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**A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM**

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## A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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**This paper proves that there is a (weak) solution  $u$  (not necessarily unique) to the generalized Dirichlet problem (with null boundary data) for the equation  $Au + pu = h$ . Here  $A$  is a strongly and uniformly elliptic operator of order  $2m$  on a bounded open set  $\Omega \subseteq R^n$ . Also  $A$  is "normal": roughly,  $AA^* = A^*A$ . The functions  $p$  and  $h$  are bounded and continuous, but are allowed to depend on  $x(x \in \Omega)$ ,  $u$ , and the generalized derivatives of  $u$  up to order  $m$ . The values of  $p$  are restricted to lie in a closed disk of the complex plane which contains the negative of no weak eigenvalue of  $A$ .**

In [4], E. Landesman and A. Lazer proved that the boundary value problem

$$Lu + p\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)u = h\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) \text{ on } D$$

$$u = 0 \text{ on } \partial D$$

has a (not necessarily unique) weak solution  $u$ . Here  $D$  is any bounded open subset of  $R^n$  with boundary  $\partial D$ . Here  $L$  is any linear, uniformly and strongly elliptic, self-adjoint, second order partial differential operator with only second order terms and with real-valued, bounded measurable coefficients for its corresponding Dirichlet bilinear form. Here  $p$  and  $h$  are any real-valued, bounded, continuous functions. It is assumed that there exist constants  $\gamma_N$  and  $\gamma_{N+1}$  such that  $\alpha_N < \gamma_N \leq p(z) \leq \gamma_{N+1} < \alpha_{N+1}$  for every  $z$  in  $D \times R^{n+1}$  (here  $\alpha_N$  and  $\alpha_{N+1}$  are the negatives of successive weak eigenvalues of  $L$ ).

The present paper may perhaps best be viewed as a generalization of [4]. Although other generalizations are made, the main result is that the assumption that  $L$  is self-adjoint can be replaced by the assumption that  $L$  is "normal": roughly,  $LL^* = L^*L$ . Two examples at the end of the present paper show in what sense the result is best-possible and show that uniqueness can not be expected.

As in [4], the final existence result is proved using Schauder's theorem. In the solving of a preliminary linear problem, a contraction mapping and the fact that the spectral radius of a normal operator is equal to its norm replace the argument in [4] based on the maximum characterization of the eigenvalues and a comparison result for self-adjoint operators.

2. NOTATION. Let  $\Omega$  be a bounded open subset of  $R^n$ . Let

$C_0^\infty(\Omega)$  denote the set of all infinitely differentiable complex-valued functions with compact support in  $\Omega$ . Let  $L_2(\Omega)$  denote the Hilbert space of all complex-valued square-integrable functions on  $\Omega$ , with inner product  $(,)$  and norm  $\| \cdot \|$ . Let  $H^{(m)}(\Omega)$  denote the Hilbert space of all complex-valued functions on  $\Omega$  whose distribution derivatives (using  $C_0^\infty(\Omega)$  test functions) of order 0 through  $m$  are in  $L_2(\Omega)$ . The inner product and norm of this space will be denoted by  $(,)_m$  and  $\| \cdot \|_m$  respectively. A multi-index is an  $n$ -tuple of nonnegative integers. If  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  is a multi-index, define

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} .$$

Here the indicated derivative is a distribution derivative. It will be used only when  $u$  is in  $H^{(|\alpha|)}(\Omega)$ . Let  $H_0^{(m)}(\Omega)$  denote the Hilbert subspace of  $H^{(m)}(\Omega)$  obtained by taking the closure of the set  $C_0^\infty(\Omega)$  in  $H^{(m)}(\Omega)$ .

Let  $A$  be the formal differential operator given by

$$Au = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta} D^\beta u) ,$$

where the complex-valued functions  $a_{\alpha\beta}$  are uniformly continuous in  $\Omega$  for  $|\alpha| = |\beta| = m$  and bounded and measurable otherwise. We assume that  $A$  is uniformly strongly elliptic and normalized, i.e., that there exists a constant  $E_0 > 0$  such that for all vectors  $\xi = (\xi_1, \dots, \xi_n)$  with real entries, and for all  $x$  in  $\Omega$ ,

$$\operatorname{Re} \left\{ \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x) \xi_1^{\alpha_1 + \beta_1} \xi_2^{\alpha_2 + \beta_2} \dots \xi_n^{\alpha_n + \beta_n} \right\} \geq E_0 |\xi|^{2m}$$

where  $\operatorname{Re}$  takes the real part of any complex number and where  $|\xi|$  denotes the length of  $\xi$  in  $R^n$ .

