Pacific Journal of Mathematics

ON GROUPS OF EXPONENT FOUR SATISFYING AN ENGEL CONDITION

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Vol. 44, No. 2 June 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 2, 1973

ON GROUPS OF EXPONENT FOUR SATISFYING AN ENGEL CONDITION

R. B. QUINTANA, JR. AND C. R. B. WRIGHT

Let $B(n)$ be the Burnside (i.e., freest) group of exponent 4 on n generators. It is known that $B(n)$ is nilpotent of class at most $3n-1$. This paper exhibits a commutator of length $3n-1$ in $B(n)$ which must be nontrivial if the class is exactly $3n-1$. The methods also yield an easy proof of the following.

THEOREM. **Let $E(n)$ be $B(n)$ reduced modulo the identical commutator relation**

$$(a_1, \cdots, a_{2n-4}, x, x, (y, z, z, z)) = 1.$$

Then $E(n)$ is nilpotent of class at most $2n+3$.

As an immediate corollary, every n-generator group of exponent 4 satisfying the Engel condition $(x, y, y, y) = 1$ identically is of class at most $2n+3$.

The theorem follows from Proposition 1 together with an elementary commutator calculation. The main point of the Proposition, however, is that it exhibits the stumbling block to a reduction in the class of $B(n)$ below $3n-1$ and at the same time suggests that perhaps if for some n the class is less than $3n-1$ then the class in general is at most $2n+k$ for some fixed k. Recent work of Gupta and others ([1], [2], [3]) has renewed interest in precise determination of the class and also in groups of exponent 4 satisfying Engel conditions. This paper updates the techniques of [4] as they appear to apply to these problems.

PRELIMINARIES. This paper may be viewed as a continuation of [4]. Notation is the same, and for $i = 1, \cdots, 9, A$ we denote formula (i) of [4] by (i) here, too. The symbol (i) in the margin at the right of an equation or congruence indicates that identity (i) justifies it. The notation $\langle x, \cdots, y\rangle$ stands for the group genarated by $\{x, \cdots, y\}$.

LEMMA. *The following commutator identities hold in a group of exponent* 4.

(B). $(x, (u, v, w)) \equiv (x, u, w, v,)(x, v, w, u) \bmod \langle x, u, w, v\rangle_5$.

(C). $(x, y, y, z, z, z) \equiv 1 \bmod \langle x, y, z\rangle_7$.

(D). $(x, y, y, y, (z, w)) \equiv 1 \bmod \langle x, y, z, w\rangle_7$.

Proof. Since

$$\begin{aligned}
&(x, (u, v, w)) \\
&\equiv (x, (u, v), w)(x, w, (u, v)) && (3) \\
&\equiv (x, u, v, w)(x, v, u, w)(x, u, (v, w))(x, v, (u, w)) && (3), (4) \\
&\equiv (x, u, w, v)(x, v, w, u)\ , && (3)
\end{aligned}$$

(B) holds.

Since

$$(x, y, y, z, z, z) \equiv (x, y, y, z)^2 \equiv ((x, y, y)^2, z) \equiv 1$$

by (2) and Theorem 2 of [4], (C) holds.

Finally, since

$$\begin{aligned}
(z, w, (x, y, y, y)) &\equiv (z, w, (x, y), y, y)(z, w, y, y, (x, y)) \\
&\equiv (z, w, x, y, y, y)(z, w, y, x, y, y) \\
&\quad \times (z, w, y, y, x, y)(z, w, y, y, y, x) \equiv 1 \qquad (3)
\end{aligned}$$

by (7) and (8), (D) holds.

LEMMA. *Let G be a group of exponent 4 with $G_{r+1} = 1$, and let a and x be in G. Then every commutator in G of length r of form*

$$(\cdots, x, x, a, x)$$

is a product of commutators of forms

$$(a, \cdots, x, x, x)$$

and

$$(a, \cdots, x, x, b, x)$$

each with the same entries as the given commutator.

Proof. By induction on r. Since $(x, x, a, x) = 1$, and

$$\begin{aligned}
(b, x, x, a, x) &\equiv (b, x^2, a, x) \\
&\equiv (a, x^2, b, x)(a, b, x^2, x) && (3) \\
&\equiv (a, x, x, b, x)(a, b, x, x, x)\ ,
\end{aligned}$$

the result is true for $r \leqq 5$. Now by (B),

$$\begin{aligned}
(c, \ldots, d, e, x, x, a, x) &\equiv (c, \ldots, d, e, x^2, a, x) \\
&\equiv (c, \ldots, d, a, x^2, e, x)(c, \ldots, d, (a, e, x^2), x) && (\mathrm{B}) \\
&\equiv (c, \ldots, d, a, x^2, e, x)(c, \ldots, d, (a, e), x, x, x) \\
&\quad \times (c, \ldots, d, x, x, (a, e), x)\ . && (3)
\end{aligned}$$

The first two factors are products of commutators of the required forms by (A). The last factor is a product of commutators of forms

$$(a, e, \ldots, x, x, x)$$

and

$$(a, e, \ldots, x, x, b, x)$$

by the inductive assumption.

