*-ACTIONS IN A*-ALGEBRAS

PAK-KEN WONG
Let $U$ be the open unit disk in the complex plane and $f$ a function defined on $U$. We show that if $A$ is an infinite dimensional dual $B^*$-algebra, then $f$ defines a $*$-action in $A$ if and only if $f$ is continuous at zero and $f(0) = 0$. We also obtain that if $A$ is commutative, then $f$ defines a continuous action in $A$ if and only if $f$ is continuous on $U$ and $f(0) = 0$.

Actions in Banach algebras were introduced and studied recently by Gulick in [1]. Most of her main results were obtained for certain subalgebras of the algebra of all completely continuous operators on a Hilbert space. By using a different approach, we generalize some results in [1].

2. Preliminaries and notation. For any set $S$ in an algebra $A$, let $L_A(S)$ and $R_A(S)$ denote the left and right annihilators of $S$ in $A$. A Banach algebra $A$ is called a dual algebra if, for every closed left ideal $I$ and every closed right ideal $J$, we have $I = L_A(R_A(I))$ and $J = R_A(L_A(J))$. For each element $x \in A$, $Sp_A(x)$ will denote the spectrum of $x$ in $A$.

Let $B$ be a commutative Banach algebra and $X_B$ its carrier space. For each $x \in B$, we let $x \rightarrow \hat{x}$ be the Gelfand map on $B$ defined by $\hat{x}(\alpha) = \alpha(x)$ for all $\alpha \in X_B$.

All algebras under consideration are over the complex field $\mathbb{C}$. Definitions not explicitly given are taken from Rickart's book [5].

3. Lemmas. In this section, we give two lemmas which are useful in § 4.

**Lemma 3.1.** Let $A$ be an $A^*$-algebra. If there exists a maximal commutative $*$-subalgebra $B$ of $A$ which is finite dimensional, then $A$ is finite dimensional.

**Proof.** Since $B$ is finite dimensional, $B$ has an identity element $e$ such that $e = \sum_{i=1}^{n} e_i$, where $\{e_i, i = 1, \ldots, n\}$ is the maximal orthogonal family of hermitian minimal idempotents in $B$. We claim that $e$ is an identity element of $A$. In fact, for each $a \in A$, let $b = a(1 - e)$. It is straightforward to show that $b^*b \in B$ and $b^*b = 0$. Therefore $b = 0$ and so $a = ae$. Similarly we can show that $a = ea$. Hence $e$ is an identity element of $A$. Clearly $A = \sum_{i=1}^{n} \sum_{j=1}^{n} e_i Ae_j$. To complete the proof, it suffices now to show that $e_i Ae_j$ is one
dimensional. We may assume $e_i A e_j \neq (0)$. Then there exists an element $x \in A$ such that $e_i x e_j \neq 0$ and so

$$0 \neq (e_i x e_j)(e_i x e_j)^* = e_i x e_j x^* e_i = \lambda e_i,$$

where $\lambda \in C$. Now for each $y \in A$, we have

$$e_i y e_j = \lambda^{-1} e_i x e_j x^* e_i y e_j = \lambda^{-1} e_i x (\lambda' e_j) = \lambda^{-1} \lambda' e_i x e_j ,$$

where $\lambda' \in C$. Hence $e_i A e_j$ is one dimensional and this completes the proof.

**Lemma 3.2.** Let $A$ be an $A^*$-algebra. If the spectrum of every hermitian element of $A$ is finite, then $A$ is finite dimensional.

**Proof.** Let $B$ be a maximal commutative *-subalgebra of $A$. It follows easily from [5, p. 111, Theorem (3.1.6)] that every element of $B$ has a finite spectrum and therefore $B$ is finite dimensional (see [3, p. 376, Lemma 7]). Hence by Lemma 3.1, $A$ is finite dimensional.

4. $A^*$-algebras and *-actions. In this section, the symbol $U$ denotes the open unit disk in the complex. For a given Banach *-algebra $A$, we let $A_1^*$ be the set $\{x \in A : xx^* = x^* x$ and $Sp_A(x) \subset U\}$.

A function $\theta$ on $U$ is said to define a *-action in $A$ if there exists a mapping $x \mapsto \theta(x)$ of $A_1^*$ into $A$ such that whenever $B$ is a maximal commutative *-subalgebra of $A$ and $x \in B \cap A_1^*$, then $\theta(x) \in B$ and $\theta(x) = \theta(x')$ on the carrier space $X_B$ of $B$.