For any  $\varphi$  and  $\psi$  in  $H_0^{(m)}(\Omega)$ , define

$$B[\varphi, \psi] = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (D^\alpha \varphi, a_{\alpha\beta} D^\beta \psi) .$$

We say that  $u$  is a solution of the generalized Dirichlet problem for  $Au = f$  if and only if  $f$  is in  $L_2(\Omega)$ ,  $u$  is in  $H_0^{(m)}(\Omega)$ , and

$$B[\varphi, u] = (\varphi, f) \text{ for every } \varphi \text{ in } H_0^{(m)}(\Omega) .$$

We say that  $\lambda$  is a weak eigenvalue for  $A$  corresponding to weak eigenfunction  $u$  if  $u \neq 0$  is a solution of the generalized Dirichlet problem for  $Au = \lambda u$ .

With the assumptions on  $A$  made above, Garding's inequality holds (see S. Agmon [1], p. 102):

$$(1) \quad \operatorname{Re} B[\phi, \phi] + \lambda_0(\phi, \phi) \geq c_0 \|\phi\|_m^2.$$

Here  $\lambda_0$  and  $c_0$  are real constants with  $c_0 > 0$ . The inequality holds for each  $\phi$  in  $C_0^\infty(\Omega)$  and hence (taking limits in  $H^{(m)}(\Omega)$ ) for each  $\phi$  in  $H_0^{(m)}(\Omega)$ . For each  $u$  in  $H_0^{(m)}(\Omega)$ , define

$$\|u\|_B = [\operatorname{Re} B(u, u) + \lambda_0(u, u)]^{1/2}.$$

An easy calculation shows that  $\|\cdot\|_B$  is bounded above by a multiple of the  $\|\cdot\|_m$  norm. Since Garding's inequality shows that it is also bounded below, these two norms on  $H_0^{(m)}(\Omega)$  are equivalent.

We are assured by [1; p. 102] that the generalized Dirichlet problem for  $Au = f - \lambda_0 u$  has for each  $f$  in  $L_2(\Omega)$  a unique solution  $T_0 f$  in  $H_0^{(m)}(\Omega)$ . The mapping  $T_0: L_2(\Omega) \rightarrow H_0^{(m)}(\Omega)$  is linear and continuous.

Let  $\mathcal{J}: H_0^{(m)}(\Omega) \rightarrow L_2(\Omega)$  denote the inclusion map and let  $I: L_2(\Omega) \rightarrow L_2(\Omega)$  denote the identity map.

3. Preliminary lemmas. Lemma 1, of interest in itself, greatly simplifies the proof of Theorem 2. Lemma 2 gives an elementary proof of the fact that the operator norm of a normal operator is equal to its spectral radius. Lemma 3 gives conditions under which a differential operator is "normal" in the sense required by this paper. Lemma 4 introduces an operator  $T$  and Lemma 5 finds an upper bound for  $\|\mathcal{J} T\|$ . These last two lemmas will be used immediately in Theorem 1.

LEMMA 1.  $T_0$  is compact as a map from  $L_2(\Omega)$  to  $H_0^{(m)}(\Omega)$ .

*Proof.* Let  $\{f_k\}$  be a sequence in  $L_2(\Omega)$  with  $\|f_k\| \leq r$ . Since  $\Omega$  is bounded, N. Dunford and J. Schwartz [3; p. 1693] assure us that  $\mathcal{J}$  is compact. There is therefore a subsequence  $\{g_i\}$  of  $\{f_k\}$  such that  $\{\mathcal{J} T_0 g_i\}$  converges in  $L_2(\Omega)$ . Use  $f = g_i - g_k$  and  $\phi = T_0 g_i - T_0 g_k$  and the definition of  $T_0$  to obtain

$$\begin{aligned} \|T_0 g_i - T_0 g_k\|_B^2 &= \operatorname{Re} B[\phi, T_0 f] + \lambda_0(\phi, \phi) \\ &\leq |B[\phi, T_0 f] + \lambda_0(\phi, \phi)| \\ &= |(\phi, f) - \lambda_0(\phi, T_0 f) + \lambda_0(\phi, \phi)| \\ &= |(\phi, f)| \leq \|f\| \|\phi\| \\ &\leq 2r \|T_0 g_i - T_0 g_k\|. \end{aligned}$$

