

# Pacific Journal of Mathematics

**ABEL SUMMABILITY OF CONJUGATE INTEGRALS**

S. P. PHILIPP

# ABEL SUMMABILITY OF CONJUGATE INTEGRALS

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It is proved here that the conjugate Fourier-Stieltjes integral of a finite-valued Borel measure  $\mu$  on Euclidean  $k$ -space,  $k \geq 1$ , taken with respect to a Calderon-Zygmund kernel in Lip  $\alpha$ ,  $0 < \alpha < 1$ , is almost everywhere (with respect to Lebesgue measure) Abel summable to the conjugate function of  $\mu$  taken with respect to the above mentioned kernel. This has been already established for  $k \leq 3$  and for  $k$  even.

We make the following assumptions:  $k$  is a positive integer;  $0 < \alpha < 1$ ;  $\Omega \in \text{Lip } \alpha(S)$ , where  $S$  denotes the  $(k - 1)$ -sphere in  $k$ -dimensional Euclidean space  $E_k$ ;  $\int_S \Omega(y) dS(y) = 0$ , where  $dS$  refers to the natural measure on  $S$ ; and  $\mu$  is a real Borel measure on  $E_k$  as defined in [3].

Let  $K(x) = \Omega(x/|x|)|x|^{-k}$  for each nonzero  $x$  in  $E_k$  (we use  $|x|$  for the usual norm and  $x \cdot y$  for the usual dot product and  $dx$  for Lebesgue measure, all in  $E_k$ ). For  $y$  in  $E_k$ , set  $\hat{\mu}(y) = (2\pi)^{-k} \int_{E_k} e^{-ix \cdot y} d\mu(x)$  and

$$\hat{K}(y) = (2\pi)^{-k} \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\varepsilon \leq |x| \leq R} e^{-ix \cdot y} K(x) dx .$$

It is known [5, p. 69] that  $\hat{K}$  is bounded on  $E_k$ . We define, for  $t > 0$  and  $x$  in  $E_k$ ,

$$I_t(x) = (2\pi)^k \int_{E_k} e^{-t|y|} \hat{K}(y) \hat{\mu}(y) e^{ix \cdot y} dy .$$

We shall prove the

**THEOREM.** 
$$\lim_{t \rightarrow 0} \left[ I_t(x) - \int_{|y-x| > t} K(x-y) d\mu(y) \right] = 0$$

*except on a set of Lebesgue measure zero in  $E_k$ .*

If  $k = 1$ , the theorem is classical (see [8, p. 103] for the essence of the matter). The case  $k = 2$  and  $1/2 < \alpha < 1$  was treated in [4]. The cases in which  $0 < \alpha < 1$  and  $k = 3$  or  $k$  is even were handled in [2]. Further references and motivation for the theorem are given in [2] and [4]. The proof given in the present paper covers all cases with  $0 < \alpha < 1$  and  $k \geq 3$ ; modifications could be made in the proof to cover the cases  $k = 2$ ,  $0 < \alpha < 1$  and  $k = 1$ , but this seems pointless.

*Proof of the theorem.* Assume  $k \geq 3$ . Define, for  $t > 0$  and  $n = 1, 2, \dots$ ,

$$(1) \quad H_n^k(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{2^{k/2}\Gamma\left(\frac{n+k}{2}\right)} \int_0^\infty e^{-ts} s^{k/2} J_{n-1+k/2}(s) ds,$$

where  $J_{n-1+k/2}(s)$  denotes a standard Bessel function of first kind. We assume throughout that  $t > 0$ . It follows from (1) and [6, p. 385] and [7, p. 282] that for  $n = 1, 2, \dots$ ,

$$(2) \quad \begin{aligned} 0 &\leq H_n^k(t) \\ &= \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+k+1}{2}\right)\pi^{-1/2}}{\Gamma\left(n+\frac{k}{2}\right)(t^2+1)^{(n+k)/2}} F\left(\frac{n+k}{2}, \frac{n-1}{2}; n+\frac{k}{2}; \frac{1}{1+t^2}\right) \\ &\leq \left(\frac{1}{1+t^2}\right)^{(k+n)/2}. \end{aligned}$$