**Theorem 4.1.** Let $A$ be an $A^*$-algebra. Then $A$ is finite dimensional if and only if any function $f$ on $U$ defines a *-action in $A$.

**Proof.** Suppose $A$ is finite dimensional. Let $x \in A_1^*$ and let $B$ be a maximal commutative *-subalgebra of $A$ containing $x$. Then $B$ is a finite dimensional dual $B^*$-algebra. Hence the carrier space $X_B$ of $B$ consists of a finite number of elements, say $\alpha_1, \ldots, \alpha_n$. Let $e_i$ be the element of $B$ corresponding to the characteristic function of the point $\alpha_i(i = 1, \ldots, n)$. Then for each $x \in B$, we have $x = \sum_{i=1}^n \alpha_i(x)e_i$ (see [4, p. 21]). By [5, p. 111, Theorem (3.1.6.)],

$$Sp_B(x) = \{\alpha_i(x); i = 1, \ldots, n\} .$$

Let $f$ be any function on $U$. Define

$$f'(x) = \sum_{i=1}^n f(\alpha_i(x))e_i .$$

Then it is easy to see that $f'(x) \in B$ and $f'(x) = f \circ \theta$. Therefore $f$ defines a *-action in $A$. 

Conversely suppose that any function $f$ on $U$ defines a *-action in $A$. If $A$ were not finite dimensional, then by Lemma 3.2 there would exist an element $x$ in $A^+_x$ such that $Sp_2(x)$ is infinite. Let $B$ be a maximal commutative *-subalgebra of $A$ containing $x$. Choose $\lambda_n \in Sp_2(x)$ such that $\lambda_n \neq 0$ ($n = 1, 2, \ldots$). Let $f$ be any function on $U$ such that $f(\lambda_n) = n$. Since $f$ defines a *-action, there exists some $f^*(x) \in B$ such that $f^*(x) = f(\lambda_n)$. But this means $n = f(\lambda_n) \in Sp_2(f^*(x))$, contradicting the boundedness of $Sp_2(f^*(x))$. Hence $A$ is finite dimensional and the proof is complete.

**Theorem 4.2.** Let $A$ be an infinite dimensional dual $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra. If a function $f$ on $U$ defines a *-action in $A$, then $f$ is continuous at $0$ and $f(0) = 0$.

**Proof.** Let $B$ be a maximal commutative *-subalgebra of $A$. By [4, p. 31, Theorem 19], $B$ is a dual algebra and so its carrier space $X_B$ is discrete. For each $\alpha \in X_B$, let $e_\alpha$ be the element of $B$ corresponding to the characteristic function of $\alpha$. Then $\{e_\alpha : \alpha \in X_B\}$ is a maximal orthogonal family of hermitian minimal idempotents in $A$. By Lemma 3.1, $B$ is infinite dimensional and so $X_B$ is infinite. Therefore we can choose a countable subset $\{\alpha_n\}$ of $X_B$ such that the complement $\{\alpha_n\}'$ of $\{\alpha_n\}$ in $X_B$ is infinite.

Let $\{\alpha_n\}$ be a sequence in $U$ such that $\alpha_n \to 0$. We want to show $f(\alpha_n) \to f(0) = 0$. By passing to a subsequence, we can assume that $|\alpha_n| \leq (n^\epsilon ||e_\alpha||)^{-1}$. Then $x = \sum_{n=1}^\infty e_\alpha e_{\alpha_n}$ is defined in $B$. Clearly $x \in A^+_x$. Hence there exists some $f^*(x) \in B$ such that $\hat{f^*(x)} = f(\alpha_n)$ on $X_B$. By [4, p. 30, Theorem 16], we have

$$f^*(x) = \sum_\alpha e_\alpha f^*(x)e_\alpha = \sum_\alpha \alpha(f^*(x))e_\alpha.$$  

Therefore $\alpha(f^*(x)) \to 0$. Since $\alpha_n(x) = \alpha_n$, we have $f(\alpha_n) = \alpha_n(f^*(x))$. Thus it follows that $f(\alpha_n) \to 0$ as $n \to \infty$. For each $\alpha \in \{\alpha_n\}'$, $\alpha(x) = 0$ and so $\alpha(f^*(x)) = f(\alpha(x)) = f(0)$. Since $\{\alpha_n\}'$ is infinite, it follows easily from (4.1) that $\alpha(f^*(x)) = 0$ for all $\alpha \in \{\alpha_n\}'$. Hence $f(0) = 0$ and so $f$ is continuous at 0. This completes the proof.

