

# Pacific Journal of Mathematics

**CODIMENSION ONE EMBEDDINGS OF MANIFOLDS WITH  
LOCALLY FLAT TRIANGULATIONS**

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## CODIMENSION ONE EMBEDDINGS OF MANIFOLDS WITH LOCALLY FLAT TRIANGULATIONS

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In this note investigations are made of the problem of deciding if a given codimension one submanifold  $M$  is locally flat in the ambient manifold  $N$ . The principal result is that if  $M$  has a locally flat triangulation in which each closed simplex is locally flat in  $N$ , then  $M$  is locally flat in  $N$ . This allows one to establish for simplicial homotopy manifolds certain local flatness criteria that had been previously known for  $PL$  manifolds.

Our definitions are the usual ones. The reader who is unfamiliar with the subject is referred to [2] and [3] for the basic definitions and results preliminary to this note. We should mention two particular results that will be used repeatedly. The first is the following. If  $D_1$  and  $D_2$  are locally flat  $(n - 1)$ -cells in  $R^n$  that intersect in an  $(n - 2)$ -cell that is locally flat in the boundary of each, then  $D_1 \cup D_2$  is locally flat [5]. The second concerns an  $m$ -cell  $D$  in  $R^n$ , that is locally flat mod a  $k$ -cell  $E$  that is locally flat in both  $\text{Bd } D$  and  $R^n$ . If  $k = n - 3$ ,  $D$  may fail to be locally flat at points of  $E$ , and for  $n = 4$ ,  $k = n - 2$  it is not presently known if  $D$  has to be locally flat. For all other possible values of  $n, m, k$  it is known that  $D$  must be locally flat [1], [3], [4], [6].

A version of the following lemma was proved in Theorem 3.2 of [2] for embeddings of cells in euclidean space. Since the result and proof are local ones, the argument given there will establish the following.

**LEMMA 0.** *Let  $M$  be an  $m$ -manifold in the interior of an  $n$ -manifold and let  $P$  be a polyhedron in  $\text{Bd } M$ . Assume that the following conditions are satisfied:*

- (a)  $M - P$  is locally flat in  $N$ ;
- (b) there is a locally finite triangulation  $T$  of  $P$  such that each open simplex of  $T$  is locally flat in  $\text{Bd } M$ ;
- (c) for  $\lambda$  a collaring of  $\text{Bd } M$  in  $M$  and  $\sigma$  a simplex of  $T$ ,  $\lambda(\text{Int } \sigma \times I)$  is locally flat in  $N$ . Then  $M$  is locally flat in  $N$ .

We will say that the triangulation  $T$  of the manifold  $M$  is locally flat if each open simplex of  $T$  is locally flat in  $M$ .

**THEOREM 1.** *Let  $M$  be an  $(n - 1)$ -manifold with a locally flat triangulation  $T$ . Suppose that  $M$  is embedded in the interior of an*

*n*-manifold  $N$  such that each closed simplex is locally flat in  $N$ . Then  $M$  is locally flat in  $N$ .

*Proof.* Special Case I:  $\text{Int } M$  is locally flat. Since each closed simplex of  $M$  is locally flat, we see that  $M$  is locally flat mod the  $(n - 3)$ -skeleton of  $\text{Bd } M$ .

We let  $\text{Bd } M$  be the polyhedron  $P$  in Lemma 0,  $\sigma$  a  $j$ -simplex in  $P$ , and  $\lambda$  a collaring of  $\text{Bd } M$  in  $M$ . Let  $\{B_i\}_{i=1}^{\infty}$  be a monotone increasing sequence of closed  $j$ -cells, each locally flat in  $\sigma$  (and hence in each of  $M$  and  $N$ ) and  $\bigcup_{i=1}^{\infty} B_i = \text{Int } \sigma$ . In case  $j = 0$ , we adopt the convention that  $\text{Int } \sigma = \sigma$ . We will show that  $\lambda(\text{Int } \sigma \times I)$  is locally flat by showing that  $\lambda(B_i \times I)$  is locally flat for each  $i$ . One easily checks that  $\lambda(B_i \times I)$  is locally flat mod  $\lambda(B_i \times 0) = B_i$  and that  $B_i$  is locally flat in  $\text{Bd } \lambda(B_i \times I)$  and in  $N$ . If  $j \leq n - 4$ , we appeal to Theorem 1.2 of [1] to conclude that  $\lambda(B_i \times I)$  is locally flat.

