REPRESENTATION OF FINITELY GENERATED NILPOTENT GROUPS

IAN DOUGLAS BROWN

REPRESENTATION OF FINITELY GENERATED NILPOTENT GROUPS

Ian D. Brown

A. I. Malcev has shown that finitely generated torsion free nilpotent groups imbed as lattices in nilpotent Lie groups, and hence their structure is similar to that of the Lie groups. Since A. A. Kirillov has classified the representations of nilpotent Lie groups and, in particular, shown that they are all monomial (induced from one dimensional representations of subgroups), one might conjecture that representations of finitely generated nilpotent groups were monomial. (A representation, here, is a weakly continuous unitary representation on separable Hilbert space.) We prove a criterion for when a representations of finitely generated nilpotent groups are monomial. We will also show that representations induced from finite dimensional ones satisfy similar equivalence and irreducibility criteria to those deduced by Kirillov for nilpotent Lie groups.

To widen the conjecture slightly, one might ask about representations induced from finite dimensional representations. Our first lemma shows that this reduces to the question of monomial representations (using the theorem on inducing in stages).

Let us establish some notation. $\chi \uparrow G$ will be the representation of G induced (definition in Mackey, [6]) from a representation χ of some subgroup. If χ is a character (one dimensional representation) on a subgroup H of G, χ^g is the character on $H^g = g^{-1}Hg$ given by $\chi^g(h) = (ghg^{-1})$. If U is a representation, $\mathscr{H}(U)$ is the space on which U acts. Let C be the complex numbers.

First of all, the wider conjecture reduces to the narrower via.

LEMMA 1. *If V is a finite dimensional irreducible representation of a nilpotent group G, then V is induced from a one dimensional representation of some subgroup of G.*

Proof. Consider V as a representation of $G_0 = G/\text{Ker } V$. If V is not one dimensional, G_0 is not abelian. Since G_0 is nilpotent, there is an abelian normal noncentral subgroup $A \subset G_0$. Then $V|A$ is a sum of characters. Call one of these χ. Let $F = \{g \in G: \chi^g = \chi\}$. If $g \in G - F$, g carries $\mathscr{H}(\chi)$, the subspace of $\mathscr{H}(V)$ on which A acts according to χ, onto $\mathscr{H}(\chi^g)$. Hence $[G_0: F] < \infty$. Since F preserves $\mathscr{H}(\chi)$, we can let the representation of F on $\mathscr{H}(\chi)$ be U. Now note that $F \neq G_0$ since V is faithful and A is not central hence not a

multiple of a single character. Hence if we can prove V is equivalent to $U \uparrow G_0$, we are done by induction on the dimension of $\mathscr{H}(V)$.

Let $f_i^v \in \mathscr{H}(U \uparrow G_0)$ be defined by $f_i^v(g_k) = \delta_{ik}v$ where $v \in \mathscr{H}(\chi)$, the g_k are representatives for G_0/F and δ_{ik} is the Kroneker δ. Then the f_i^v span $\mathscr{H}(U \uparrow G_0)$. Let $Wf_i^v = V(g_i^{-1})v$. Then $W: \mathscr{H}(U \uparrow G_0) \to \mathscr{H}(V)$. One can check that W commutes with the action of G_0 and is unitary, using the fact that $V(g_i)\mathscr{H}(\chi) \perp \mathscr{H}(\chi)$ for $g_i \neq e$. Thus we conclude that V is induced as a representation of G_0, from a faithful representation of some subgroup of G_0, hence, by induction, from a character χ_0 on some subgroup H of G_0. Then, as a representation of G, V is induced from χ_0' on $p^{-1}(H)$ where $p: G \to G_0$ is the canonical projection and $\chi_0'(g) = \chi_0(p(g))$.

Next, we need some information about finitely generated nilpotent groups.

DEFINITIONS. (1) If N is a group, let $N^i = [N, N^{i-1}]$, $N^2 = [N, N]$ where $[A, B]$ is the subgroup generated by all commutators $aba^{-1}b^{-1}$, $a \in A$, $b \in B$. (2) A nilpotent group is nilpotent of class k if $N^{k+1} = 1$, $N^k \neq 1$.

LEMMA 2. *A subgroup of a finitely generated nilpotent group is finitely generated.*

Proof. Let H be a subgroup of the finitely generated nilpotent group G. If G is nilpotent of class 1, G is abelian and the result is quite easy.

Assume the result for all nilpotent groups of class k. If G is nilpotent, of class $k + 1$, $H \cap [G, G]$ is finitely generated as a subgroup of $[G, G]$. Also $H/H \cap [G, G]$, as a subgroup of $G/[G, G]$ is finitely generated since $G/[G, G]$ is abelian.

A set of representatives for the generators of $H/H \cap [G, G]$ plus the generators for $H \cap [G, G]$ will serve as a set of generators for H.

