 > 0, e^{-2\alpha x^2 + \beta x} > 0$ for all $x, (x^2 + y^2)^r$ is a polynomial in y^2 with positive coefficients that are functions of x , and $e^{2\alpha y^2}$ is an absolutely convergent series in y^2 with positive coefficients. Combining all of these facts with (2.12) we have for any $z = x + iy \neq \infty$.

$$(2.13) \quad |F(x + iy)|^2 = \sum_{k=0}^{\infty} F_k(x) y^{2k}$$

where

$$F_k(x) \geq 0 \quad \text{for } k = 0, 1, 2, \dots$$

But (2.1) and (2.13) are identical so we have $L_k(F; x) = F_k(x) \geq 0$ for $k = 0, 1, 2, \dots$, and (2.3) is proved for $n = 0$.

As in the proof of (2.1)–(2.2), the proof of (2.3) for $n \geq 1$ follows immediately by replacing $F(z)$ by $F^{(n)}(z)$ in the above considerations.

For the sake of better understanding of the functions $L_k(F^{(n)}; x)$ we note that

$$(2.14) \quad L_0(F^{(n)}; x) = (F^{(n)}(x))^2$$

$$(2.15) \quad L_1(F^{(n)}; x) = (F^{(n+1)}(x))^2 - F^{(n)}(x)F^{(n+2)}(x)$$

$$(2.16) \quad L_2(F^{(n)}; x) = \frac{(F^{(n+2)}(x))^2}{(2!)^2} - \frac{2F^{(n+1)}(x)F^{(n+3)}(x)}{1! 3!} \\ + \frac{2F^{(n)}(x)F^{(n+4)}(x)}{4!}$$

and, in general,

$$(2.17) \quad L_k(F^{(n)}; x) = \frac{(F^{(n+k)}(x))^2}{(k!)^2} - \frac{2F^{(n+k-1)}(x)F^{(n+k+1)}(x)}{(k-1)! (k+1)!} \\ + \dots + (-1)^k \frac{2F^{(n)}(x)F^{(n+2k)}(x)}{(2k)!}.$$

We further note that (2.15) is the well-known Laguerre expression and that (2.3) with $k = 1$ the corresponding Laguerre inequality. For this reason we call (2.3) the "extended" Laguerre inequalities.

In [8] we show, among other things, that $0 \leq L_j(F^{(n)}; x) \leq L_j(F^{(n)}; 1)$, $j = 0, 1, 2, 3$, for $-1 \leq x \leq 1$ when $F(z)$ is restricted to the class of ultraspherical polynomials. We conjecture that a similar inequality holds for $j > 4$ over the same class of functions.

3. "Extended" Turán inequalities. Using Theorem 1 of § 2 and the results of Skovgaard [10] we can easily prove the following theorem.

THEOREM 2. Let $u_n = u_n(x)$, $n = 0, 1, 2, \dots$, be a system of real functions which for certain values of x have a generating function of the type

$$(3.1) \quad \sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z)$$

where $F(z)$ is of the form (1.1). Then for those values of x

$$(3.2) \quad \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} u_{n+j}(x) u_{n+2k-j}(x) \geq 0 \quad \text{for } k = 0, 1, 2, \dots$$

Proof. As indicated by Skovgaard [10], we have for those values of x where $u_n = u_n(x)$ satisfies (3.1) that $u_n = F^{(n)}(0)$. That is for such x inequality (3.2) is a special case of "extended" Laguerre inequality (2.3).

We note for $k = 1$ that (3.2) is the interesting Turán inequality

$$(3.3) \quad u_{n+1}^2(x) - u_n(x)u_{n+2}(x) \geq 0.$$

For this reason we call the inequalities (3.2) the "extended" Turán inequalities.

Skovgaard used (2.3) with $k = 1$ to obtain (3.2) with $k = 1$ for certain classical functions by indicating their generating functions of the type (1.1). These same generating functions, since they are of type (1.1), will, by Theorem 1, satisfy (2.3) for $k = 0, 1, 2, \dots$. Therefore, by Theorem 2, the corresponding set of classical functions will satisfy the "extended" Turan inequalities (3.2). The following is a list of these classical functions:

(3.4) Ultraspherical polynomials

$$P_n^{(\lambda)}(x), \quad -1 \leq x \leq 1, \lambda \geq \frac{1}{2},$$

(3.5) p th derivative of $P_n^{(\lambda)}(x)$

$$D^p P_n^{(\lambda)}(x), \quad p \leq n, \quad -1 \leq x \leq 1, \lambda \geq \frac{1}{2} - p,$$

(3.6) Laguerre polynomials

$$L_n^{(\alpha)}(x), \quad -\infty < x < \infty, \alpha > -1,$$

(3.7) p th derivative of $L_n^{(\alpha)}(x)$

$$D^p L_n^{(\alpha)}(x), \quad p \leq n, \quad -\infty < x < \infty, \alpha \geq -p,$$

(3.8) Hermite polynomials

$$H_n(x), \quad -\infty < x < \infty,$$

(3.9) p th derivative of $H_n(x)$

$$D^p H_n(x), \quad -\infty < x < \infty,$$

(3.10) Sine and cosine functions

$$\begin{cases} \cos \\ \sin \end{cases} nx, \quad -\infty < x < \infty,$$

(3.11) Bessel functions

$$J_n(x), \quad -\infty < x < \infty,$$

(3.12) Derivative of Bessel function

$$J'_n(x), \quad -\infty < x < \infty,$$

(3.13) Tchebychef function of first kind

$$T_n(x), \quad -1 \leq x \leq 1,$$

(3.14) Tchebychef function of second kind

$$U_n(x), \quad -1 \leq x \leq 1.$$

We point out that (3.4) and (3.5) includes Legendre polynomials and their derivatives.

