ON GROUPS OF EXPONENT FOUR SATISFYING AN ENGEL CONDITION

R. B. QUINTANA AND CHARLES R. B. WRIGHT
ON GROUPS OF EXPONENT FOUR Satisfying An Engel Condition

R. B. Quintana, Jr. And C. R. B. Wright

Let $B(n)$ be the Burnside (i.e., freest) group of exponent 4 on $n$ generators. It is known that $B(n)$ is nilpotent of class at most $3n - 1$. This paper exhibits a commutator of length $3n - 1$ in $B(n)$ which must be nontrivial if the class is exactly $3n - 1$. The methods also yield an easy proof of the following.

Theorem. Let $E(n)$ be $B(n)$ reduced modulo the identical commutator relation

$$(a_1, \cdots, a_{2n-4}, x, x, (y, z, z, z)) = 1.$$

Then $E(n)$ is nilpotent of class at most $2n + 3$.

As an immediate corollary, every $n$-generator group of exponent 4 satisfying the Engel condition $(x, y, y, y) = 1$ identically is of class at most $2n + 3$.

The theorem follows from Proposition 1 together with an elementary commutator calculation. The main point of the Proposition, however, is that it exhibits the stumbling block to a reduction in the class of $B(n)$ below $3n - 1$ and at the same time suggests that perhaps if for some $n$ the class is less than $3n - 1$ then the class in general is at most $2n + k$ for some fixed $k$. Recent work of Gupta and others ([1], [2], [3]) has renewed interest in precise determination of the class and also in groups of exponent 4 satisfying Engel conditions. This paper updates the techniques of [4] as they appear to apply to these problems.

Preliminaries. This paper may be viewed as a continuation of [4]. Notation is the same, and for $i = 1, \cdots, 9$, $A$ we denote formula (i) of [4] by (i) here, too. The symbol (i) in the margin at the right of an equation or congruence indicates that identity (i) justifies it. The notation $\langle x, \cdots, y \rangle$ stands for the group generated by $\{x, \cdots, y\}$.

Lemma. The following commutator identities hold in a group of exponent 4.

(B). $(x, (u, v, w)) \equiv (x, u, w, v, x, v, w, u) \mod \langle x, u, w, v \rangle$.

(C). $(x, y, y, z, z, z) \equiv 1 \mod \langle x, y, z \rangle$.

(D). $(x, y, y, y, (z, w)) \equiv 1 \mod \langle x, y, z, w \rangle$.

Proof. Since
\[(x, (u, v, w))\]
\[\equiv (x, (u, v), w)(x, w, (u, v)) \quad (3)\]
\[\equiv (x, u, v, w)(x, v, u, w)(x, u, (v, w))(x, v, (u, w)) \quad (3), (4)\]
\[\equiv (x, u, w, v)(x, v, w, u) , \quad (3)\]

(B) holds.

Since
\[(x, y, y, z, z, z) = (x, y, y, z, z, z) = 1 \text{ by (2) and Theorem 2 of [4], (C) holds.}\]

Finally, since
\[(z, w, (x, y, y, y)) = (z, w, (x, y, y, y))\]
\[\equiv (z, w, x, y, y, y)(z, w, y, x, y, y) \equiv (z, w, y, y, x, y)(z, w, y, y, y, x) = 1 \quad (3)\]
by (7) and (8), (D) holds.

**Lemma.** Let \(G\) be a group of exponent 4 with \(G_{r+1} = 1\), and let \(a\) and \(x\) be in \(G\). Then every commutator in \(G\) of length \(r\) of form
\[(\cdots, x, x, a, x)\]
is a product of commutators of forms
\[(a, \cdots, x, x, x)\]
and
\[(a, \cdots, x, x, b, x)\]
each with the same entries as the given commutator.

**Proof.** By induction on \(r\). Since \((x, x, a, x) = 1\), and
\[\equiv (b, x^2, a, x) \quad (3)\]
\[\equiv (a, x^2, b, x)(a, b, x^2, x) \quad (3)\]
the result is true for \(r \leq 5\). Now by (B),
\[(c, \ldots, d, e, x, x, a, x) = (c, \ldots, d, e, x^2, a, x) \quad (B)\]
\[\equiv (c, \ldots, d, x, x^2, e, x)(c, \ldots, d, a, e, x^3, x) \quad (B)\]
\[\equiv (c, \ldots, d, a, x^2, e, x)(c, \ldots, d, (a, e), x, x, x) \quad (3)\]
\[\times (c, \ldots, d, x, x, (a, e), x) \quad (3)\]
The first two factors are products of commutators of the required forms by (A). The last factor is a product of commutators of forms
\[(a, e, \ldots, x, x, x)\]

and

\[(a, e, \ldots, x, x, b, x)\]

by the inductive assumption.

A consequence of this result is that Lemma 2 of [4] can be strengthened by the additional conclusion that \(y_1 = x_1\), i.e., that the first entry in \((x_1, \ldots, x_n)\) can be held fixed. It is clear from the proof of Lemma 2 that each commutator which arises has \(x_1, \ldots, x_n\) in some order as its entries.