LEMMA 3. *If G is a finitely generated nilpotent group for any fixed n there are only a finite number of subgroups of G of index n.*

Proof. The result is due to Howe [3] for a torsion free finitely generated nilpotent group. We show here how the result extends to any finitely generated nilpotent group.

Hirsch [2] has shown that G is isomorphic to a subgroup of finite index in the direct product $A \times B$ where A is a finite group and B is a torsion free finitely generated nilpotent group. Then if $[G: H] \leq r$ and $[A \times B: G] = n$, then $[A \times B: H] \leq n + r$. Since $H/H \cap B \cong BH/B$ $[H: H \cap B] = [BH: B] \leq |A|$. Thus $[A \times B: H \cap B] \leq (n + r) \cdot |A|$ and

$[B: H \cap B] \leq n + r$. By Howe's result there are only a finite number of choices for $H \cap B$ and then, to complete the choice of H, one is left to select among less than $(n + r) \cdot |A|$ cosets of $H \cap B$ in G. Hence there are only a finite number of possibilities for H given r and G.

We now introduce, the

DEFINITION. A subgroup H of a group G is called isolated if it contains all its roots, i.e., if $g^n \in H$ for some positive integer n, then $g \in H$.

We will denote the smallest isolated subgroup containing H by H^*. Then H^* has the following properties.

LEMMA 4.
(1) $H^{g^*} = H^{*g}$ for all $g \in G$
(2) $(H_1 \cap H_2)^* = H_1^* \cap H_2^*$
(3) $N_G(H^*) = N_G(H)^*$
(4) $H/H^g \cap H$ and $H^g/H^g \cap H$ are finite $\Leftrightarrow g \in N_G(H^*)$.

Proof. (1) $k \in H^{g^*} \Rightarrow k^n \in H^g$ for some n
$\Rightarrow k^n = g^{-1}hg$ for some n, h
$\Rightarrow (gkg^{-1})^n = h$ for some n, h
$\Rightarrow gkg^{-1} \in H^*$
$\Rightarrow k \in H^{*g}$

(2) $g \in (H_1 \cap H_2)^* \Rightarrow g^n \in H_1 \cap H_2 \Rightarrow g \in H_1^* \cap H_2^*$
$g \in H_1^* \cap H_2^* \Rightarrow g^{n_1} \in H_1$ and $g^{n_2} \in H_2$
$\Rightarrow g^{n_1 n_2} \in H_1 \cap H_2 \Rightarrow g \in (H_1 \cap H_2)^*$

(3) $[H^*: H]$ is finite (from Baumslag [1], Lemma 2.8). If $g \in N_G(H^*)$, $gHg^{-1} \subset H^*$ of index $[H^*: H]$. Thus only a finite number of these gHg^{-1} are different by Lemma 3. Thus $g^n \in N_G(H)$ for some n, i.e., $g \in N_G(H)^*$ and $N_G(H^*) \subset N_G(H)^*$. But $N_G(H^*)$ is isolated (Kuros [5], p. 249). Hence $N_G(H^*)^* = N_G(H^*)$. Also for $h \in H^*$ $g \in N_G(H)$, if $h^n \in H$, $gh^ng^{-1} = (ghg^{-1})^n$ is in H. Hence $ghg^{-1} \in H^*$ and $g \in N_G(H^*)$. Thus $N_G(H) \subset N_G(H^*)$, so we have

$$N_G(H^*) \subset N_G(H)^* \subset N_G(H^*)^* = N_G(H^*) .$$

Hence the result (3)

(4) $(H^g \cap H)^* = H^{*g} \cap H^*$ $H/H^g \cap H$ is finite $\to H^*/H^g \cap H$ is finite $\Rightarrow (H^g \cap H)^* = H^{*g} \cap H^* = H^*$ (this because if $H \supset K$ are both isolated they may be considered as subgroups of torsion free $G/$(torsion subgroup of G) and there it is easy to see that either $[H: K] = 1$ or $[H: K]$ is infinite). Similarly $H^g/H^g \cap H$ finite $\Rightarrow H^{*g} \cap H^* = H^{*g}$. Hence the two finiteness conditions yield $g \in N_G(H^*)$. If $g \in N_G(H^*)$,

$(H^g \cap H)^* = H^*$ and the two quotients are finite.

1. To develop our criterion mentioned above, we need a couple of lemmas, and for these we need the

DEFINITION. If L, M are representations of a group G, and intertwining operator for L and M is a bounded linear operator $T \colon \mathscr{H}(L) \to \mathscr{H}(M)$ such that $TL(g) = M(g)T$ for all $g \in G$.
The intertwining number, $i(L, M)$, of L and M is the dimension of the space of intertwining operators for L and M.
The first lemma centres around a situation which will be the chief concern of this section. A subgroup H preserves a one dimensional subspace of a certain representation space $\mathscr{H}(\rho)$. This action describes a character of H. If this character is induced to G, the lemma describes its relationship to the original representation ρ.