Since  $\{T_0 g_i\}$  is a Cauchy sequence in  $L_2(\Omega)$ ,  $\{T_0 g_i\}$  is a Cauchy sequence

in  $H_0^{(m)}(\Omega)$  with the  $\|\cdot\|_B$  norm. Therefore it is Cauchy under the  $\|\cdot\|_m$  norm.<sup>1</sup>

**LEMMA 2.** *If  $N$  is a normal operator in a Hilbert space with inner product  $(\cdot, \cdot)$  and norm  $\|\cdot\|$ , then  $\|N\|$ , the operator norm of  $N$ , is equal to its spectral radius.*

*Proof.* For any  $x$  in the Hilbert space,  $(N^2x, N^2x) = (N^*Nx, N^*Nx)$  and thus  $\|N^2\| = \|N^*N\|$ . But for any operator in a Hilbert space,  $\|N^*N\| = \|N\|^2$  (see [3], p. 874). Thus  $\|N^2\| = \|N\|^2$ . By induction  $\|N^p\| = \|N\|^p$  whenever  $p$  is a power of 2. The spectral radius of  $N$  is given by the expression

$$\lim_{p \rightarrow \infty} \|N^p\|^{1/p} \quad (\text{see [3], p. 864}).$$

Considering the subsequence involving only those  $p$  which are powers of 2, the result follows.<sup>2</sup>

**LEMMA 3.** *Let  $A$  be a differential operator with coefficients having enough continuous derivatives so that  $A^*$ ,  $AA^*$ , and  $A^*A$  make sense classically on  $C_0^\infty(\Omega)$ . Suppose that  $AA^* = A^*A$ . Then  $\mathcal{S}T_0$  is a normal operator.*

*Proof.* The discussion in [1; pp. 97–103] shows that the generalized Dirichlet problem for  $A^*u = f - \lambda_0 u$  has for every  $f$  in  $L_2(\Omega)$  a unique solution  $T_0^*f$  in  $H_0^{(m)}(\Omega)$ , where  $\lambda_0$  is the same constant as was used to define  $T_0$ . For  $\varphi$  and  $\psi$  in  $C_0^\infty(\Omega)$  the Dirichlet form for  $A$  is given by  $B[\varphi, \psi] = B_A[\varphi, \psi] = (\varphi, A\psi)$ . Similarly  $B_{A^*}[\varphi, \psi] = (\varphi, A^*\psi)$ . It follows easily that  $\mathcal{S}T_0^*$  is the adjoint of  $\mathcal{S}T_0$ .

The Dirichlet form for  $(A + \lambda_0)^*(A + \lambda_0)$  is given by

$$B_{(A+\lambda_0)^*(A+\lambda_0)}[\varphi, \psi] = (\varphi, (A + \lambda_0)^*(A + \lambda_0)\psi) = ([A + \lambda_0]\varphi, [A + \lambda_0]\psi).$$

An easy calculation shows that the Dirichlet form for  $(A + \lambda_0)(A + \lambda_0)^*$  is the same since  $AA^* = A^*A$ . If  $u$  is a solution of the generalized Dirichlet problem for  $(A + \lambda_0)^*(A + \lambda_0)u = 0$ , then

$$([A + \lambda_0]u, [A + \lambda_0]u) = 0,$$

so  $(A + \lambda_0)u = 0$  and hence finally  $u = 0$ . By the Fredholm alternative the generalized Dirichlet problem for  $(A + \lambda_0)^*(A + \lambda_0)u = f$  has a unique solution  $u$  in  $H_0^{(2m)}(\Omega)$ . It is easy to see that  $\mathcal{S}T_0^*\mathcal{S}T_0f = u = \mathcal{S}T_0\mathcal{S}T_0^*f$ . Thus  $\mathcal{S}T_0^*\mathcal{S}T_0 = \mathcal{S}T_0\mathcal{S}T_0^*$ .

<sup>1</sup> The proof of this lemma is motivated by a similar calculation in [4; pp. 321, 322].

<sup>2</sup> The author wishes to thank Dr. S. Ebenstein for his elementary proof of Lemma 2.