A consequence of this result is that Lemma 2 of [4] can be strengthened by the additional conclusion that $y_1 = x_1$, i.e., that the first entry in $(x_1, \ldots, x_n)$ can be held fixed. It is clear from the proof of Lemma 2 that each commutator which arises has $x_1, \ldots, x_n$ in some order as its entries.

The main results.

PROPOSITION 1. *Let G be a group of exponent 4, and let $r \geqq 3n \geqq 6$. Modulo G_{r+1}, every commutator $(a_1, \ldots, a_r)$ in which some n entries each appear three or more times is a product of commutators of form*

$$(a, b, \ldots, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}, c, x_{n-1}, \ldots, x_1)$$

with entries some permutation of $a_1, \ldots, a_r$.

Proof. We may assume that $G_{r+1} = 1$, that $r > 3n$, by Theorem 3 of [4], and that no entry in $(a_1, \ldots, a_r)$ occurs more than three times, by Theorem 1 of [4]. Say each of $x_1, \ldots, x_n$ appears three times among $a_1, \ldots, a_r$. Since $r > 3_n$, we may suppose that $a_1 = a \notin \{x_1, \ldots, x_n\}$, by (A) of [4]. Since $n \geqq 2$, some x_i (say x_1) appears three times among $a_3, \ldots, a_r$. By Lemma 2 of [4] as just strengthened, we need only consider the forms

$$\text{(I)} \qquad (a, \ldots, x_1, x_1, x_1)$$

and

$$\text{(II)} \qquad (a, \ldots, x_1, x_1, b, x_1)\,.$$

Case (I). By (7), (I) is equivalent to

$$(a, b, x_1, x_1, x_1, \ldots)\,.$$

At least two of the last $r - 5$ entries here are the same, say x_2, since $n \geqq 2$ and $a \neq x_2$. By repeated use of (D) and (3) these entries can be brought forward to give

$$(a, b, x_1, x_1, x_1, x_2, x_2, \ldots)\,.$$

By (7), $(a, b, x, x, x, y, y) \equiv (a, b, y, y, x, x, x,)$, and now (C) applies. So $(a_1, \ldots, a_r)$ is trivial in this case.

Case (II). We have

$$\begin{aligned}
&(a, c, \ldots, x_1, x_1, b, x_1) \\
&= (a, c, x_1, x_1, d, x_1, \ldots, b) && (9) \\
&= (a, c, x_1, d, x_1^2, \ldots, b) && (8) \\
&\equiv (a, c, (x_1, d), x_1^2, \ldots, b)(a, c, d, x_1, x_1, x_1, \ldots, b) && (3) \\
&= (a, x_1^2, (x_1, c), d, \ldots, b) && (5) \text{ and Case (I).} \\
&= (a, x_1^2, c, x_1, d, \ldots, b)(a, x_1, x_1, x_1, c, d, \ldots, b) && (3) \\
&= (a, x_1^2, c, x_1, d, \ldots, b),
\end{aligned}$$

the last step by the argument of Case (I).

Suppose inductively that we have reached the form

$$(a, x_1^2, \ldots, x_i^2, c, x_i, \ldots, x_1, \ldots)$$

with $1 \leqq i < n$. Some three of the last $r - 3i - 2$ entries are the same, say x_{i+2}, and the argument just given yields the form

$$(a, x_1^2, \ldots, x_i^2, x_{i+1}^2, c, x_{i+1}, x_i, \ldots, x_1, \ldots) ,$$

where the improved Lemma 2 is used to keep the starting block of length $3i - 2$ at the front. The proposition follows by finite induction, using (9).

Together with (D), Proposition 1 shows in particular that $B(n)_{3n-1} = 1$ precisely if all commutators of form

$$(a^2, x_1^2, x_2^2, x_3^2, \ldots, x_{n-1}^2, x_1, x_{n-1}, \ldots, x_3, x_2)$$

are trivial.

PROPOSITION 2. *Let G be a group of exponent 4. Let $m \geqq 9$. If every commutator of length $m - 1$ in G of form*

$$(\ldots, x, x, (w, y, y, y))$$

is in G_{m+1}, then every commutator of length m in G of form

$$(\ldots, x, x, y, y, z, y, x)$$

is in G_{m+1}.

Proof. We may assume that $G_{m+1} = 1$. Now for $a \in G_{m-7}$

$$\begin{aligned}
&(a, x, x, y, y, z, y, x) \\
&= (a, x, x, z, y, y, x, y) && (9) \\
&= (a, x, x, z, y^2, (x, y))(a, x, x, z, y, y, y, x) && (3) \\
&= (a, x, x, (x, y), y^2, z)(a, x, x, (x, y, z, y^2)) \\
&\quad \times (a, x, x, y, y, y, z, x) && (\mathrm{B}), (7)
\end{aligned}$$

$$
\begin{aligned}
&= (a, x, x, x, y, y, y, z)(a, x, x, y, x, y, y, z) \\
&\quad \times (a, x, x, (x, y, z, y^2) && (3), (C) \\
&= (a, x, x, x, y, z, y, y)(a, x, x, (x, y, z, y^2)) && (C), (8), (9) \\
&= (a, x, x, (y, x, z, y^2)) && (C) \\
&= (a, x, x, (y, x, y^2)(y, z, y^2)(y, xz, y^2)) = 1
\end{aligned}
$$

by hypothesis.