Theorem 4.2 is a generalization of [1, p. 668, Proposition 5.1], since $Cp(1 \leq p < \infty)$ and their *-subalgebras are dual $A^*$-algebras which are dense two-sided ideals of their completions in the auxiliary norm (see [6]).

We remark that the converse of Theorem 4.2 does not hold as is shown by the following example.

**Example.** Let $A$ be an infinite dimensional proper $H^*$-algebra. Then $A$ is a dual $A^*$-algebra which is a dense two-sided ideal of its
completion in an auxiliary norm (see [4, p. 31]). Let $B$ be a maximal commutative *-subalgebra of $A$ and let $\{e_\alpha : \alpha \in \mathcal{X}_B\}$ be the maximal orthogonal family of hermitian minimal idempotents given in the proof of Theorem 4.2. Let $\{e_\alpha : \alpha \in \mathcal{X}_B\}$ be a countable subset of $\{e_\alpha : \alpha \in \mathcal{X}_B\}$ and let $\alpha_n = (n || e_{\alpha_n} ||)^{-1}$. Then $x = \sum_{n=1}^\infty \alpha_n e_{\alpha_n}$ is defined in $B$ and $|| x ||^2 = \sum_{n=1}^\infty n^{-2}$. Define a function $f$ on $U$ by $f(z) = (\sqrt{n} || e_{\alpha_n} ||)^{-1}$ if $z = \alpha_n$ and $f(z) = 0$ otherwise. Then $f$ is continuous at 0. If $f$ defines a *-action in $A$, then there exists an element $f'(x) \in B$ such that $f'(x) = f \circ \hat{x}$. But

$$|| f'(x) ||^2 = \sum_{n=1}^\infty | f'(\alpha_n) ||^2 || e_{\alpha_n} ||^2 = \sum_{n=1}^\infty n^{-1}. $$

This is a contradiction. Therefore $f$ does not define a *-action in $A$.

**Theorem 4.3.** Let $A$ be an infinite dimensional dual $B^*$-algebra. Then a function $f$ on $U$ defines a *-action in $A$ if and only if $f$ is continuous at 0 and $f(0) = 0$.

**Proof.** Suppose $f$ is continuous at 0 and $f(0) = 0$. Let $x \in A^*_+$ and let $B$ be a maximal commutative *-subalgebra of $A$ containing $x$. By the proof of Theorem 4.2, $x = \sum_{n=1}^\infty \alpha_n e_{\alpha_n}$, where $\alpha_n \in \mathcal{X}_B$ and $e_{\alpha_n}$ is the element of $B$ corresponding to the characteristic function of $\alpha_n$. Since $\alpha_n(x) \to 0$, $f(\alpha_n(x)) \to 0$. For any two positive integers $m$, $n (m \leq n)$, it follows easily from the commutativity of $B$ that

$$|| \sum_{i=m}^n f(\alpha_i(x)) e_{\alpha_i} || = \max \{| f(\alpha_i(x)) | : i = m, \ldots, n \}. $$

Therefore $\sum_{n=1}^\infty f(\alpha_n(x)) e_{\alpha_n}$ is defined in $B$. Now let $f'(x) = \sum_{n=1}^\infty f(\alpha_n(x)) e_{\alpha_n}$. Then $f'(x) = f \circ \hat{x}$. Hence $f$ defines a *-action in $A$. The converse of the theorem follows from Theorem 4.2 and the proof is complete.

Since the algebra of all completely continuous operators on a Hilbert space is a dual $B^*$-algebra, Theorem 4.3 generalizes [1, p. 668, Theorem 5.2].

**Theorem 4.4.** Let $A$ be an infinite dimensional commutative dual $B^*$-algebra and $f$ a function on $U$. Then $f$ defines a continuous action in $A$ (see [2, p. 109, Definition 5.1]) if and only if $f$ is a continuous function on $U$ and $f(0) = 0$.