LEMMA 4. *If  $\gamma_0$  is a complex number such that  $-\gamma_0$  is not a weak eigenvalue of  $A$ , then we may set  $T = T_0[(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}$  and have for every  $f$  in  $L_2(\Omega)$  and every  $\varphi$  in  $H_0^{(m)}(\Omega)$  that*

$$B[\varphi, Tf] + \overline{\gamma_0}(\varphi, Tf) = (\varphi, f).$$

(Thus  $Tf$  is the unique weak solution of  $Au + \gamma_0 u = f$ .)

*Proof.* Since  $-\gamma_0$  is not a weak eigenvalue of  $A$ ,  $(\lambda_0 - \gamma_0)^{-1}$  is not an eigenvalue of  $\mathcal{S}T_0$ . Since  $\mathcal{S}T_0$  is compact, every nonzero complex number in its spectrum must be an eigenvalue. Therefore  $(\lambda_0 - \gamma_0)^{-1}$  is not in the spectrum of  $\mathcal{S}T_0$ , so  $[\mathcal{S}T_0 - (\lambda_0 - \gamma_0)^{-1}I]^{-1}$  (and hence  $(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}$ ) exists and is continuous.

$$\begin{aligned} B[\varphi, Tf] + \overline{\gamma_0}(\varphi, Tf) &= -\lambda_0(\varphi, T_0[(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}f) + (\varphi, [(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}f) \\ &\quad + \overline{\gamma_0}(\varphi, T_0[(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}f) \\ &= (\varphi, [(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I][(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}f) = (\varphi, f). \end{aligned}$$

LEMMA 5. *Assume that  $\mathcal{S}T_0$  is a normal operator and that  $|z - \gamma_0| \leq c$  is a disk in the complex plane which contains the negative of no weak eigenvalue of  $A$ . Then  $\|\mathcal{S}T\|c < 1$ , where  $T$  is the map of the above lemma.*

*Proof.* Since  $\mathcal{S}T_0$  is a normal operator, so is  $[(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}$ . Since  $\mathcal{S}T_0$  and this operator commute,

$$\mathcal{S}T = \mathcal{S}T_0[(\gamma_0 - \lambda_0)\mathcal{S}T_0 + I]^{-1}$$

is normal. Therefore  $\|\mathcal{S}T\|$  is the same as the spectral radius of  $\mathcal{S}T$ . Since  $\mathcal{S}T$  is compact, the spectral radius is the supremum of the norms of the eigenvalues of  $\mathcal{S}T$ . But  $\lambda$  is a weak eigenvalue of  $A$  if and only if  $(\lambda + \gamma_0)^{-1}$  is an eigenvalue of  $\mathcal{S}T$ . Thus the weak eigenvalues of  $A$  have no accumulation point in the (finite) complex plane. Since  $|\lambda + \gamma_0| \geq c + \varepsilon$  for some  $\varepsilon > 0$  and every weak eigenvalue  $\lambda$  of  $A$ ,  $|(\lambda + \gamma_0)^{-1}| \leq (c + \varepsilon)^{-1}$  so that every eigenvalue of  $\mathcal{S}T$  has norm  $\leq (c + \varepsilon)^{-1}$ . Thus  $\|\mathcal{S}T\|c < 1$  as claimed.

#### 4. The preliminary linear problem.

THEOREM 1. *Let  $D$  be a closed disk  $\{z \in \mathbb{C}; |z - \gamma_0| \leq c\}$  in the complex plane which contains the negative of no weak eigenvalue of  $A$ . Let  $h$  be in  $L_2(\Omega)$  and let  $p$  be a measurable function on  $\Omega$  whose values lie in the disk  $D$ . Suppose that the operator  $\mathcal{S}T_0$  associated with  $A$  is normal. Then the generalized Dirichlet problem*

for  $Au + pu = h$  has a unique solution  $u$  in  $H_0^{(m)}(\Omega)$ . Moreover, there exists a constant  $M$  independent of  $p$  such that

$$\operatorname{Re} B[u, u] + \lambda_0(u, u) \leq M(h, h) .$$

*Proof.* We want  $Au + pu = h$ , or equivalently  $Au + \gamma_0 u = h - (p - \gamma_0)u$ . Thus we want  $u = T(h - (p - \gamma_0)u)$ , where  $T$  is the map of Lemmas 4 and 5. We prove that the map from  $L_2(\Omega)$  into itself given by  $u \rightarrow \mathcal{S}T[h - (p - \gamma_0)u]$  is a contraction map.