Generating functions for (3.4), (3.6), (3.8), (3.10), (3.11), and (3.12) can be found in Skovgaard [10]. As in Skovgaard's paper we have analogously, since

$$D^p P_n^{(\lambda)}(x) = 2^p \frac{\Gamma(\lambda + p)}{\Gamma(\lambda)} P_{n-p}^{(\lambda+p)}(x), \quad p \leq n,$$

that (3.5) satisfies (3.2) because (3.4) does. Also

$$D^p L_n^{(\alpha)}(x) = (-1)^p L_{n-p}^{(\alpha+p)}(x), \quad p \leq n,$$

so (3.7) satisfies (3.2) because (3.6) does. Similarly

$$D^p H_n(x) = 2^p \frac{n!}{(n-p)!} H_{n-p}(x)$$

and (3.9) satisfies (3.2) because (3.8) does. Finally, (3.13) and (3.14) satisfy (3.2) because (3.10) does.

Using an inductive proof we showed that if $F^{(n)}(z)$ is of the form

$$(3.15) \quad F^{(n)}(z) = Ce^{\beta z}$$

then

$$(3.16) \quad L_k(F^{(n)}; z) = 0$$

for $k = 1, 2, \dots$. Also, (3.16) holds at multiple zeros of order $k + 1$ of $F^{(n)}(z)$. Consequently, given the assumptions of Theorem 2, equality in (3.2) holds for values of x for which $F^{(n)}(z)$ is of the form (3.15), or for which $z = 0$ is a zero of multiplicity $k + 1$ of $F^{(n)}(z)$.

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Tsuyoshi Andô, <i>Closed range theorems for convex sets and linear liftings</i>	393
Richard David Bourgin, <i>Conically bounded sets in Banach spaces</i>	411
Robert Jay Buck, <i>Hausdorff dimensions for compact sets in R^n</i>	421
Henry Cheng, <i>A constructive Riemann mapping theorem</i>	435
David Fleming Dawson, <i>Summability of subsequences and stretchings of sequences</i>	455
William Thomas Eaton, <i>A two sided approximation theorem for 2-spheres</i>	461
Jay Paul Fillmore and John Herman Scheuneman, <i>Fundamental groups of compact complete locally affine complex surfaces</i>	487
Avner Friedman, <i>Bounded entire solutions of elliptic equations</i>	497
Ronald Francis Gariepy, <i>Multiplicity and the area of an $(n - 1)$ continuous mapping</i>	509
Andrew M. W. Glass, <i>Archimedean extensions of directed interpolation groups</i>	515
Morisuke Hasumi, <i>Extreme points and unicity of extremum problems in H^1 on polydiscs</i>	523
Trevor Ongley Hawkes, <i>On the Fitting length of a soluble linear group</i>	537
Garry Arthur Helzer, <i>Semi-primary split rings</i>	541
Melvin Hochster, <i>Expanded radical ideals and semiregular ideals</i>	553
Keizō Kikuchi, <i>Starlike and convex mappings in several complex variables</i>	569
Charles Philip Lanski, <i>On the relationship of a ring and the subring generated by its symmetric elements</i>	581
Jimmie Don Lawson, <i>Intrinsic topologies in topological lattices and semilattices</i>	593
Roy Bruce Levow, <i>Counterexamples to conjectures of Ryser and de Oliveira</i>	603
Arthur Larry Lieberman, <i>Some representations of the automorphism group of an infinite continuous homogeneous measure algebra</i>	607
William George McArthur, <i>G_δ-diagonals and metrization theorems</i>	613
James Murdoch McPherson, <i>Wild arcs in three-space. II. An invariant of non-oriented local type</i>	619
H. Millington and Maurice Sion, <i>Inverse systems of group-valued measures</i>	637
William James Rae Mitchell, <i>Simple periodic rings</i>	651
C. Edward Moore, <i>Concrete semispaces and lexicographic separation of convex sets</i>	659
Jingyal Pak, <i>Actions of torus T^n on $(n + 1)$-manifolds M^{n+1}</i>	671
Merrell Lee Patrick, <i>Extensions of inequalities of the Laguerre and Turán type</i>	675
Harold L. Peterson, Jr., <i>Discontinuous characters and subgroups of finite index</i>	683
S. P. Philipp, <i>Abel summability of conjugate integrals</i>	693
R. B. Quintana and Charles R. B. Wright, <i>On groups of exponent four satisfying an Engel condition</i>	701
Marlon C. Rayburn, <i>On Hausdorff compactifications</i>	707
Martin G. Ribe, <i>Necessary convexity conditions for the Hahn-Banach theorem in metrizable spaces</i>	715
Ryōtarō Satō, <i>On decomposition of transformations in infinite measure spaces</i>	733
Peter Drummond Taylor, <i>Subgradients of a convex function obtained from a directional derivative</i>	739
James William Thomas, <i>A bifurcation theorem for k-set contractions</i>	749
Clifford Edward Weil, <i>A topological lemma and applications to real functions</i>	757
Stephen Andrew Williams, <i>A nonlinear elliptic boundary value problem</i>	767
Pak-Ken Wong, <i>*-actions in A^*-algebras</i>	775