Here and throughout  $F(a, b; c; x)$  refers to Gauss' hypergeometric series. Let the series of spherical harmonics associated with  $\Omega(\xi)$  be  $\sum_{n=1}^\infty Y_n(\xi)$ . Let  $\omega$  denote the  $(k-1)$ -dimensional volume of  $S$  and let  $\xi$  be a point of  $S$ . We shall use the symbol  $A$  generically to denote any positive constant which depends only on  $\Omega$  and  $k$ . Let  $\lambda = (k-2)/2$ . The Gegenbauer polynomials  $P_n^\lambda(\cos \theta)$  are defined in [5]. By (2) and the boundedness of  $\Omega$  and the fact [1, p. 245] that

$$(3) \quad |P_n^\lambda(\cos \theta)| \leq P_n^\lambda(1) \leq An^{2\lambda-1}$$

for  $n = 0, 1, 2, \dots$  and  $0 \leq \theta \leq \pi$ , we may write

$$(4) \quad \begin{aligned} \sum_{n=1}^\infty H_n^k(t) Y_n(\xi) &= \sum_{n=1}^\infty H_n^k(t) \frac{n+\lambda}{\lambda\omega} \int_S \Omega(y) P_n^\lambda(\xi \cdot y) dS(y) \\ &= \frac{1}{\omega} \int_S \Omega(y) \left[ \sum_{n=1}^\infty \frac{n+\lambda}{\lambda} H_n^k(t) P_n^\lambda(\xi \cdot y) \right] dS(y). \end{aligned}$$

By the concluding argument given in [2], we see that in order to prove the theorem it suffices to prove that

$$(5) \quad \left| \sum_{n=1}^\infty H_n^k(t) Y_n(\xi) - \Omega(\xi) \right| \leq At^\alpha.$$

So we define a kernel  $K_t(\theta)$  by

$$(6) \quad K_t(\theta) = 1 + \sum_{n=1}^\infty \frac{n+\lambda}{\lambda} H_n^k(t) P_n^\lambda(\cos \theta)$$

for  $0 \leq \theta \leq \pi$ . Now

$$(7) \quad \frac{1}{\omega} \int_s K_t(\xi \cdot y) dS(y) = 1,$$

as may be seen from (4). It follows from (4), (6), and (7) that

$$(8) \quad \sum_{n=1}^{\infty} H_n^k(t) Y_n(\xi) - \Omega(\xi) = \frac{1}{\omega} \int_s [\Omega(y) - \Omega(\xi)] K_t(\xi \cdot y) dS(y).$$

Notice that if we can establish that

$$(9) \quad |K_t(\theta)| \leq \frac{At}{\left(t^2 + \sin^2 \frac{\theta}{2}\right)^{k/2}}$$

for  $0 \leq \theta \leq \pi$ , then it will follow from (8), (9), and the Lipschitz condition on  $\Omega$  that

$$\begin{aligned} \left| \sum_{n=1}^{\infty} H_n^k(t) Y_n(\xi) - \Omega(\xi) \right| &\leq A \int_s |K_t(\xi \cdot y)| \cdot |y - \xi|_\alpha dS(y) \\ &\leq At \int_0^\pi \frac{(1 - \cos\theta)^{\alpha/2} \sin^{k-2} \theta}{\left(t^2 + \sin^2 \frac{\theta}{2}\right)^{k/2}} d\theta \leq A \int_0^\pi \frac{t\theta^{k-2+\alpha}}{(t^2 + \theta^2)^{k/2}} d\theta \leq At^\alpha. \end{aligned}$$

That is, once we prove (9), then (5) follows and we are done.

Now define

$$(10) \quad P_\nu(s, t) = \int_0^\infty e^{-ty} y s^{1/2} J_\nu(sy) J_\nu(y) dy$$

for  $\nu > -(1/2)$  and  $s > 0$ . The relations

$$(11) \quad J_\nu(x) = O(x^{-1/2}) \text{ as } x \rightarrow \infty, \quad J_\nu(x) = O(x^\nu) \text{ as } x \rightarrow 0+,$$

which are valid for each  $\nu > -(1/2)$  and are proved in [6], will be useful to us. Using (10), (11), Fubini's theorem, the change of variable  $sy = r$  and [6, p. 391], we obtain that

$$(12) \quad \begin{aligned} \int_0^\infty s^{-\lambda-3/2} P_{n+\lambda}(s, t) ds \\ = \int_0^\infty e^{-ty} y^{\lambda+1} J_{n+\lambda}(y) dy \int_0^\infty r^{-\lambda-1} J_{n+\lambda}(r) dr = H_n^k(t) \end{aligned}$$

for  $n \geq 1$ . It follows from (6) and (12) that

$$(13) \quad K_t(\theta) = 1 + \sum_{n=1}^{\infty} \frac{n + \lambda}{\lambda} P_n^\lambda(\cos \theta) \int_0^\infty s^{-\lambda-3/2} P_{n+\lambda}(s, t) ds$$