For any  $u_1$  and  $u_2$  in  $L_2(\Omega)$ ,

$$\begin{aligned} & \|\mathcal{S}T[h - (p - \gamma_0)u_1] - \mathcal{S}T[h - (p - \gamma_0)u_2]\| \\ &= \|\mathcal{S}T(p - \gamma_0)(u_1 - u_2)\| \leq \|\mathcal{S}T\|c \|u_1 - u_2\| . \end{aligned}$$

Since  $\|\mathcal{S}T\|c < 1$  by Lemma 5, the map is a contraction as claimed. Thus there exists a unique  $v$  in  $L_2(\Omega)$  such that  $v = \mathcal{S}T[h - (p - \gamma_0)v]$ .

Let  $Q = \|\mathcal{S}T\|(1 - \|\mathcal{S}T\|c)^{-1}$ . Then  $Q = \|\mathcal{S}T\| + \|\mathcal{S}T\|cQ$ . Since  $\|u\| \leq Q\|h\|$  implies that

$$\begin{aligned} \|\mathcal{S}T[h - (p - \gamma_0)u]\| &\leq \|\mathcal{S}T\| \|h\| + c\|\mathcal{S}T\| \|u\| \\ &\leq \|\mathcal{S}T\| \|h\| + c\|\mathcal{S}T\|Q\|h\| \\ &= Q\|h\| , \end{aligned}$$

it follows that for fixed  $h$  the ball  $\{u \in L_2(\Omega); \|u\| \leq Q\|h\|\}$  is mapped into itself by our contraction map. Therefore the fixed point  $v$  satisfies  $\|v\| \leq Q\|h\|$ . Since the  $\|\cdot\|_m$  norm and the  $\|\cdot\|_B$  norm are equivalent, and since

$$\|v\|_m = \|T[h - (p - \gamma_0)v]\|_m \leq \|T\| \|h - (p - \gamma_0)v\| ,$$

(here  $\|T\|$  is the operator norm of  $T: L_2(\Omega) \rightarrow H_0^{(m)}(\Omega)$ ) it follows easily that there exists an  $M$  such that  $\|v\|_B^2 \leq M\|h\|^2$ .

### 5. The nonlinear problem.

**THEOREM 2.** *Let  $D$  be a closed disk in the complex plane which contains the negative of no weak eigenvalue of  $A$ . Let  $h(x, u, \partial u/\partial x_1, \dots)$  and  $p(x, u, \partial u/\partial x_1, \dots)$  be continuous functions of their arguments, allowed to involve derivatives of  $u$  up to order  $m$ . Let  $|h(x, u, \dots)| \leq r$  and assume that the values of  $p$  are always in the disk  $D$ . Assume that the operator  $\mathcal{S}T_0$  associated with  $A$  is normal. Then the generalized Dirichlet problem for*

$$(3) \quad Au + p\left(x, u, \frac{\partial u}{\partial x_1}, \dots\right)u = h\left(x, u, \frac{\partial u}{\partial x_1}, \dots\right)$$

has a (not necessarily unique) solution  $u$  in  $H_0^{(m)}(\Omega)$ .

*Proof.* Define a map  $G: H_0^{(m)}(\Omega) \rightarrow H_0^{(m)}(\Omega)$  as follows: for every  $u$  in  $H_0^{(m)}(\Omega)$ , let  $G(u)$  be the unique solution  $v$  in  $H_0^{(m)}(\Omega)$  of

$$v = \mathcal{S} T \left[ h \left( x, u, \frac{\partial u}{\partial x_1}, \dots \right) - \left( p \left( x, u, \frac{\partial u}{\partial x_1}, \dots \right) - \gamma_0 \right) v \right],$$

where  $\gamma_0$  is the center of the disk  $D$  and  $T$  is the operator of Lemmas 4 and 5. It is clear that a fixed point of  $G$  would furnish a solution for the generalized Dirichlet problem for (3). We will show that  $G$  is continuous and compact from a bounded, closed, convex subset  $S$  of  $H_0^{(m)}(\Omega)$  into itself. Schauder's theorem (see, for example, J. Cronin [2], p. 131) then assures us a fixed point.