**Proof.** Suppose $f$ is continuous and $f(0) = 0$. Then by Theorem 4.3, $f$ defines an action in $A$. Let $x_n$ and $x \in A^*_+$ such that $x_n \to x$ in $A$. By the proof of Theorem 4.2, we have

$$x = \sum_\alpha \alpha(x_\alpha) e_\alpha \quad \text{and} \quad x = \sum_\alpha \alpha(x) e_\alpha,$$
where \( \{e_\alpha : \alpha \in X_A\} \) is the maximal orthogonal family of hermitian minimal idempotents in \( A \). Since \( A \) is commutative, we have
\[
\| x_n - x \| = \sup\{|\alpha(x_n) - \alpha(x) | : \alpha \in X_A\}
\]
and
\[
\| f(x_n) - f(x) \| = \sup\{|f(\alpha(x_n)) - f(\alpha(x)) | : \alpha \in X_A\}
\]
Therefore it is now easy to see that \( f(x_n) \to f(x) \) in \( A \). Hence \( f \) defines a continuous action in \( A \). The converse of the theorem follows from [2, p. 109, Proposition 5.2] and Theorem 4.3.

**Remark.** If \( A \) is noncommutative, then Theorem 4.4 is not true as is shown in [2, p. 110, Example 5.3].

**References**


Received October 29, 1971.

McMaster University, Hamilton, Canada

AND

Stetson Hall University, South Orange, N. J.
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBOY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH
B. H. NEUMANN
F. WOLF
K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.
Tsuyoshi Andô, Closed range theorems for convex sets and linear liftings
Richard David Bourgin, Conically bounded sets in Banach spaces
Robert Jay Buck, Hausdorff dimensions for compact sets in \( R^n \)
Henry Cheng, A constructive Riemann mapping theorem
David Fleming Dawson, Summability of subsequences and stretchings of sequences
William Thomas Eaton, A two sided approximation theorem for 2-spheres
Jay Paul Fillmore and John Herman Scheuneman, Fundamental groups of compact complete locally affine complex surfaces
Avner Friedman, Bounded entire solutions of elliptic equations
Ronald Francis Gariepy, Multiplicity and the area of an \((n - 1)\) continuous mapping
Andrew M. W. Glass, Archimedean extensions of directed interpolation groups
Morisuke Hasumi, Extreme points and unicity of extremum problems in \( H^1 \) on polydiscs
Trevor Ongley Hawkes, On the Fitting length of a soluble linear group
Garry Arthur Helzer, Semi-primary split rings
Melvin Hochster, Expanded radical ideals and semiregular ideals
Keizō Kikuchi, Starlike and convex mappings in several complex variables
Charles Philip Lanski, On the relationship of a ring and the subring generated by its symmetric elements
Jimmie Don Lawson, Intrinsic topologies in topological lattices and semilattices
Roy Bruce Levow, Counterexamples to conjectures of Ryser and de Oliveira
Arthur Larry Lieberman, Some representations of the automorphism group of an infinite continuous homogeneous measure algebra
William George McArthur, \( G_3 \)-diagonals and metrization theorems
James Murdoch McPherson, Wild arcs in three-space. II. An invariant of non-oriented local type
H. Millington and Maurice Sion, Inverse systems of group-valued measures
William James Rae Mitchell, Simple periodic rings
C. Edward Moore, Concrete semispaces and lexicographic separation of convex sets
Jingyal Pak, Actions of torus \( T^n \) on \((n + 1)\)-manifolds \( M^{n+1} \)
Merrell Lee Patrick, Extensions of inequalities of the Laguerre and Turán type
Harold L. Peterson, Jr., Discontinuous characters and subgroups of finite index
S. P. Philipp, Abel summability of conjugate integrals
R. B. Quintana and Charles R. B. Wright, On groups of exponent four satisfying an Engel condition
Marlon C. Rayburn, On Hausdorff compactifications
Martin G. Ribe, Necessary convexity conditions for the Hahn-Banach theorem in metrizable spaces
Ryōtarō Satō, On decomposition of transformations in infinite measure spaces
Peter Drummond Taylor, Subgradients of a convex function obtained from a directional derivative
James William Thomas, A bifurcation theorem for \( k \)-set contractions
Clifford Edward Weil, A topological lemma and applications to real functions
Stephen Andrew Williams, A nonlinear elliptic boundary value problem
Pak-Ken Wong, \(*\)-actions in \( A^*-algebras \)