for  $0 \leq \theta \leq \pi$ . By Gegenbauer's addition theorem [6, p. 363] we may write

$$\begin{aligned}
 (14) \quad & \frac{J_\lambda(y\sqrt{s^2 - 2s \cos \theta + 1}) s^{\lambda+1/2} y^{\lambda+1} e^{-ty}}{(s^2 - 2s \cos \theta + 1)^{\lambda/2} 2^\lambda \Gamma(\lambda + 1)} \\
 & = \sum_{n=0}^{\infty} \frac{n + \lambda}{\lambda} y s^{1/2} J_{n+\lambda}(y) J_{n+\lambda}(sy) P_n^\lambda(\cos \theta) e^{-ty}
 \end{aligned}$$

for  $y > 0, s > 0, 0 \leq \theta \leq \pi$ . Because of (3) and the inequality [6, p. 49]

$$|J_{n+\lambda}(x)| \leq \frac{Ax^{n+\lambda}}{\Gamma(n + \lambda)},$$

valid for  $x > 0$  and  $n \geq 0$ , we may integrate the right member of (14) term by term with respect to  $y$  over  $(0, \infty)$ . So, by integrating both sides of (14) with respect to  $y$  over  $(0, \infty)$  and then using (10) in the right member and [6, p. 391] in the left member, we obtain that

$$\begin{aligned}
 (15) \quad & \sum_{n=1}^{\infty} \frac{n + \lambda}{\lambda} P_{n+\lambda}(s, t) P_n^\lambda(\cos \theta) \\
 & = \frac{\Gamma\left(\lambda + \frac{3}{2}\right) t(z - \cos \theta)^{-\lambda-3/2}}{2^{\lambda+1/2} \Gamma(\lambda + 1) \pi^{1/2} s} - P_\lambda(s, t)
 \end{aligned}$$

for  $0 \leq \theta \leq \pi, s > 0$ , where

$$(16) \quad z = \frac{s^2 + t^2 + 1}{2s}$$

for  $s > 0$ . We shall adhere to the notation (16). It follows from [6, p. 389] and [7, pp. 317 and 281] that

$$\begin{aligned}
 (17) \quad & P_\nu(s, t) \\
 & = \frac{\Gamma\left(\nu + \frac{3}{2}\right) t s^{-1}}{2^{\nu+1/2} \pi^{1/2} \Gamma(\nu + 1)} z^{-\nu-3/2} F\left(\frac{\nu + \frac{1}{2}}{2}, \frac{\nu + \frac{3}{2}}{2}; \nu + 1; z^{-2}\right) \\
 & + \frac{\Gamma\left(\nu + \frac{5}{2}\right) t s^{-1}}{2^{\nu+3/2} \pi^{1/2} \Gamma(\nu + 2)} z^{-\nu-7/2} F\left(\frac{\nu + \frac{5}{2}}{2}, \frac{\nu + \frac{7}{2}}{2}; \nu + 2; z^{-2}\right)
 \end{aligned}$$

for  $\nu > 1/2$  and  $s > 0$ . Proceeding formally, we multiply both sides of (15) by  $s^{-\lambda-3/2}$  and integrate with respect to  $s$  over  $(0, \infty)$  term by term in the left member; using (13), the result is that

$$\begin{aligned}
 (18) \quad & K_i(\theta) \\
 & = 1 + \int_0^\infty s^{-\lambda-3/2} \left[ \frac{\Gamma\left(\lambda + \frac{3}{2}\right) t(z - \cos \theta)^{-\lambda-3/2}}{2^{\lambda+1/2} \Gamma(\lambda + 1) \pi^{1/2} s} - P_\lambda(s, t) \right] ds
 \end{aligned}$$

for  $0 \leq \theta \leq \pi$ . To justify this formal procedure, we observe that, by (17),  $P_{n+\lambda}(s, t) \geq 0$  for  $n \geq 1, s > 0$ ; therefore, by (3), (15) and Fubini's theorem, we can be sure that (18) holds if the integral in the right member of (18) is a finite Lebesgue integral when  $\theta = 0$ . But it follows from (17) that for fixed  $t$  and  $\theta = 0$ , the integrand in the right member of (18) is  $O(1)$  as  $s \rightarrow 0$ ; and by (10) and (11), this same integrand is  $O(s^{-\lambda-3/2})$  as  $s \rightarrow \infty$ . Therefore (18) holds.