Since  $|h(x, u, \dots)| \leq r$ ,  $(h, h) \leq R = r^2 \text{meas}(\Omega) < \infty$ . Using the constant  $M$  of Theorem 1,  $\|G(u)\|_B^2 \leq MR$  for all  $u$  in  $H_0^{(m)}(\Omega)$ . Thus if we take  $S = \{u \in H_0^{(m)}(\Omega); \|u\|_B^2 \leq MR\}$ ,  $S$  is a bounded, closed, convex set of  $H_0^{(m)}(\Omega)$  and  $G(S) \subseteq S$ .

Now we show that  $G$  is continuous. Let  $\{u_k\}$  be a sequence in  $H_0^{(m)}(\Omega)$  converging to  $u$ . The sequence  $\{h(x, u_k, \dots) - (p(x, u_k, \dots) - \gamma_0)G(u_k)\}$  is clearly bounded in  $L_2(\Omega)$ , so since  $T$  is compact (Lemma 1 shows that  $T_0$  is compact, and  $T$  is  $T_0$  composed with a continuous map) there is a subsequence of  $\{G(u_k)\}$  which converges in  $H_0^{(m)}(\Omega)$  to a limit  $v$ . Then taking limits with the corresponding subsequence of  $\{u_k\}$ ,

$$v = \mathcal{S} T [h(x, u, \dots) - (p(x, u, \dots) - \gamma_0)v],$$

so that  $v = G(u)$ . Since any subsequence of  $\{G(u_k)\}$  has a subsequence converging in  $H_0^{(m)}(\Omega)$  to  $G(u)$ ,  $\{G(u_k)\}$  itself converges in  $H_0^{(m)}(\Omega)$  to  $G(u)$ , proving continuity.

Now we show that  $G$  is compact. Let  $\{u_k\}$  be a bounded sequence in  $H_0^{(m)}(\Omega)$ . Then the sequence  $\{h(x, u_k, \dots) - (p(x, u_k, \dots) - \gamma_0)G(u_k)\}$  is bounded in  $L_2(\Omega)$ , so the fact that  $T$  is compact assures us a subsequence of  $\{G(u_k)\}$  which converges in  $H_0^{(m)}(\Omega)$ .

### 6. Examples and a remark.

**EXAMPLE 1.** If the disk  $D$  includes the negative of a weak eigenvalue  $\lambda$  of  $A$ , let  $v$  be a weak eigenfunction of  $A^*$  corresponding to the weak eigenvalue  $\bar{\lambda}$ . If  $h(x)$  is any bounded continuous function on  $\Omega$  such that  $(h, v) \neq 0$ , then the generalized Dirichlet problem for  $Au + \lambda u = h$  has no solution, since the Fredholm alternative applies [1, p. 102]. It is in this sense that Theorem 2 is best possible.

**EXAMPLE 2.** Suppose that there is a weak eigenvalue  $\lambda$  of  $A$  which corresponds to a continuous weak eigenfunction  $v$  with  $|v(x)| \leq 1$  for every  $x$  in  $\Omega$ . Let  $\gamma_0$  be the center of the disk  $D$  and let  $p = \gamma_0$



identically. Let  $h = h(u)$  be a bounded  $C^\infty$  function of  $u$  with  $h(u) = \gamma_0 u + \lambda u$  for  $|u| \leq 1$ . Then  $v$  and  $v/2$  are two distinct solutions of the generalized Dirichlet problem for  $Au + pu = h$ . This shows that we cannot expect a unique solution to problems of the type discussed in this paper.

REMARK. Consider the generalized Dirichlet problem for  $Au = f(x, u, \partial u/\partial x_i, \dots)$ , where  $f$  is a continuous function of its arguments, involving derivatives of  $u$  up to order  $m$ . Under what circumstances can we write  $f = -pu + h$ , where  $|h| \leq r$  and the values of  $p$  lie in a closed disk  $D$  with center  $\gamma_0$  and radius  $c$ ? Clearly  $|f + \gamma_0 u| \leq c|u| + r$  is a necessary condition. It is interesting to note that this condition is also sufficient. To see this, given an  $f$  satisfying this growth condition, define  $p$  to be the closest point in  $D$  to  $-f/u$  for any values of the arguments with  $|u| \geq 1$ . Then extend  $p$  so as to be defined also for  $|u| < 1$ , so as to be continuous overall, and so as to have each of its values in  $D$ . Then set  $h = f + pu$ . (For  $|u| \geq 1$  we have  $|h| \leq r$ , but for  $|u| < 1$ , although  $h$  as given in the above construction is bounded, we are not assured that  $|h| \leq r$ .)

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