LEMMA 5. *Let ρ be a representation of G, $v \in \mathscr{H}(\rho)$ such that $\rho(H)v \subset Cv$ where H is some subgroup, and $\rho(g)v \perp v$ for all $g \in G - H$. Then if $\chi(h)v \equiv \rho(h)v$, $i(\chi \uparrow G, \rho) > 0$.*

Proof. We merely have to construct one nonzero intertwining operator.
Let $U = \chi \uparrow G$, g_i be right coset representatives of H and f_i^v be in $\mathscr{H}(U)$ defined by $f_i^v(g_j) = v$ $i = j$ and $f_i^v(g_j) = 0$ $i \neq j$. Then $U(g)f_i^v = f_k^w$ where $w = \chi(g_k g g_i^{-1})v$ and g_k is the representative of $Hg_i g^{-1}$. To see this,

$$U(g)f_i^v(hg_j) = f_i^v(hg_j g)$$
$$= \chi(h)f_i^v(g_j g)$$
$$= \chi(h)f_i^v(g_j g g_1^{-1} g_1) \text{ where } g_1$$

represents

$$Hg_j g = \begin{cases} 0 & 1 \neq i \\ \chi(h)\chi(g_k g g_i^{-1})v & 1 = i \end{cases} \text{ since}$$

$1 = i$ if and only if $g_j g \in Hg_i \Leftrightarrow g_j \in Hg_i g^{-1}$

$$= \chi(h)f_k^w(g_j) \text{ as claimed. Now define}$$

$W \colon \mathscr{H}(U) \to \mathscr{H}(\rho)$ be

$$W(f_i^v) = \rho(g_i^{-1})v .$$

Then one can show that $WU(g) = \rho(g)W$ for all $g \in G$, after defining W on all of $\mathscr{H}(U)$ by linearity. Then if W is bounded it is an intertwining operator. In fact, W is an isometry. First note that for $f \in \mathscr{H}(U)$,

$$f = \sum f_i^{f(g_i)} \; .$$

Thus

$$(Wf, \, Wf_0) = (\sum \rho(g_i^{-1})f(g_i), \; \Sigma \; \rho(g_j^{-1})f_0(g_j))$$

$$= \sum_i \, (\rho(g_i^{-1})f(g_i), \, \rho(g_i^{-1})f_0(g_i))$$

by the perpendicularity hypothesis of the lemma,

$$= \sum_i \, (f(g_i), f_0(g_i))$$

$$= (f, f_0) \; .$$

The other lemma is concerned with the multiplicity of characters in the restriction of a representation.

LEMMA 6. *If ρ is a representation of a nilpotent group H and χ is a one dimensional subrepresentation of ρ, and H_0 is a normal subgroup of H such that $[H: H_0]$ is finite with the property that $\chi \,|\, H_0$ occurs an infinite number of times in $\rho \,|\, H_0$ (i.e., $i(\chi \,|\, H_0, \rho \,|\, H_0)$ is infinite), then χ occurs an infinite number of times in ρ.*

Proof. Choose an integer k such that $H^k \subset H_0$, $H^{k-1} \not\subset H_0$ ($H^i = [H, H^{i-1}]$). Choose $h \in H^{k-1} - H_0$. Then the group H_1 generated by h and H_0 is normal in H. Also, for some positive integer n, $h^n \in H_0$. Let $v_i \in \mathscr{H}(\rho)$ be an infinite independent set on which H_0 acts as $\chi \,|\, H_0$. Then the subspace spanned by the $\{h^i v_r : i = 1, \cdots, n\}$ contains a vector on which H_1 acts as $\chi \,|\, H_1$. Namely, if $a = \chi(h)$ and

$$w = \sum_{i=0}^{n-1} a^{n-i} \rho(h^i) v_r \; , \quad \text{then}$$

$h^p w = \chi(h)^p w$ for all integers p. Since there is an infinite number of independent v_r, there are an infinite number of independent such w-hence $\chi \,|\, H_1$ is of infinite multiplicity in $\rho \,|\, H_1$. Continuing this construction a finite number of times, we reach the conclusion of the lemma.

These two lemmas will be very useful in what follows. We now come to our criterion.

DEFINITION. We will say a representation of a group G has the finite multiplicity property if there is a subgroup H which preserves a one dimensional subspace of $\mathscr{H}(\rho)$ with the additional property that if χ is the character of H so defined, $\rho \,|\, H$ contains χ as a discrete direct summand only a finite number of times.

With this we can state.

THEOREM 1. *If G is a finitely generated nilpotent group, an irreducible representation ρ of G has the finite multiplicity property if and only if ρ is induced from a one dimensional representation of some subgroup of G.*

Proof of Theorem 1. Assume ρ has the finite multiplicity property.

Claim. There is a subgroup $H \subset G$ and a vector v in $\mathscr{H}(\rho)$ such that $\rho(h)v = \chi(h)v$ for all $h \in H$ ($\chi(h) \in C$) with the additional property that $\chi \uparrow G$ is irreducible.

