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**ON GROUPS OF EXPONENT FOUR SATISFYING AN ENGEL
CONDITION**

R. B. QUINTANA AND CHARLES R. B. WRIGHT

ON GROUPS OF EXPONENT FOUR SATISFYING AN ENGEL CONDITION

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Let $B(n)$ be the Burnside (i.e., freest) group of exponent 4 on n generators. It is known that $B(n)$ is nilpotent of class at most $3n - 1$. This paper exhibits a commutator of length $3n - 1$ in $B(n)$ which must be nontrivial if the class is exactly $3n - 1$. The methods also yield an easy proof of the following.

THEOREM. Let $E(n)$ be $B(n)$ reduced modulo the identical commutator relation

$$(a_1, \dots, a_{2n-4}, x, x, (y, z, z, z)) = 1.$$

Then $E(n)$ is nilpotent of class at most $2n + 3$.

As an immediate corollary, every n -generator group of exponent 4 satisfying the Engel condition $(x, y, y, y) = 1$ identically is of class at most $2n + 3$.

The theorem follows from Proposition 1 together with an elementary commutator calculation. The main point of the Proposition, however, is that it exhibits the stumbling block to a reduction in the class of $B(n)$ below $3n - 1$ and at the same time suggests that perhaps if for some n the class is less than $3n - 1$ then the class in general is at most $2n + k$ for some fixed k . Recent work of Gupta and others ([1], [2], [3]) has renewed interest in precise determination of the class and also in groups of exponent 4 satisfying Engel conditions. This paper updates the techniques of [4] as they appear to apply to these problems.

PRELIMINARIES. This paper may be viewed as a continuation of [4]. Notation is the same, and for $i = 1, \dots, 9$, A we denote formula (i) of [4] by (i) here, too. The symbol (i) in the margin at the right of an equation or congruence indicates that identity (i) justifies it. The notation $\langle x, \dots, y \rangle$ stands for the group generated by $\{x, \dots, y\}$.

LEMMA. *The following commutator identities hold in a group of exponent 4.*

- (B). $(x, (u, v, w)) \equiv (x, u, w, v,)(x, v, w, u) \pmod{\langle x, u, w, v \rangle_5}$.
- (C). $(x, y, y, z, z, z) \equiv 1 \pmod{\langle x, y, z \rangle_7}$.
- (D). $(x, y, y, y, (z, w)) \equiv 1 \pmod{\langle x, y, z, w \rangle_7}$.

Proof. Since

$$\begin{aligned}
 &(x, (u, v, w)) \\
 &\equiv (x, (u, v), w)(x, w, (u, v)) && (3) \\
 &\equiv (x, u, v, w)(x, v, u, w)(x, u, (v, w))(x, v, (u, w)) && (3), (4) \\
 &\equiv (x, u, w, v)(x, v, w, u), && (3)
 \end{aligned}$$

(B) holds.

Since

$$(x, y, y, z, z, z) \equiv (x, y, y, z)^2 \equiv ((x, y, y)^2, z) \equiv 1$$

by (2) and Theorem 2 of [4], (C) holds.

Finally, since

$$\begin{aligned}
 (z, w, (x, y, y, y)) &\equiv (z, w, (x, y), y, y)(z, w, y, y, (x, y)) \\
 &\equiv (z, w, x, y, y, y)(z, w, y, x, y, y) \\
 &\quad \times (z, w, y, y, x, y)(z, w, y, y, y, x) \equiv 1 && (3)
 \end{aligned}$$

by (7) and (8), (D) holds.

LEMMA. *Let G be a group of exponent 4 with $G_{r+1} = 1$, and let a and x be in G . Then every commutator in G of length r of form*

$$(\dots, x, x, a, x)$$

is a product of commutators of forms

$$(a, \dots, x, x, x)$$

and

$$(a, \dots, x, x, b, x)$$

each with the same entries as the given commutator.

Proof. By induction on r . Since $(x, x, a, x) = 1$, and

$$\begin{aligned}
 (b, x, x, a, x) &\equiv (b, x^2, a, x) \\
 &\equiv (a, x^2, b, x)(a, b, x^2, x) && (3) \\
 &\equiv (a, x, x, b, x)(a, b, x, x, x),
 \end{aligned}$$

the result is true for $r \leq 5$. Now by (B),

$$\begin{aligned}
 (c, \dots, d, e, x, x, a, x) &\equiv (c, \dots, d, e, x^2, a, x) \\
 &\equiv (c, \dots, d, a, x^2, e, x)(c, \dots, d, (a, e, x^2), x) && (B) \\
 &\equiv (c, \dots, d, a, x^2, e, x)(c, \dots, d, (a, e), x, x, x) \\
 &\quad \times (c, \dots, d, x, x, (a, e), x). && (3)
 \end{aligned}$$

The first two factors are products of commutators of the required forms by (A). The last factor is a product of commutators of forms

$$(a, e, \dots, x, x, x)$$

and

$$(a, e, \dots, x, x, b, x)$$

by the inductive assumption.

