

# Pacific Journal of Mathematics

**ENUMERATION OF UP-DOWN PERMUTATIONS BY NUMBER  
OF RISES**

L. CARLITZ

## ENUMERATION OF UP-DOWN PERMUTATIONS BY NUMBER OF RISES

L. CARLITZ

It is well known that  $A(n)$ , the number of up-down permutations of  $\{1, 2, \dots, n\}$  satisfies

$$\sum_{n=0}^{\infty} A(2n) \frac{z^{2n}}{(2n)!} = \sec z,$$

$$\sum_{n=0}^{\infty} A(2n+1) \frac{z^{2n+1}}{(2n+1)!} = \tan z.$$

In the present paper generating functions are obtained for the number of up-down permutations counting the number of rises among the "peaks".

1. If  $(a_1, a_2, \dots, a_n)$  denotes an arbitrary up-down permutation, then  $(b_1, b_2, \dots, b_n)$ , where

$$b_i = n - a_i + 1 \quad (i = 1, 2, \dots, n),$$

is a down-up permutation and vice versa.

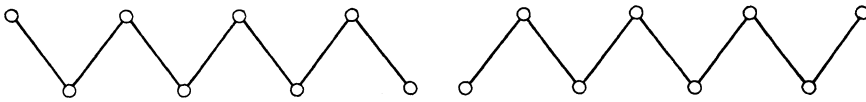


FIGURE 1

Thus, for  $n > 1$ , there is a one-to-one correspondence between up-down and down-up permutations so that it suffices to consider the former.

Let  $A(n, r)$  denote the number of up-down permutations of  $Z_n = \{1, 2, \dots, n\}$  with  $r$  rises on the top line.

Let  $C(n, r)$  denote the number of down-up permutations with  $r$  rises on the top line.

A *rise* is a pair of consecutive elements  $a, b$  with  $a < b$ . Also we agree to count a conventional rise on the left. For example

$$132546, \quad 426153$$

have 3 and 2 rises, respectively.

It will be instructive first to derive the generating functions for  $A(2n+1)$  and  $A(2n)$ . We have  $A(1) = 1$  and

$$(1.1) \quad A(2n+1) = \sum_{k=0}^{n-1} \binom{2n}{2k+1} A(2k+1) A(2n-2k-1) \quad (n > 0).$$

Hence if we put

$$F(z) = \sum_{n=0}^{\infty} A(2n+1) \frac{z^{2n+1}}{(2n+1)!},$$

it follows from (1.1) that

$$F'(z) = 1 + F^2(z).$$

Since  $F(0) = 0$ , we get  $F(z) = \tan z$ .

Next

$$(1.2) \quad A(2n) = \sum_{k=0}^{n-1} \binom{2n-1}{2k+1} A(2k+1)A(2n-2k-2),$$

where  $A(0) = 1$ . Hence if

$$G(z) = \sum_{n=0}^{\infty} A(2n) \frac{z^{2n}}{(2n)!},$$

it follows from (1.2) that

$$G'(z) = F(z)G(z).$$

Since  $G(0) = 1$ , this gives  $G(z) = \sec z$ . Thus we have proved that [1], [2, pp. 105-112]

$$(1.3) \quad \sum_{n=0}^{\infty} A(n) \frac{z^n}{n!} = \sec z + \tan z.$$

2. Turning next to  $A(2n+1, r)$  we take

$$A(1, 0) = 1, A(1, r) = 0 \quad (r > 0).$$

Corresponding to (1.1) we have the recurrence

$$(2.1) \quad A(2n+1, r) = \sum_{k=0}^{n-1} \sum_{s=0}^r \binom{2k+1}{2n} A(2k+1, s)A^*(2n-2k-1, r-s),$$

where

$$A^*(2n+1, r) = A(2n+1, r) \quad (n > 0)$$

but

$$A^*(1, 0) = 0, A^*(1, 1) = 1.$$

Put

$$\begin{aligned} A_{2n+1}(x) &= \sum_r A(2n+1, r)x^r, A_1(x) = 1 \\ A_{2n+1}^*(x) &= A_{2n+1}(x) \quad (n > 0), A_1^*(x) = x. \end{aligned}$$

Then (2.1) gives

$$(2.2) \quad A_{2n+1}(x) = \sum_{k=0}^{n-1} \binom{2k+1}{2n} A_{2k+1}(x) A_{2n-2k-1}^*(x) \quad (n > 0).$$