If  $j = n - 3$ , we observe that  $\lambda(B_i \times I)$  can be pushed into one (either) of the two  $(n - 2)$ -simplexes in  $\text{Bd } M$  that has  $\sigma$  as a face, and that local flatness of this simplex then implies local flatness of  $\lambda(B_i \times I)$ . This argument is written out in detail in Lemma 8 of [3].

Special Case II:  $\text{Bd } M$  is empty.

The proof will be downward induction on the dimension of the skeletons of  $M$ . Since each  $(n - 1)$ -simplex is locally flat, it is immediate that  $M$  is locally flat mod the  $(n - 2)$ -skeleton.

For  $\sigma$  an  $(n - 2)$ -simplex, the link of  $\sigma$  in  $M$  is a 0-sphere (this follows from the fact that  $M$  is a manifold) and the star of  $\sigma$  is the union of two  $(n - 1)$ -simplexes having  $\sigma$  as a common face. Since each of these  $(n - 1)$ -simplexes is locally flat, and  $\sigma$  is locally flat in the boundary of each and in  $M$ , it follows that the star of  $\sigma$  is locally flat. Thus,  $M$  is locally flat at each point of  $\text{Int } \sigma$ .

For  $\sigma$  an  $(n - 3)$ -simplex, we again observe that, since  $M$  is a manifold, the link of  $\sigma$  is a 1-sphere. We let  $\tau$  be a 1-simplex of  $lk\sigma$  and let  $B$  be the closure of  $lk\sigma - \tau$ . Then  $\text{St } \sigma = \sigma^*\tau \cup \sigma^*B$ .  $\sigma^*\tau$  is locally flat by hypothesis and  $\sigma^*B$  is locally flat by Special Case I and the preceding paragraph. Hence,  $\text{St } \sigma$  must be locally flat at each interior point. In particular,  $M$  is locally flat at each point of  $\text{Int } \sigma$ .

We could continue this line of proof, viewing  $\text{St } \sigma$  as the union of two locally flat cells  $\sigma^*\tau$  and  $\sigma^*(lk\sigma - \tau)$ , as long as we knew that  $lk\sigma$  was a  $PL$  sphere. If  $T$  were a  $PL$  triangulation of  $M$ , we would know this for all  $\sigma$ , but for arbitrary triangulations we only know this for  $\dim lk\sigma \leq 2$  or, equivalently,  $\dim \sigma \geq n - 4$ . After this an-

other type of proof would be needed. Fortunately these larger codimension cases have been handled (see Theorem 6.1 of [2]). For completeness, we will include the proof for the skeletons of dimension  $\leq n - 4$ .

Suppose then that  $M$  is locally flat mod the  $j$ -skeleton,  $j \leq n - 4$ , let  $\sigma$  be a  $j$ -simplex and let  $p$  be an interior point of  $\sigma$ . Let  $E$  be a  $j$ -cell with  $x \in \text{Int } E \subset E \subset \sigma$  such that  $E$  is locally flat in  $\sigma$ . Since locally flat cells lie trivially in some euclidean neighborhood [8], it is easy to find two  $(n - 1)$ -cells  $D_1$  and  $D_2$  in  $M$  with the following properties:

- (1)  $D_1$  and  $D_2$  are locally flat in  $M$  and intersect the  $j$ -skeleton only in  $E$ ;
- (2)  $D_1 \cap D_2 = \text{Bd } D_1 \cap \text{Bd } D_2 = D_3$  is an  $(n - 2)$ -cell which is locally flat in  $\text{Bd } D_1$  and  $\text{Bd } D_2$ ;
- (3)  $(D_3, E) \approx (B^{n-2}, B^j)$ .

From (1) and transitivity of the relation "is locally flat in" it follows that each of  $D_1$  and  $D_2$  is locally flat mod  $E$  in  $N$ . Then, since  $\dim E \leq n - 4$ ,  $D_1$  and  $D_2$  are locally flat in  $N$  and, hence,  $D_1 \cup D_2$  is locally flat in  $N$ . This implies that  $M$  is locally flat at  $p$ . In this way we see that  $M$  is locally flat mod the  $(j - 1)$ -skeleton, and the induction is complete.