If this is so, $\chi \uparrow N_G(H^*)$ is irreducible, where $N_G(H^*)$ is the normalizer of H^* in G. But this means, by Mackey [7], Theorem 3', that $\chi^g \neq \chi$ on $H^g \cap H$ for any $g \in N_G(H^*) - H$. Choose $h \in H^g \cap H$ such that $\chi^g(h) \neq \chi(h)$. Then

$$
\begin{aligned}
(\rho(g)v, v) &= (\rho(hg)v, \rho(h)v) \\
&= (\rho(g)\chi^g(h)v, \chi(h)v) \\
&= \chi^g(h)\overline{\chi(h)}(\rho(g)v, v) \ .
\end{aligned}
$$

Since $\chi^g(h) \neq \chi(h)$, then $(\rho(g)v, v) = 0$. Let $K = N_G(H^*)$. Then $\chi \uparrow K$ is an irreducible subrepresentation of the representation of K given by the closed linear span of $\rho(K)v$, by Lemma 5.

Let $N_G(K) = K_1$. Let $\chi \uparrow K = U$ and choose $g \in K_1 - K$. Then by Mackey [7], Theorem 3' U^g is not equivalent to U since $U \uparrow G$ is irreducible.

Let $\mathscr{H}_0$ be a subspace of $\mathscr{H}(\rho)$ on which K acts according to U. Then $g\mathscr{H}_0 \perp \mathscr{H}_0$ for $g \in K_1 - K$, since, if not, the projection of $g\mathscr{H}_0$ to $\mathscr{H}_0$ is an intertwining operator for U^g and U.

By the form of U there is a vector $v_0 \in \mathscr{H}_0$ such that $\rho(k)v_0 \perp v_0$ for all $k \in K - H$ and $\rho(h)v_0 = \chi(h)v_0$ for all $h \in H$. Thus $\rho(g)v_0 \perp v_0$ for all $g \in K_1 - H$. Hence, by Lemma 5, $\chi \uparrow K_1$ is a subrepresentation of the representation of K_1 on the closed linear span of $\rho(K_1)v_0$.

Since G is nilpotent, a finite number of applications of this argument shows that $\chi \uparrow G$ is a subrepresentation of ρ. But ρ is irreducible. Thus $\chi \uparrow G$ is equivalent to ρ.

Thus the "only if" part of the theorem is proven if we can establish the claim.

By Theorem 3' of Mackey [7] we would be done if $\chi^g \neq \chi$ on $H^g \cap H$ for all $g \in N_G(H^*)$, where H, χ are as in the definition of the finite multiplicity property. We will produce a new triple H', χ', v' with this property.

The first step is to satisfy this requirement for $g \in N_G(H^*) - H^*$. Let $H_0 = \bigcap_{g \in N_G(H^*)} g^{-1} H g$. By (3) of Lemma 4 this intersection is finite. Hence, by (2) and (1) of Lemma 4, $H_0^* = H^*$. Thus $[H : H_0]$ is finite.

Let $\chi_0 = \chi\,|\,H_0$ and $L = \{g \in N_G(H^*): \chi_0^g = \chi_0\}$. Then, using Lemma 6, the dimension of the space spanned by Lv is finite, since $\rho\,|\,H_0$ acts as χ_0 thereon.

Then $\rho(L)$ on this subspace is a nilpotent unitary subgroup of $GL(n, C)$ for some n. Hence the connected component of its algebraic closure is diagonalizable. Thus L has a subgroup L_0 such that $[L:L_0]$ is finite and there is a v_0 in the span of $\rho(L)v$ such that $tv_0 \in Cv_0$ for all $t \in L_0$. Now if there is a $g \in N_G(H^*) - H^*$ such that $\chi^g = \chi$ on $H^g \cap H$, then $\chi^g = \chi$ on H_0 and $g \in L$.

Let t.f. rank $K =$ torsion free rank $K =$ dimension of the simply connected nilpotent Lie group in which $K^*/$(torsion subgroup of K) imbeds as a uniform subgroup.

Then t.f. rank $L >$ t.f. rank H and t.f. rank $L_0 >$ t.f. rank H. Since $G/$(torsion subgroup of G) imbeds as a uniform subgroup of a simply connected nilpotent Lie group, the torsion free ranks of the subgroups are bounded, so we can begin with an H with the required properties and maximum rank, and the above argument shows that there is no $g \in N_G(H^*) - H^*$ such that $\chi^g = \chi$ on $H^g \cap H$. In fact, our argument is stronger—there is no $g \in N_G(H^*) - H^*$ such that $\chi^g = \chi$ on H_0.