**The main results.**

**Proposition 1.** Let \(G\) be a group of exponent 4, and let \(r \geq 3n \geq 6\). Modulo \(G_{r+1}\), every commutator \((a_1, \ldots, a_r)\) in which some \(n\) entries each appear three or more times is a product of commutators of form

\[(a, b, \ldots, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}, c, x_{n-1}, \ldots, x_1)\]

with entries some permutation of \(a_1, \ldots, a_r\).

**Proof.** We may assume that \(G_{r+1} = 1\), that \(r > 3n\), by Theorem 3 of [4], and that no entry in \((a_1, \ldots, a_r)\) occurs more than three times, by Theorem 1 of [4]. Say each of \(x_1, \ldots, x_n\) appears three times among \(a_1, \ldots, a_r\). Since \(r > 3n\), we may suppose that \(a_1 = a \in \{x_1, \ldots, x_n\}\), by (A) of [4]. Since \(n \geq 2\), some \(x_i\) (say \(x_i\)) appears three times among \(a_2, \ldots, a_r\). By Lemma 2 of [4] as just strengthened, we need only consider the forms

(I) \[(a, \ldots, x_1, x_1, x_i)\]

and

(II) \[(a, \ldots, x_1, x_1, b, x_i)\].

**Case (I).** By (7), (I) is equivalent to

\[(a, b, x_1, x_1, x_i, \ldots)\].

At least two of the last \(r - 5\) entries here are the same, say \(x_2\), since \(n \geq 2\) and \(a \neq x_2\). By repeated use of (D) and (3) these entries can be brought forward to give

\[(a, b, x_1, x_1, x_1, x_2, x_2, \ldots)\].

By (7), \((a, b, x, x, x, y, y) \equiv (a, b, y, y, x, x, x, x, \ldots)\), and now (C) applies. So \((a_1, \ldots, a_r)\) is trivial in this case.

**Case (II).** We have
Suppose inductively that we have reached the form

\[(a, x_1^i, \ldots, x_i^2, c, x_i, \ldots, x_i, \ldots)\]

with \(1 \leq i < n\). Some three of the last \(r - 3i - 2\) entries are the same, say \(x_{i+2}\), and the argument just given yields the form

\[(a, x_1^i, \ldots, x_i^2, x_i^2, c, x_{i+1}, x_i, \ldots, x_i, \ldots),\]

where the improved Lemma 2 is used to keep the starting block of length \(3i - 2\) at the front. The proposition follows by finite induction, using (9).

Together with (D), Proposition 1 shows in particular that

\[B(n)_{3n-1} = 1\]

precisely if all commutators of form

\[(a^2, x_1^2, x_2^2, x_i^2, \ldots, x_{n-1}^2, x_1, x_{n-1}, \ldots, x_3, x_2)\]

are trivial.

**PROPOSITION 2.** Let \(G\) be a group of exponent 4. Let \(m \geq 9\). If every commutator of length \(m - 1\) in \(G\) of form

\[(\ldots, x, x, (w, y, y, y))\]

is in \(G_{m+1}\), then every commutator of length \(m\) in \(G\) of form

\[(\ldots, x, x, y, y, z, y, x)\]

is in \(G_{m+1}\).

**Proof.** We may assume that \(G_{m+1} = 1\). Now for \(a \in G_{m-7}\)

\[(a, x, x, y, y, z, y, x)\]

\[= (a, x, x, z, y, y, x, y)\] (9)

\[= (a, x, x, z, y, y, x, y, y, y, x)\] (3)

\[= (a, x, x, (x, y), y, z, (a, x, x, (x, y, z, y^2))) \times (a, x, x, y, y, y, z, x)\] (B), (7)
\[(a, x, x, x, y, y, y, z)(a, x, x, y, x, y, y, z) \times (a, x, x, (x, y, z, y^2)) = (a, x, x, y, z, y, y, y) (a, x, x, (y, x, z, y^2)) (y, z, y^2)(y, xz, y^2) = 1\]

by hypothesis.

Now let \(n \geq 3\) and let \(E(n)\) be \(B(n)\) reduced modulo the identical relation

\[(a_1, \ldots, a_{2n-4}, x, x, (y, z, z, z)) = 1 .\]

By Proposition 2 with \(m = 2n + 3\), every commutator of length \(2n + 3\) in \(E(n)\) of form \((\ldots, x, x, y, y, z, y, x)\) is in \(E(n)_{2n+4}\). Hence, by Proposition 1, every commutator of length \(2n+3\) in \(E(n)\) in which three or more entries each appear three times is in \(E(n)_{2n+4}\). Finally, by Theorem 1 of [4], every commutator of length \(2n + 3\) in \(E(n)\) in which some entry appears four or more times is in \(E(n)_{2n+4}\). The theorem stated in the introduction now follows.

Added in proof. By substituting \(uv\) for \(y\) in (C) and linearizing, one obtains \((u, v, x, z, z, z) = 1 \mod \langle u, v, x, z \rangle_7\), which shortens some of the arguments given above.

I. D. Įvanjuta [Certain groups of exponent four, Dopovidi Akad. Nauk Ukrain RSR Ser. A (1969), 787-790] has shown that every \(n\)-generator group of exponent 4 satisfying \((x, y, y, y) = 1\) identically has class at most \(2n\). His methods are specific to such groups, however, and do not apply readily to \(B(n)\) or \(E(n)\).

References


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