Hence if

$$(2.3) \quad A(z) = A(x, z) = \sum_{n=0}^{\infty} A_{2n+1}(x) \frac{z^{2n+1}}{(2n+1)!},$$

it follows from (2.2) that

$$\begin{aligned} A'(z) &= \sum_{n=0}^{\infty} A_{2n+1}(x) \frac{z^{2n}}{(2n)!} \\ &= 1 + \sum_{k=0}^{\infty} A_{2k+1}(x) \frac{z^{2k+1}}{(2k+1)!} \sum_{n=1}^{\infty} A_{2n-1}^*(x) \frac{z^{2n-1}}{(2n-1)!}, \end{aligned}$$

so that

$$(2.4) \quad \begin{aligned} A'(z) &= 1 + A(z)(A(z) - (1-x)z) \\ &= 1 - (1-x)zA(z) + A^2(z). \end{aligned}$$

If we put

$$A(z) = \frac{1}{U} \frac{dU}{dz}, \quad \frac{dA}{dz} = \frac{1}{U^2} \left( \frac{dU}{dz} \right)^2 - \frac{1}{U} \frac{d^2U}{dz^2},$$

(2.4) becomes

$$(2.5) \quad \frac{d^2U}{dz^2} + (1-x)z \frac{dU}{dz} + U = 0.$$

It is clear that  $U$  is an even function of  $z$ . We accordingly put

$$U = \sum_{n=0}^{\infty} (-1)^n a_n(x) \frac{z^{2n}}{(2n)!} \quad (a_0(x) = 1).$$

Substituting in (2.5) we get

$$-a_{n+1}(x) + 2n(1-x)a_n(x) + a_n(x) = 0,$$

so that

$$(2.6) \quad a_{n+1}(x) = (1 + 2n(1-x))a_n(x).$$

It follows at once from (2.6) that

$$a_n(x) = \prod_{k=0}^{n-1} (1 + 2k(1-x)).$$

Hence

$$(2.7) \quad U = \sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^{n-1} (1 + 2k(1-x)) \cdot \frac{z^{2n}}{(2n)!},$$

and

$$(2.8) \quad A(z) = \frac{\sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^n (1 + 2k(1-x)) \cdot \frac{z^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^{n-1} (1 + 2k(1-x)) \cdot \frac{z^{2n}}{(2n)!}}.$$

The first few coefficients are given by

$$A_1(x) = 1, A_3(x) = 2x, A_5(x) = 8x + 8x^2, A_7(x) = 48x + 176x^2 + 48x^3.$$

It follows by induction from

$$A_{2n+1}(x) = \sum_{k=1}^{n-2} \binom{2n+1}{2k} A_{2k+1}(x) A_{2n-2k-1}(x) + 2n(1+x) A_{2n-1}(x) \quad (n > 1)$$

that

$$(2.9) \quad x^{n+1} A_{2n+1}\left(\frac{1}{x}\right) = A_{2n+1}(x).$$

This implies

$$(2.10) \quad A(2n+1, r) = A(2n+1, n-r+1) \quad (1 \leq r \leq n).$$

Also, using the fuller notation  $A(x, z)$ , we have

$$(2.11) \quad x^{1/2} A\left(\frac{1}{x}, x^{1/2}z\right) = (x-1)z + A(z, x).$$

3. Now we consider the case  $A(2n, r)$ . We take

$$A(0, 0) = 1, A(0, r) = 0 \quad (r > 0).$$

Corresponding to (1.2) we have the recurrence

$$(3.1) \quad A(2n+2, r) = \sum_{k=0}^n \sum_s \binom{2n+1}{2k+1} A(2k+1, s) A^*(2n-2k, r-s) \quad (n \geq 0),$$

where

$$A^*(2n, r) = A(2n, r) \quad (n > 0),$$

but

$$A^*(0, 0) = 0, A^*(0, 1) = 1.$$

Now put

$$A_{2n}(x) = \sum_r A(2n, r)x^r, A_0(x) = 1,$$

$$A_{2n}^*(x) = A_{2n}(x)(n > 0), A_0^*(x) = x.$$

Then (3.1) gives

$$(3.2) \quad A_{2n+2}(x) = \sum_{k=0}^n \binom{2n+1}{2k+1} A_{2k+1}(x) A_{2n-2k}^*(x) \quad (n \geq 0).$$

Hence if

$$(3.3) \quad B(z) = B(x, z) = \sum_{n=0}^{\infty} A_{2n}(x) \frac{z^{2n}}{(2n)!},$$

we have

$$B'(z) = A(z)(B(z) - 1 + x).$$

Replacing  $A(z)$  by  $U'/U$ , we get

$$(3.4) \quad UB' + U'B = (1 - x)U'.$$

Since  $B(0) = 1$ ,  $U(0) = 1$ , it follows from (3.4) that

$$UB = x + (1 - x)U.$$

Therefore

$$(3.5) \quad B(z) = 1 - x + \frac{x}{U}.$$

The first few coefficients are

$$A_0(x) = A_2(x) = x, A_4(x) = 3x + 2x^2, A_6(x) = 15x + 38x^2 + 8x^3.$$

4. We turn now to  $C(2n, r)$ . We take

$$C(0, 0) = 1, C(0, r) = 0 \quad (r > 0).$$

We have the recurrence

$$(4.1) \quad C(2n+2, r) = \sum_{k=0}^n \sum_s \binom{2n+1}{2k} C(2k, s) A^*(2n-2k+1, r-s),$$

where  $A^*(2k+1, s)$  has the same meaning as in §2.