We proceed to estimate the right member of (18) by using the expression for  $P_\lambda(s, t)$  given by (17). Observe that if  $s > 0$  and  $|s - 1| \geq 1/2$ , then  $z \geq 13/12$ ; and recall that the radius of convergence of the hypergeometric series is at least one. Therefore, using [7, p. 299, Ex. 18], we may write

$$(19) \quad F\left(\frac{\lambda + \frac{1}{2}}{2}, \frac{\lambda + \frac{3}{2}}{2}; \lambda + 1; z^{-2}\right) = 1 + \varphi(z),$$

where

$$(20) \quad |\varphi(z)| \leq Az^{-2} \quad \text{for } |s - 1| \geq \frac{1}{2}, \quad s > 0,$$

and

$$(21) \quad |\varphi(z)| \leq A \log(1 - z^{-2})^{-1} \quad \text{for } |s - 1| < \frac{1}{2}.$$

It follows from (20) and (21) that

$$(22) \quad \int_0^\infty s^{-\lambda-3/2} t z^{-\lambda-3/2} s^{-1} \varphi(z) ds = O(t) \quad \text{as } t \rightarrow 0.$$

We conclude from (17), (18), (19), and (22) that, as  $t \rightarrow 0$ ,

$$(3) \quad K_\lambda(\theta) = O(t) + B(t, \theta) + \psi(t)$$

for  $0 \leq \theta \leq \pi$ , where we have set

$$(24) \quad B(t, \theta) = \frac{\Gamma\left(\lambda + \frac{3}{2}\right)t}{2^{\lambda+1/2}\pi^{1/2}\Gamma(\lambda + 1)} \int_0^\infty s^{-\lambda-5/2} [(z - \cos \theta)^{-\lambda-3/2} - z^{-\lambda-3/2}] ds$$

for  $0 \leq \theta \leq \pi$  and

$$(25) \quad \psi(t) = 1 - \frac{\Gamma\left(\lambda + \frac{5}{2}\right)t}{2^{\lambda+3/2}\pi^{1/2}\Gamma(\lambda + 2)} \times \int_0^\infty s^{-\lambda-5/2} z^{-\lambda-7/2} F\left(\frac{\lambda + \frac{5}{2}}{2}, \frac{\lambda + \frac{7}{2}}{2}, \lambda + 2; z^{-2}\right) ds.$$

By the mean value theorem we may write, for  $0 \leq \theta \leq \pi$ ,

$$\begin{aligned}
 (26) \quad B(t, \theta) &= At \int_0^\infty s^{-\lambda-5/2} \left( \frac{1}{z - \cos \theta} - \frac{1}{z} \right) w^{\lambda+1/2} ds \\
 &= At \int_0^\infty \frac{s^{-\lambda-5/2} (\cos \theta) w^{\lambda+1/2}}{z(z - \cos \theta)} ds,
 \end{aligned}$$

where  $w$  lies between  $z^{-1}$  and  $(z - \cos \theta)^{-1}$ . If  $\pi/2 \leq \theta \leq \pi$ , we obtain from (26) that

$$|B(t, \theta)| \leq At \int_0^\infty \frac{s^{-\lambda-5/2} \left( \frac{1}{z} \right)^{\lambda+1/2}}{z^2} ds \leq At.$$

If  $0 \leq \theta \leq \pi/2$ , we obtain from (26) that

$$\begin{aligned}
 |B(t, \theta)| &\leq At \int_0^\infty \frac{s^{-\lambda-5/2}}{z(z - \cos \theta)^{\lambda+3/2}} ds \\
 &\leq At \int_0^\infty \frac{ds}{[(s - \cos \theta)^2 + t^2 + \sin^2 \theta]^{\lambda+3/2}} \\
 &\leq At \int_{-\infty}^\infty \frac{ds}{[(s - \cos \theta)^2 + t^2 + \sin^2 \theta]^{\lambda+3/2}} \\
 &= \frac{At}{(t^2 + \sin^2 \theta)^{k/2}}.
 \end{aligned}$$