Now let $L = \{g \in H^*: \chi^g\,|\,H_0 = \chi\,|\,H_0\}$. Let $L^i = [L, L^{i-1}]$, $L^m \subset H_0$ and $L^{m-1} \not\subset H_0$. Choose $g \in L^{m-1} - H_0$ and let H_1 be the group generated by g and H_0. Then H_1 is normal in L since H_0 is. Since $L \subset H^*$, $g^k \in H_0$ for some positive integer k. Let a be a kth root of $\chi(g^k)$ and define w as in the proof of Lemma 6. Then $\rho(g)w = aw$ and $\rho(h)w = \chi(h)w$ for all $h \in H_0$. Thus χ extends to a character χ_1 of H_1. Let $L_1 = \{t \in L: \chi_1^t = \chi_1\}$ and continue this process. Since $[H^*:H_0]$ is finite, this terminates with $L_k = H_k$, say.

Then for $g \in H^* - L_k$, $\chi^g \neq \chi$ on H_0 if $g \in H^* - L$, $\chi_1^g \neq \chi_1$ on $H_1 \supset H_0$ if $g \in L - L_1, \cdots, \chi_k^g \neq \chi_k$ on $H_k \supset H_{k-1}$ if $g \in L_{k-1} - L_k$. Thus $\chi_k^g \neq \chi_k$ on H_k for any $g \in H^* - L_k = H^* - H_k$. From the above paragraph, $\chi_k^g \neq \chi_k$ on $H_0 \subset H_k$ for any $g \in N_g(H^*) - H^*$. Thus $\chi_k^g \neq \chi_k$ on $H_k^g \cap H_k$ for $g \in N_G(H^*) - H_k$. Since $H^* = H_k^*$ this is precisely the subgroup-representation pair we were seeking. Hence we have the claim, and thus the "only if" part of the theorem.

Suppose $f \in \mathscr{H}(\chi \uparrow G)$ and $hf = \chi(h)f$. As in the proof of Lemma 2, let g_i be the right coset representatives of H. Let

$$f_i(g_j) = 1 \quad i = j$$
$$= 0 \quad i \neq j$$

and define f_i elsewhere so that it is in $\mathscr{H}(\chi \uparrow G)$. Then $f = \Sigma f(g_i)f_i$ and $hf_i = \chi(g_k h g_i^{-1})f_k$ where g_k is the representative of Hg_ih^{-1}. Then

$$(1) \quad \begin{aligned} hf &= \Sigma f(g_i)\chi(g_k h g_i^{-1})f_k \\ &= \Sigma \chi(h)f(g_s)f_s \,. \end{aligned}$$

The f_i are an orthonormal basis of $\mathscr{H}(\chi \uparrow G)$. Thus $|f(g_i)| = |f(g_k)|$ for all $h \in H$. Thus for f to be in $\mathscr{H}(\chi \uparrow G)$, Hg_i must have a finite orbit in the cosets where $f(g_i) \neq 0$ for f to have a finite norm. Thus there is a finite integer n such that $g_i h^n \in Hg_i$. i.e., $g_i h^n g_i^{-1} \in H$. Thus $g_i h g_i^{-1} \in H^*$ for each $h \in H$. Thus $g_i H g_i^{-1} \subset H^*$. Taking $*$ on both sides, $g_i H^* g_i^{-1} = H^*$, hence $g_i \in N_G(H^*)$.

But $\chi \uparrow G$ is irreducible if and only if $i(\chi \uparrow G, \chi \uparrow G) = 1$ and by Theorem 3' of Mackey [7], this is if and only if

$$\sum_{\substack{d \in D_f \\ d \neq e}} i(\chi^d | H^d \cap H, \chi | H^d \cap H) = 0 \,.$$

Now $d \in D_f$ if and only if $d \in N_G(H^*)$. Hence for $d \in N_G(H^*)$ there is an $h \in H^d \cap H$ such that $\chi^d(h) \neq \chi(h)$. Now

$$\begin{aligned} hf(d) = f(dh) &= f(dhd^{-1}d) \\ &= \chi^d(h)f(d) \quad \text{for} \quad h \in H^d \cap H \\ &\neq \chi(h)f(d) \quad \text{unless} \quad f(d) = 0 \,. \end{aligned}$$

Hence f can only be nonzero on one right H coset $- H$ itself. There it is determined once its value at e is. Hence χ has multiplicity 1 in $\chi \uparrow G | H$.

2. Now let use this criterion to show there is representation of a nilpotent group which is not induced from any finite dimensional representation (or, equivalently, from any one-dimensional representation).

Our example will be a representation of the group of 3×3 matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \text{ where } x, y, z \text{ are integers}$$

We will denote this matrix by (x, y, z) so that

$$(x, y, z)(x', y', z') = (x + y', y + y', z + z' + xy') \,.$$

First we describe a general form for a representation of G. A particular example of this will turn out to be the required counter-example. For f in $L^2(S_1, \lambda)$ where S_1 is the circle and λ is Lebesgue measure, let

$U_s(x, y, z)f(t) = e^{i(ty+bz)}s(x, t)f(t + xb)$ where b is not in $Q\pi$ (Q is the rational numbers). This is a representation for any measurable

$s: Z \times S_1 \to S_1$, where the image circle is the unit circle of the complex numbers, if s satisfies the property

$$((1)) \qquad s(x + x', t) = s(x, t)s(x', t + xb)$$

Note that, in fact, $s(1, t)$ can be an arbitrary measurable function, and then $s(x, t)$ is completely defined by $((1))$.