Thus, if

$$C_{2n}(x) = \sum_r C(2n, r)x^r,$$

we get

$$(4.2) \quad C_{2n+2}(x) = \sum_{k=0}^n \binom{2n+1}{2k} C_{2k}(x) A_{2n-2k+1}^*(x).$$

Put

$$(4.3) \quad C(z) = C(x, z) = \sum_{n=0}^{\infty} C_{2n}(x) \frac{z^{2n}}{(2n)!}.$$

Then it follows from (4.2) that

$$(4.4) \quad C'(z) = C(z)(A(z) - (1-x)z),$$

so that

$$\frac{C'(z)}{C(z)} = \frac{U'}{U} - (1-x)z.$$

Since  $C(0) = 1$ , this yields

$$(4.5) \quad C(z) = \frac{1}{U} e^{-1/2(1-x)z^2}.$$

The first few coefficients are

$$C_0(x) = 1, C_2(x) = x, C_4(x) = 2x + 3x^2, C_6(x) = 8x + 38x^2 + 15x^3.$$

We shall now show that  $U = U(x, z)$  satisfies the functional equation

$$(4.6) \quad U\left(\frac{1}{x}, x^{1/2}z\right) e^{-1/2(1-x)z^2} = U(x, z),$$

or

$$\sum_0^{\infty} (-1)^n a_n \left(\frac{1}{x}\right) \frac{x^n z^{2n}}{(2n)!} \sum_0^{\infty} (-1)^k \frac{(1-x)^k z^{2k}}{2^k \cdot k!} = \sum_0^{\infty} (-1)^n a_n(x) \frac{z^{2n}}{(2n)!}.$$

This is equivalent to

$$(4.7) \quad \sum_{k=0}^n \frac{(2n)!}{(2k)!(n-k)!} \left(\frac{1-x}{2}\right)^{n-k} a_k \left(\frac{1}{x}\right) = a_n(x).$$

The left hand side of (4.7) is equal to

$$\begin{aligned} & \frac{(2n)!}{n!} \left(\frac{1-x}{2}\right)^n \sum_{k=0}^n (-1)^k \frac{(-n)_k}{k! \left(\frac{1}{2}\right)_k} \left(\frac{x}{2(1-x)}\right)^k \prod_{j=0}^{k-1} \left(1 + 2j\left(1 - \frac{1}{x}\right)\right) \\ &= \frac{(2n)!}{n!} \left(\frac{1-x}{2}\right)^n \sum_{k=0}^n \frac{(-n)_k}{k! \left(\frac{1}{2}\right)_k} \left(\frac{x}{2(k-1)}\right)_k \\ &= \frac{(2n)!}{n!} \left(\frac{1-x}{2}\right)^n \frac{\left(\frac{1}{2(1-x)}\right)_n}{\left(\frac{1}{2}\right)_n} = 2^n (1-x)^n \left(\frac{1}{2(1-x)}\right)_n = a_n(x), \end{aligned}$$

by Vandermonde's theorem.

It evidently follows from (3.5), (4.5), and (4.6) that

$$(4.8) \quad C_{2n}(x) = x^{n+1} A_{2n} \left( \frac{1}{x} \right) \quad (n > 0)$$

and therefore

$$(4.9) \quad C(2n, r) = A(2n, n - r + 1) \quad (1 \leq r \leq n).$$

5. Finally we consider  $C(2n + 1, r)$ . We now take

$$C(1, 1) = 1, C(1, r) = 0 \quad (r \neq 1).$$

We have the recurrence

$$(5.1) \quad C(2n + 1, r) = \sum_{k=0}^n \sum_s \binom{2n}{2k} C(2k) A^*(2n - 2k, r - s).$$

Thus, if

$$C_{2n+1}(x) = \sum_r C(2n + 1, r) x^r,$$

it follows that

$$(5.2) \quad C_{2n+1}(x) = \sum_{k=0}^n \binom{2n}{2k} C_{2k}(x) A_{2n-2k}^*(x).$$

Put

$$D(z) = D(x, z) = \sum_{n=0}^{\infty} C_{2n+1}(x) \frac{z^{2n+1}}{(2n + 1)!}.$$

Then, by (5.2),

$$(5.3) \quad D'(z) = C(z)(B(z) - 1 + x).$$

It follows that

$$\begin{aligned} D'(z) &= \frac{x}{U^2(x, z)} e^{-1/2(1-x)z^2} \\ &= \frac{x}{U(x, z)U(x^{-1}, x^{1/2}z)} \\ &= xC(x, z)C(x^{-1}, x^{1/2}z). \end{aligned}$$