It follows from these estimates that

$$(27) \quad |B(t, \theta)| \leq \frac{At}{\left( t^2 + \sin^2 \frac{\theta}{2} \right)^{k/2}}$$

for  $0 \leq \theta \leq \pi$ . Now we wish to estimate  $\psi(t)$ . It is shown in [7, pp. 286 and 282] that

$$\begin{aligned}
 (28) \quad &F\left(\frac{\lambda + \frac{5}{2}}{2}, \frac{\lambda + \frac{7}{2}}{2}; \lambda + 2; z^{-2}\right) \\
 &= (1 - z^{-2})^{-1} F\left(\frac{\lambda + \frac{1}{2}}{2}, \frac{\lambda + \frac{3}{2}}{2}; \lambda + 2; z^{-2}\right)
 \end{aligned}$$

and

$$(29) \quad \lim_{x \rightarrow -0} F\left(\frac{\lambda + \frac{1}{2}}{2}, \frac{\lambda + \frac{3}{2}}{2}; \lambda + 2; x\right) = L,$$

where

$$L = \frac{\Gamma(\lambda + 2)}{\Gamma\left(\frac{\lambda + 5/2}{2}\right)\Gamma\left(\frac{\lambda + 1/2}{2}\right)} .$$

So, by (25) and (28), we have that

$$\begin{aligned} \psi(t) = & \left[ 1 - \frac{L\Gamma\left(\lambda + \frac{5}{2}\right)t}{2^{\lambda+3/2}\pi^{1/2}\Gamma(\lambda + 2)} \int_0^\infty s^{-\lambda-5/2}z^{-\lambda-7/2}(1 - z^{-2})^{-1}ds \right] \\ (30) \quad & + \frac{\Gamma\left(\lambda + \frac{5}{2}\right)t}{2^{\lambda+3/2}\pi^{1/2}\Gamma(\lambda + 2)} \int_0^\infty s^{-\lambda-5/2}z^{-\lambda-7/2} \\ & \times \left[ \frac{L - F\left(\frac{\lambda + 1/2}{2}, \frac{\lambda + 3/2}{2}; \lambda + 2; z^{-2}\right)}{1 - z^{-2}} \right] ds . \end{aligned}$$

Using (29), [7, p. 281], the mean value theorem, and then estimates of the type (20) and (21), all in the second term of the right member of (30), we obtain after simplifying the gamma functions which occur in the first term of the right member of (30) that

$$(31) \quad \psi(t) = \left[ 1 - \frac{t}{\pi} \int_0^\infty s^{-\lambda-5/2}z^{-\lambda-7/2}(1 - z^{-2})^{-1}ds \right] + O(t)$$

as  $t \rightarrow 0$ . It follows from (31) and (16) and the change of variable  $s = 1 + xt$  that

$$\begin{aligned} \psi(t) = & O(t) + 1 - 2^{\lambda+3/2} \frac{t}{\pi} \int_0^\infty (s^2 + t^2 + 1)^{-\lambda-3/2} \\ & \times \left[ \frac{1}{(s-1)^2 + t^2} - \frac{1}{(s+1)^2 + t^2} \right] ds \\ (32) \quad = & O(t) + 1 - 2^{\lambda+3/2} \frac{t}{\pi} \int_0^\infty (s^2 + t^2 + 1)^{-\lambda-3/2} [(s-1)^2 + t^2]^{-1} ds \\ = & O(t) + 1 - 2^{\lambda+3/2} \frac{t}{\pi} \int_{1/2}^{3/2} (s^2 + t^2 + 1)^{-\lambda-3/2} [(s-1)^2 + t^2]^{-1} ds \\ = & O(t) + 1 - \frac{1}{\pi} \int_{-1/2t}^{1/2t} \left( 1 + \frac{t^2 + x^2t^2 + 2xt}{2} \right)^{-\lambda-3/2} \frac{dx}{1 + x^2} \end{aligned}$$

as  $t \rightarrow 0$ . So, assuming as we may that  $0 < t < 1/4$  and using the binomial theorem, we obtain from (32) that

$$\begin{aligned} (33) \quad \psi(t) = & O(t) + \left[ 1 - \frac{1}{\pi} \int_{-1/2t}^{1/2t} \frac{dx}{1 + x^2} \right] \\ & + \frac{(\lambda + 3/2)}{2\pi} \int_{-1/2t}^{1/2t} \frac{t^2 + x^2t^2 + 2xt}{1 + x^2} dx \end{aligned}$$



$$\begin{aligned}
& + \frac{1}{\pi} \int_{-1/2t}^{1/2t} \frac{O((t^2 + x^2t^2 + 2xt)^2) dx}{1 + x^2} \\
& = O(t) + O(t) + O(t) + O(t) = O(t)
\end{aligned}$$

as  $t \rightarrow 0$ . Finally, (9) follows from (23), (27), and (33).

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Received March 3, 1971.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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