Let us now find out which U_s are equivalent, and at the same time show that they are irreducible.

LEMMA 7.   *The U_s are irreducible and U_{s_1} is equivalent to U_{s_2} if and only if there is measurable $f: S_1 \to S_1$ such that*

$$s_1(1, t) = f(t)\overline{f(t + b)}s_2(1, t) \ .$$

Proof.   Suppose $V: L^2(S_1, \lambda) \to L^2(S_1, \lambda)$ is unitary and $U_{s_1}V = VU_{s_2}$ — i.e., $U_{s_1}(x, y, z)V = VU_{s_2}(x, y, z)$ for all integers x, y, z.   Then

$$VU_{s_2}(0, y, 0)f = U_{s_1}(0, y, 0)Vf \ .$$

Thus V commutes with multiplication by e^{iyt} for any integer y.   Thus, by a known theorem of functional analysis (cf. Naimark, [10], p. 499, I) $Vf(t) = v(t)f(t)$ for some measurable $v: S_1 \to S_1$.   Then $VU_{s_2}(x, 0, 0) = U_{s_1}(x, 0, 0)V$ implies $v(t)s_2(x, t)f(t + xb) = s_1(x, t)v(t + xb)f(t + xb)$ for any f in $L^2(S_1, \lambda)$ for all $t \in S_1$ and $x \in Z$.   Thus

$$v(t)s_2(x, t) = s_1(x, t)v(t + xb)$$

is necessary for equivalence, and by the decomposition of V above it can easily be shown to be sufficient.   It is also easy to see that if $s(x, t)$ is constructed from $s(1, t)$ by $((1))$, then this equation is equivalent to $v(t)s_2(1, t) = s_1(1, t)v(t + b)$, as required.   If $s_1 = s_2$, then this means that $v(t) = v(t + b)$ — i.e. v is invariant under translation by b.   But this translation is an ergodic action on the circle.   Hence, by a standard theorem of ergodic theory, v is constant a.e., with respect to λ.   Thus V is in fact a scalar multiple of the identity operator, which shows that U_s is irreducible.

Now let us look at a character χ and subgroup H. U_s is infinite dimensional.   Hence if $\chi \uparrow G$ is equivalent to U_s, $[G: H]$ is infinite.   Further, by Theorem 3' of Mackey [7]

$$i(\chi \uparrow G, \chi \uparrow G) = \sum_{d \in D_f} i(\chi^d \,|\, H^d \cap H, \chi \,|\, H^d \cap H) \ .$$

If $d = (0, 0, z), d$ is central so that $H^d = H$ and $\chi^d = \chi$.   Thus if $i(\chi \uparrow G, \chi \uparrow G) = 1$, there can be only one $H: H$ double coset representative of the form $d = (0, 0, z)$.   Hence $(0, 0, Z) \subset H$.   From this

and the fact that $[G: H]$ is infinite, we can see that $H = \{(nk, np, z):$ n, z in $Z\}$. Call this subgroup H_{kp}. If χ is a character on H, since $(0, 0, Z)$ is central, $\chi((0, 0, z)) = e^{iwz}$ for some w for all $z \in Z$. If I is the identity operator, then $\chi \uparrow G((0, 0, z)) = e^{iwz}I$. But $U_s((0, 0, z)) = e^{ibz}$. Hence $w = b$ if $\chi \uparrow G$ is equivalent to U_s.

LEMMA 8. *If U_s is equivalent to $\chi \uparrow G$ for χ a character on H_{kp}, then $k = 1$.*

Proof. Let $K = H_{01}$. Parameterize the elements of H_{kp} by n and z. Let $[n, z] = (nk, np, z)$. Then $[n, z][n', z'] = (n + n', z + z' + nn'kp)$. Then one can check that any character on H_{kp} is of the form χ_{ab} where

$$\chi_{ab}(nk, np, z) = \exp i(na - bn(n - 1)kp/2 + bz) \,.$$

First let us examine $\chi_{ab} \uparrow G \,|\, K$ where χ_{ab} is a character on H_{kp} and $k \neq 0$. Then the $H_{kp}: K$ double coset representatives may be chosen $\{(x, 0, 0): 0 \leqq x < k\}$. One can check that if $d = (x, 0, 0)$, $\chi_{ab}^d = \chi_{a+xpb,b}$. Then by Mackey's theorem (Mackey, [6], p. 117, Theorem 7.1)

$$\chi_{ab} \uparrow G \,|\, K = \sum_{x=0}^{k-1} [\chi_{ab}^d \,|\, (0, 0, Z)] \uparrow K$$
$$= \sum_{x=0}^{k-1} [\chi_{0b} \,|\, (0, 0, Z)] \uparrow K$$
$$= k$$