A consequence of this result is that Lemma 2 of [4] can be strengthened by the additional conclusion that $y_1 = x_1$, i.e., that the first entry in (x_1, \dots, x_n) can be held fixed. It is clear from the proof of Lemma 2 that each commutator which arises has x_1, \dots, x_n in some order as its entries.

The main results.

PROPOSITION 1. *Let G be a group of exponent 4, and let $r \geq 3n \geq 6$. Modulo G_{r+1} , every commutator (a_1, \dots, a_r) in which some n entries each appear three or more times is a product of commutators of form*

$$(a, b, \dots, x_1, x_1, x_2, x_2, \dots, x_{n-1}, x_{n-1}, c, x_{n-1}, \dots, x_1)$$

with entries some permutation of a_1, \dots, a_r .

Proof. We may assume that $G_{r+1} = 1$, that $r > 3n$, by Theorem 3 of [4], and that no entry in (a_1, \dots, a_r) occurs more than three times, by Theorem 1 of [4]. Say each of x_1, \dots, x_n appears three times among a_1, \dots, a_r . Since $r > 3n$, we may suppose that $a_1 = a \notin \{x_1, \dots, x_n\}$, by (A) of [4]. Since $n \geq 2$, some x_i (say x_1) appears three times among a_3, \dots, a_r . By Lemma 2 of [4] as just strengthened, we need only consider the forms

$$(I) \quad (a, \dots, x_1, x_1, x_1)$$

and

$$(II) \quad (a, \dots, x_1, x_1, b, x_1).$$

Case (I). By (7), (I) is equivalent to

$$(a, b, x_1, x_1, x_1, \dots).$$

At least two of the last $r - 5$ entries here are the same, say x_2 , since $n \geq 2$ and $a \neq x_2$. By repeated use of (D) and (3) these entries can be brought forward to give

$$(a, b, x_1, x_1, x_1, x_2, x_2, \dots).$$

By (7), $(a, b, x, x, x, y, y) \equiv (a, b, y, y, x, x, x)$, and now (C) applies. So (a_1, \dots, a_r) is trivial in this case.

Case (II). We have

$$\begin{aligned}
 &(a, c, \dots, x_1, x_1, b, x_1) \\
 &= (a, c, x_1, x_1, d, x_1, \dots, b) \tag{9} \\
 &= (a, c, x_1, d, x_1^2, \dots, b) \tag{8} \\
 &\equiv (a, c, (x_1, d), x_1^2, \dots, b)(a, c, d, x_1, x_1, x_1, \dots, b) \tag{3} \\
 &= (a, x_1^2, (x_1, c), d, \dots, b) \tag{5} \text{ and Case (I).} \\
 &= (a, x_1^2, c, x_1, d, \dots, b)(a, x_1, x_1, x_1, c, d, \dots, b) \tag{3} \\
 &= (a, x_1^2, c, x_1, d, \dots, b),
 \end{aligned}$$

the last step by the argument of Case (I).

Suppose inductively that we have reached the form

$$(a, x_1^2, \dots, x_i^2, c, x_i, \dots, x_1, \dots)$$

with $1 \leq i < n$. Some three of the last $r - 3i - 2$ entries are the same, say x_{i+2} , and the argument just given yields the form

$$(a, x_1^2, \dots, x_i^2, x_{i+1}^2, c, x_{i+1}, x_i, \dots, x_1, \dots),$$

where the improved Lemma 2 is used to keep the starting block of length $3i - 2$ at the front. The proposition follows by finite induction, using (9).

Together with (D), Proposition 1 shows in particular that $B(n)_{3n-1} = 1$ precisely if all commutators of form

$$(a^2, x_1^2, x_2^2, x_3^2, \dots, x_{n-1}^2, x_1, x_{n-1}, \dots, x_3, x_2)$$

are trivial.

PROPOSITION 2. *Let G be a group of exponent 4. Let $m \geq 9$. If every commutator of length $m - 1$ in G of form*

$$(\dots, x, x, (w, y, y, y))$$

is in G_{m+1} , then every commutator of length m in G of form

$$(\dots, x, x, y, y, z, y, x)$$

is in G_{m+1} .

Proof. We may assume that $G_{m+1} = 1$. Now for $a \in G_{m-7}$

$$\begin{aligned}
 &(a, x, x, y, y, z, y, x) \\
 &= (a, x, x, z, y, y, x, y) \tag{9} \\
 &= (a, x, x, z, y^2, (x, y))(a, x, x, z, y, y, y, x) \tag{3} \\
 &= (a, x, x, (x, y), y^2, z)(a, x, x, (x, y, z, y^2)) \\
 &\quad \times (a, x, x, y, y, z, x) \tag{B), (7)}
 \end{aligned}$$

$$\begin{aligned}
 &= (a, x, x, x, y, y, y, z)(a, x, x, y, x, y, y, z) \\
 &\quad \times (a, x, x, (x, y, z, y^2)) \qquad (3), (C) \\
 &= (a, x, x, x, y, z, y, y)(a, x, x, (x, y, z, y^2)) \quad (C), (8), (9) \\
 &= (a, x, x, (y, x, z, y^2)) \qquad (C) \\
 &= (a, x, x, (y, x, y^2)(y, z, y^2)(y, xz, y^2)) = 1
 \end{aligned}$$

by hypothesis.