*General Case.* Apply Special Case II to  $M - \text{Bd } M$  and then apply Special Case I to  $M$ .

**LEMMA 1.** *Let  $\sigma$  be an  $(n - 1)$ -simplex (topologically) embedded in the interior of an  $n$ -manifold  $N$ ,  $n \neq 4$ , and such that  $\text{Int } \sigma$  is locally flat in  $N$  and each closed  $(n - 2)$ -simplex of  $\text{Bd } \sigma$  is locally flat in  $N$ . Then  $\sigma$  is locally flat.*

*Proof.* Note that  $(n - 2)$ -simplexes being locally flat will imply that all lower dimensional simplexes are locally flat. The proof is the same as the proof of Special Case I of Theorem 1, except that if  $\dim B_i = n - 2$ , we get local flatness of  $\lambda(B_i \times I)$  from Theorem 2 of [4].

**COROLLARY 1.** *Let  $T$  be a locally flat triangulation of an  $(n - 1)$ -manifold  $M$  and  $M$  embedded in the interior of an  $n$ -manifold  $N$ . If  $n \neq 4$ , and each open  $(n - 1)$ -simplex and each closed  $(n - 2)$ -simplex is locally flat in  $N$ , then  $M$  is locally flat.*

There is another interesting observation that comes from Theorem 1 and its proof. This deals with the question of local flatness of an

$(n - 1)$ -sphere  $S$  in  $S^n$ ,  $n \neq 4$ , that is known to be locally flat mod an  $(n - 2)$ -cell  $E$  that is locally flat in both  $S$  and  $S^n$ . We can establish local flatness of  $S$  as follows. We give  $S$  the triangulation of the boundary of an  $n$ -simplex with  $E$  being one of the  $(n - 2)$ -simplexes. To meet the requirements of Corollary 1, we need to see that each closed  $(n - 2)$ -simplex is locally flat. Let  $\sigma$  be an  $(n - 2)$ -simplex different from  $E$ . Then  $\sigma$  and  $E$  have a common  $(n - 3)$ -face and lie on an  $(n - 1)$ -simplex  $\tau$  that is locally flat mod  $\text{Bd } E$ . As in the proof of Special Case I of Theorem 1, we can push  $\sigma$  across  $\tau$  into the locally flat cell  $E$  and conclude that  $\sigma$  is locally flat.

Let  $N$  be an  $n$ -manifold triangulated so that the link of each  $i$ -simplex ( $0 \leq i < n$ ) has the homotopy of an  $(n - i - 1)$ -sphere or ball. We will then call  $N$  (with this triangulation) a simplicial homotopy manifold. Glaser has shown that triangulations that make  $N$  a simplicial homotopy manifold are locally flat [7, Theorem 2]. This fact together with Theorem 1 and Corollary 1 gives the following extension of Theorem 12 of [3] to include simplicial homotopy manifolds.

**THEOREM 2.** *Let  $M$  be a simplicial homotopy  $(n - 1)$ -manifold in the interior of an  $n$ -manifold  $N$ . (1) If  $n = 4$ , suppose that each closed simplex of  $M$  is locally flat in  $N$ . (2) If  $n \neq 4$ , suppose that each open  $(n - 1)$ -simplex and each closed  $(n - 2)$ -simplex of  $M$  is locally flat in  $N$ . Then  $M$  is locally flat in  $N$ .*

In codimensions three and greater, we have a better criterion than Theorem 2 gives for codimension one embeddings. The above mentioned result of Glaser, together with Theorem 5.3 of [1] or Theorem 6.1 of [2] gives the following.

**THEOREM 3.** *Let  $M$  be an  $m$ -dimensional simplicial homotopy manifold in the interior of an  $n$ -dimensional manifold  $N$  with  $n - m \geq 3$ . If each open simplex of  $M$  is locally flat in  $N$ , then  $M$  is locally flat in  $N$ .*

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Received October 8, 1971 and in revised form January 6, 1972. This research was supported in part by NSF Grant GP-19961.

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