copies of the regular representation of $(0, Z, 0)$ direct product with ψ_b where $\psi_b(0, 0, z) = e^{ibz}$. If $k \neq 1$, this has a commuting algebra which is not abelian. A projection onto one of the copies does not commute with the isomorphism between the copies. On the other hand,

$$U_s(0, y, z)f(t) = e^{i(yt+bz)}f(t) \,,$$

which is precisely the regular representation of $(0, Z, 0)$ direct product with ψ_b. Using the theorem in Naimark ([10], p. 499, I), it is easy to see that this representation has an abelian commuting algebra. Hence to prove the result it remains to dispose of the case where $k = 0$. Then $\chi_{ab} \uparrow G \,|\, K = \sum_{x=-\infty}^{\infty} \chi_{ab}^d$ by Mackey's theorem where $d = (x, 0, 0)$ if $p = 1$. For $k = 0$ and $p \neq 1$ or -1 it is not hard to show, using Theorem 3' of Mackey [7], that $\chi_{ab} \uparrow G$ is not irreducible. Now if U_s and $\chi_{ab} \uparrow G$ were equivalent for χ_{ab} defined on H_{01}, then $U_s | K$ and $\chi_{ab} \uparrow G \,|\, K$ would be equivalent, and by a theorem of Mackey (Mackey, [8], p. 103) Lebesgue measure would be measure isomorphic to the measure which weights each point of the orbit of a as 1. But this is a contradiction, since Lebesgue measure would then be atomic.

This concludes the lemma.

Now some U_s are equivalent to $\chi \uparrow G$ for some χ. In fact we have the following lemma.

LEMMA 9. *If χ_{ab} is a character on $H_{1p}, \chi_{ab} \uparrow G$ is equivalent to U_s where $s(1, t) = e^{i(a-tp)}$.*

Proof. For $f \in \mathscr{H}(\chi_{ab} \uparrow G)$ let $f'(m) = f(0, m, 0)$. Then $f \to f'$ is an isomorphism $\mathscr{H}(\chi_{ab} \uparrow G) \to l_2$ and

$$\chi_{ab} \uparrow G(x, y, z) f((0, m, 0)$$
$$= f(x, m + y, z)$$
$$= f((x, xp, z - xm - xy + x^2p)(0, m + y - xp, 0))$$
$$= \chi_{ab}(x, xp, z - xm - xy + x^2p)f'(m + y - xp) \, .$$

Taking Fourier transform,

$$\chi_{ab} \uparrow G(x, y, z)\hat{f}'(t)$$
$$= \sum_m e^{-itm}\chi_{ab} \uparrow G(x, y, z)f'(m)$$
$$= \sum_m e^{-itm}\chi_{ab}(x, xp, z - xm - xy + x^2p)f'(m + y - xp)$$
$$= \sum \exp i[t(m - y + xp) - bx(m - y + xp) + xa - bx(x - 1)p/2$$
$$\qquad\qquad + b(z - xy + x^2p)]f'(m)$$
$$= e^{i(ty+bz)} \exp i[xa - bx(x - 1)p/2 - txp]\hat{f}'(t + xb).$$

Thus comparing to the form of U_s,

$$s(1, t) = e^{i(a-tp)} \, .$$

COROLLARY. *U_s is equivalent to a representation induced from a character on some subgroup if and only if there is some real a and integer p and measurable $f: S_1 \to S_1$ such that*

$$s(1, t) = e^{i(a-pt)}f(t)\overline{f(t + b)} \, .$$

What we are going to show is that we can choose $s(1, t)$ so that this equation is impossible.

To do this, we need

LEMMA 7. *If χ_{ab} is a character on H_{10} and χ_{uv} a character on H_{1p} for $p \neq 0$, then $\chi_{ab} \uparrow G$ is not equivalent to $\chi_{uv} \uparrow G$ for any choice of a, b, u, v.*

Proof.

$$i(\chi_{ab} \uparrow G, \chi_{uv} \uparrow G)$$
$$= \sum_{d \varepsilon Df} i(\chi_{ab}^{d} | H_{10}^{d} \cap H_{1p}, \chi_{uv} | H_{10}^{d} \cap H_{1p}) \, ,$$

by Mackey [7], Theorem 3'. But H_{10} is normal and $H_{10} \cap H_{1p} = (0, 0, Z)$. Hence for no d is $[H_{10}^{d} : H_{10}^{d} \cap H_{1p}]$ finite. Thus D_f is empty and the representations are not equivalent.

Let us note what this says according to the equivalence formula for the U_s. Substituting $s_1(1, t) = e^{ia}$ and $s_2(i, t) = e^{i(u-pt)}$ we find that

$$(2) \qquad\qquad e^{ia} = e^{i(u-pt)} f(t) \overline{f(t + b)}$$

is false for any measurable $f \colon S_1 \to S_1$, $p \neq 0$, and any a, u.