Now let $n \geq 3$ and let $E(n)$ be $B(n)$ reduced modulo the identical relation

$$(a_1, \dots, a_{2n-4}, x, x, (y, z, z, z)) = 1 .$$

By Proposition 2 with $m = 2n + 3$, every commutator of length $2n + 3$ in $E(n)$ of form $(\dots, x, x, y, y, z, y, x)$ is in $E(n)_{2n+4}$. Hence, by Proposition 1, every commutator of length 2_{n+4} in $E(n)$ in which three or more entries each appear three times is in $E(n)_{2n+4}$. Finally, by Theorem 1 of [4], every commutator of length $2n + 3$ in $E(n)$ in which some entry appears four or more times is in $E(n)_{2n+4}$. The theorem stated in the introduction now follows.

Added in proof. By substituting uv for y in (C) and linearizing, one obtains $(u, v, x, z, z, z) \equiv 1 \pmod{\langle u, v, x, z \rangle_7}$, which shortens some of the arguments given above.

I. D. Ivanjuta [Certain groups of exponent four, *Dopovidĭ Akad. Nauk Ukrain RSR Ser. A* (1969), 787-790] has shown that every n -generator group of exponent 4 satisfying $(x, y, y, y) = 1$ identically has class at most $2n$. His methods are specific to such groups, however, and do not apply readily to $B(n)$ or $E(n)$.

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Pacific Journal of Mathematics

Vol. 44, No. 2

June, 1973

Tsuyoshi Andô, <i>Closed range theorems for convex sets and linear liftings</i>	393
Richard David Bourgin, <i>Conically bounded sets in Banach spaces</i>	411
Robert Jay Buck, <i>Hausdorff dimensions for compact sets in R^n</i>	421
Henry Cheng, <i>A constructive Riemann mapping theorem</i>	435
David Fleming Dawson, <i>Summability of subsequences and stretchings of sequences</i>	455
William Thomas Eaton, <i>A two sided approximation theorem for 2-spheres</i>	461
Jay Paul Fillmore and John Herman Scheuneman, <i>Fundamental groups of compact complete locally affine complex surfaces</i>	487
Avner Friedman, <i>Bounded entire solutions of elliptic equations</i>	497
Ronald Francis Gariepy, <i>Multiplicity and the area of an $(n - 1)$ continuous mapping</i>	509
Andrew M. W. Glass, <i>Archimedean extensions of directed interpolation groups</i>	515
Morisuke Hasumi, <i>Extreme points and unicity of extremum problems in H^1 on polydiscs</i>	523
Trevor Ongley Hawkes, <i>On the Fitting length of a soluble linear group</i>	537
Garry Arthur Helzer, <i>Semi-primary split rings</i>	541
Melvin Hochster, <i>Expanded radical ideals and semiregular ideals</i>	553
Keizō Kikuchi, <i>Starlike and convex mappings in several complex variables</i>	569
Charles Philip Lanski, <i>On the relationship of a ring and the subring generated by its symmetric elements</i>	581
Jimmie Don Lawson, <i>Intrinsic topologies in topological lattices and semilattices</i>	593
Roy Bruce Levow, <i>Counterexamples to conjectures of Ryser and de Oliveira</i>	603
Arthur Larry Lieberman, <i>Some representations of the automorphism group of an infinite continuous homogeneous measure algebra</i>	607
William George McArthur, <i>G_δ-diagonals and metrization theorems</i>	613
James Murdoch McPherson, <i>Wild arcs in three-space. II. An invariant of non-oriented local type</i>	619
H. Millington and Maurice Sion, <i>Inverse systems of group-valued measures</i>	637
William James Rae Mitchell, <i>Simple periodic rings</i>	651
C. Edward Moore, <i>Concrete semispaces and lexicographic separation of convex sets</i>	659
Jingyal Pak, <i>Actions of torus T^n on $(n + 1)$-manifolds M^{n+1}</i>	671
Merrell Lee Patrick, <i>Extensions of inequalities of the Laguerre and Turán type</i>	675
Harold L. Peterson, Jr., <i>Discontinuous characters and subgroups of finite index</i>	683
S. P. Philipp, <i>Abel summability of conjugate integrals</i>	693
R. B. Quintana and Charles R. B. Wright, <i>On groups of exponent four satisfying an Engel condition</i>	701
Marlon C. Rayburn, <i>On Hausdorff compactifications</i>	707
Martin G. Ribe, <i>Necessary convexity conditions for the Hahn-Banach theorem in metrizable spaces</i>	715
Ryōtarō Satō, <i>On decomposition of transformations in infinite measure spaces</i>	733
Peter Drummond Taylor, <i>Subgradients of a convex function obtained from a directional derivative</i>	739
James William Thomas, <i>A bifurcation theorem for k-set contractions</i>	749
Clifford Edward Weil, <i>A topological lemma and applications to real functions</i>	757
Stephen Andrew Williams, <i>A nonlinear elliptic boundary value problem</i>	767
Pak-Ken Wong, <i>$*$-actions in A^*-algebras</i>	775