Now let $n \geqq 3$ and let $E(n)$ be $B(n)$ reduced modulo the identical relation

$$(a_1, \ldots, a_{2n-4}, x, x, (y, z, z, z)) = 1 \,.$$

By Proposition 2 with $m = 2n + 3$, every commutator of length $2n + 3$ in $E(n)$ of form $(\ldots, x, x, y, y, z, y, x)$ is in $E(n)_{2n+4}$. Hence, by Proposition 1, every commutator of length 2_{n+4} in $E(n)$ in which three or more entries each appear three times is in $E(n)_{2n+4}$. Finally, by Theorem 1 of [4], every commutator of length $2n + 3$ in $E(n)$ in which some entry appears four or more times is in $E(n)_{2n+4}$. The theorem stated in the introduction now follows.

Added in proof. By substituting uv for y in (C) and linearizing, one obtains $(u, v, x, z, z, z) \equiv 1 \bmod \langle u, v, x, z\rangle_7$, which shortens some of the arguments given above.

Ī. D. Īvanjuta [Certain groups of exponent four, Dopovīdī Akad. Nauk Ukrain RSR Ser. A (1969), 787–790)] has shown that every n-generator group of exponent 4 satisfying $(x, y, y, y) = 1$ identically has class at most $2n$. His methods are specific to such groups, however, and do not apply readily to $B(n)$ or $E(n)$.

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Received October 1, 1971 and in revised form January 10, 1972. The second author is indebted to the National Science Foundation for contract support of his work on this paper.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 44, No. 2 June, 1973

Tsuyoshi Andô, *Closed range theorems for convex sets and linear liftings* 393
Richard David Bourgin, *Conically bounded sets in Banach spaces* 411
Robert Jay Buck, *Hausdorff dimensions for compact sets in R^n* 421
Henry Cheng, *A constructive Riemann mapping theorem* 435
David Fleming Dawson, *Summability of subsequences and stretchings of sequences* 455
William Thomas Eaton, *A two sided approximation theorem for 2-spheres* 461
Jay Paul Fillmore and John Herman Scheuneman, *Fundamental groups of compact complete locally affine complex surfaces* 487
Avner Friedman, *Bounded entire solutions of elliptic equations* 497
Ronald Francis Gariepy, *Multiplicity and the area of an $(n-1)$ continuous mapping* 509
Andrew M. W. Glass, *Archimedean extensions of directed interpolation groups* 515
Morisuke Hasumi, *Extreme points and unicity of extremum problems in H^1 on polydiscs* 523
Trevor Ongley Hawkes, *On the Fitting length of a soluble linear group* 537
Garry Arthur Helzer, *Semi-primary split rings* 541
Melvin Hochster, *Expanded radical ideals and semiregular ideals* 553
Keizō Kikuchi, *Starlike and convex mappings in several complex variables* 569
Charles Philip Lanski, *On the relationship of a ring and the subring generated by its symmetric elements* 581
Jimmie Don Lawson, *Intrinsic topologies in topological lattices and semilattices* 593
Roy Bruce Levow, *Counterexamples to conjectures of Ryser and de Oliveira* 603
Arthur Larry Lieberman, *Some representations of the automorphism group of an infinite continuous homogeneous measure algebra* 607
William George McArthur, *G_δ-diagonals and metrization theorems* 613
James Murdoch McPherson, *Wild arcs in three-space. II. An invariant of non-oriented local type* 619
H. Millington and Maurice Sion, *Inverse systems of group-valued measures* 637
William James Rae Mitchell, *Simple periodic rings* 651
C. Edward Moore, *Concrete semispaces and lexicographic separation of convex sets* 659
Jingyal Pak, *Actions of torus T^n on $(n+1)$-manifolds M^{n+1}* 671
Merrell Lee Patrick, *Extensions of inequalities of the Laguerre and Turán type* 675
Harold L. Peterson, Jr., *Discontinuous characters and subgroups of finite index* 683
S. P. Philipp, *Abel summability of conjugate integrals* 693
R. B. Quintana and Charles R. B. Wright, *On groups of exponent four satisfying an Engel condition* 701
Marlon C. Rayburn, *On Hausdorff compactifications* 707
Martin G. Ribe, *Necessary convexity conditions for the Hahn-Banach theorem in metrizable spaces* 715
Ryōtarō Satō, *On decomposition of transformations in infinite measure spaces* 733
Peter Drummond Taylor, *Subgradients of a convex function obtained from a directional derivative* 739
James William Thomas, *A bifurcation theorem for k-set contractions* 749
Clifford Edward Weil, *A topological lemma and applications to real functions* 757
Stephen Andrew Williams, *A nonlinear elliptic boundary value problem* 767
Pak-Ken Wong, *$*$-actions in A^*-algebras* 775