But by Theorem 1, if U_s is induced from χ there must be an $f \in \mathscr{H}(U_s)$ such that $U_s(h)f = \chi(h)f$ for all $h \in H$, the subgroup on which χ is defined. Thus there must be an $f \in L^2(S_1, \lambda)$ such that

$$U_s(n, np, z)f(t) = \exp i[na - bn(n - 1)p/2 + bz]f(t) \, .$$

i.e.

$$e^{i(ntp+bz)} s(n, t)f(t + nb) = \exp i[na - bn(n - 1)p/2 + bz]f(t) \, .$$

Taking absolute values, this means $|f|$ is constant on orbits under translation by b. Hence $|f|$ is constant a.e., by ergodicity. Hence we can assume $f \colon S_1 \to S_1$. Then for $n = 1$, this says

$$s(1, t) = e^{ia} f(t) \overline{f(t + b)} e^{-itp} \, .$$

U_s is an irreducible representation if $s(1, t) = e^{it/2}$. Then

$$e^{it/2} = e^{ia} e^{-itp} f(t) \overline{f(t + b)}$$

Then

$$e^{it(1+2p)} = e^{i2a} f(t)^2 \overline{f(t + b)^2} \, .$$

Since $1 + 2p \neq 0$ for any p, this contradicts (2). Hence this U_s is not induced from a character.

3. Although every irreducible representation of a finitely generated nilpotent group is not induced from a character, it is useful to know exactly how to obtain those that are. In fact Mackey's Theorem 3' of [7] leads quickly to a Kirillov type result. We have

THEOREM 2. *Let G be a finitely generated nilpotent group. Then*
(1) *If H is a subgroup of G, χ a character (one dimensional representation) on H, and there is no $g \in N_G(H^*) - H$ such that $\chi^g = \chi$ on $H^g \cap H$, then $\chi \uparrow G$ is irreducible.*

(2) *If H is a subgroup of G, V a finite dimensional representation H such that $V \uparrow G$ is irreducible then $V \uparrow G$ is equivalent to a representation obtained as in (1).*

(3) *If χ_1, χ_2 are characters on subgroups H_1, H_2 respectively satisfying the condition in (1), then $\chi_1 \uparrow G$ is equivalent to $\chi_2 \uparrow G$ if and only if there is a $g \in G$ such that*

$$H_1^{*g} = H_2^* \quad and \quad \chi_1^g = \chi_2 \quad on \quad H_1^g \cap H_2 \,.$$

Proof. By the theorem of Mackey,

$$i(\chi_1 | G, \chi_2 | G) = \sum_{d \in D_f} i(\chi_1^d | H_1^d \cap H_2, \chi_2 | H_1^d \cap H_2)$$

where $d \in D_f$ if and only if both $H_1^d / H_1^d \cap H_2$ and $H_2 / H_1^d \cap H_2$ are finite

$$\Leftrightarrow H_1^{d*} / H_1^d \cap H_2 \quad and \quad H_2^* / H_1^d \cap H_2$$

are finite

$$\Leftrightarrow H_1^{d*} / (H_1^d \cap H_2)^* \quad and \quad H_2^* / (H_1^d \cap H_2)^*$$

are finite

$$\Leftrightarrow H_1^{d*} = H_1^{d*} \cap H_2^* \quad and \quad H_2^* = H_1^{d*} \cap H_2^*$$
$$\Leftrightarrow H_1^{d*} = H_1^{*d} = H_2^* \,.$$

From this (3) is immediate.

If we let $\chi_1 = \chi_2 = \chi$, then $d \in D_f$ if and only if $d \in N_G(H^*)$, and (1) is immediate.

(2) is merely Lemma 1.

REFERENCES

1. G. Baumslag, *Lecture Notes on Nilpotent Groups*, Conference Board of Math. Sciences, Washington, 1971.
2. K. A. Hirsch, *On infinite soluble groups III*, Proc. London Math. Soc., (2) **49** (1946), 184-194.
3. R. Howe, *On Representations of Discrete, Finitely Generated Torsion Free Nilpotent Groups*, Preprint 1970.
4. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian. Math. Survey, **17** No. 3, (1962), 53-104.
5. A. G. Kurosh, *The of Groups*, Vol. II, Chelsea, New York 1956.
6. G. W. Mackey, *Induced representations of locally compact groups I*, Ann. Math, **55** (2nd ser.) (1952), 101-139.
7. ———, *On induced representations of groups*, Amer. J. Math., **73** (1951), 576-592.
8. ———, *The Theory of Group Representations*, Notes by Fell and Lowdenslager, Dept. of Math. of University of Chicago, 1955.
9. A. I. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc., Trans., Ser. 1, **9** (1962), 277-307.
10. M. A. Naimark, *Normed Rings*, 2nd. ed., P. Noordhoff N. V. Gronigen, 1964.

UNIVERSITÉ DE RENNES
35-RENNES, FRANCE

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

Pacific Journal of Mathematics
Vol. 45, No. 1   September, 1973