Therefore

$$(5.4) \quad C_{2n+1}(x) = \sum_{k=0}^n \binom{2n}{2k} x^{n-k+1} C_{2k}(x) C_{2n-2k}(x^{-1}).$$

It follows from (5.4) that



$$x^{n+2}C_{2n+1}(x^{-1}) = C_{2n+1}(x),$$

so that

$$(5.5) \quad C(2n+1, r) = C(2n+1, n-r+2) \quad (1 \leq r \leq n+1).$$

The first few values of  $C_{2n+1}(x)$  are given by

$$\begin{aligned} C_1(x) &= x, C_3(x) = x + x^2, C_5(x) = 3x + 10x^2 + 3x^3, C_7(x) \\ &= 15x + 121x^2 + 121x^3 + 15x^4. \end{aligned}$$

Note that  $C_{2n+1}(x)$  is of degree  $n+1$ .

6. A number of special values can be obtained. It follows first from

$$A_{2n+1}(x) = \sum_{k=1}^{n-2} \binom{2n}{2k+1} A_{2k+1}(x) + 2n(1+x)A_{2n-1}(x) \quad (n > 1)$$

and

$$x \mid A_{2k+1}(x) \quad (k > 0)$$

that

$$A'_{2n+1}(0) = 2nA'_{2n-1}(0).$$

This yields

$$(6.1) \quad A(2n+1, 1) = 2nA(2n-1, 1) = 2^n n!.$$

Next, it follows from

$$A_{2n+2}(x) = (2n+1)A_{2n}(x) + \sum_{k=1}^{n-1} \binom{2n+1}{2k+1} A_{2k+1}(x)A_{2n-2k}(x) + xA_{2n+1}(x)$$

and

$$x \mid A_{2k}(x) \quad (k > 0)$$

that

$$A'_{2n+2}(0) = (2n+1)A'_{2n}(0).$$

This gives

$$(6.2) \quad A(2n, 1) = (2n-1)(2n-3) \cdots 3 \cdot 1.$$

It follows from

$$\begin{aligned} C_{2n+2}(x) &= A_{2n+1}(x) + \sum_{k=1}^{n-1} \binom{2n+1}{2k} C_{2k}(x)A_{2n-2k+1}(x) \\ &\quad + (2n+1)x C_{2n}(x) \end{aligned}$$

and

$$x \mid C_{2k}(x) \quad (k > 0)$$

that

$$C'_{2n+2}(x) = A'_{2n+1}(0) \quad (n > 0) .$$

Hence

$$(6.3) \quad C(2n + 2, 1) = 2^n n! .$$

Finally, from

$$C_{2n+1}(x) = A_{2n}(x) + \sum_{k=1}^{n-1} \binom{2n}{2k} C_{2k}(x) A_{2n-2k}(x) + x C_{2n}(x)$$

and

$$x \mid C_{2k+1}(x) \quad (k \geq 0) ,$$

we get

$$C'_{2n+1}(0) = A'_{2n}(0) ,$$

so that

$$(6.4) \quad C(2n + 1, 1) = (2n - 1)(2n - 3) \cdots 3 \cdot 1 .$$

In view of (4.9),

$$C(2n, r) = A(2n, n - r + 1) \quad (1 \leq r \leq n) ,$$

we have also

$$(6.5) \quad A(2n + 2, n + 1) = 2^n n! ,$$

$$(6.6) \quad C(2n, n) = (2n - 1)(2n - 3) \cdots 3 \cdot 1 .$$

7. We remark that the differential equation

$$(7.1) \quad U'' + (1 - x)zU' + U = 0$$

has the second solution  $W = UV$ , where

$$(7.2) \quad V' = e^{-1/2(1-x)z^2} U^{-2} .$$

Thus, by (5.3), we have

$$D'(z) = xV'(z) ,$$

so that

$$(7.3) \quad D(z) = xV(z) .$$

Since  $W$  is an odd function of  $z$  we may put

$$W(z) = \sum_0^{\infty} (-1)^n b_n(x) \frac{z^{2n+1}}{(2n+1)!}.$$

It follows from the differential equation that

$$b_{n+1}(x) = (1 + (2n+1)(1-x))b_n(x),$$

so that

$$(7.4) \quad b_n(x) = \prod_{k=0}^{n-1} (1 + (2k+1)(1-x)).$$

Finally we have

$$(7.5) \quad D(z) = x \frac{\sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^{n-1} (1 + (2k+1)(1-x)) \cdot \frac{z^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n-1} (1 + 2k(1-x)) \cdot \frac{z^{2n}}{(2n)!}}.$$

We may, if we prefer, express both  $U$  and  $V$  as hypergeometric functions of the type  ${}_1F_1$ .

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