SURGERY AND HANDLE STRAIGHTENING IN HILBERT CUBE MANIFOLDS

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The main result of this paper establishes an infinite dimensional version of the finite-dimensional handle straightening result of Kirby-Siebenmann. In order to do this a procedure is developed for doing surgery on infinite-dimensional manifolds. These results are used elsewhere to prove that every compact Hilbert cube manifold can be triangulated and to establish the topological invariance of Whitehead torsion.

1. **Introduction.** A *Hilbert cube manifold* (or Q-manifold) is a separable metric space which has an open cover by sets which are homeomorphic to open subsets of the Hilbert cube Q. Some obvious examples of Q-manifolds are (1) open subsets of Q and (2) $M \times Q$, for any finite-dimensional manifold M. Some nonobvious examples of Q-manifolds are provided by the work of West [17], where it is shown that $P \times Q$ is a Q-manifold, for any locally compact polyhedron P. We say that a Q-manifold can be *triangulated* (or is triangulable) provided that it is homeomorphic to $P \times Q$, for some locally compact polyhedron P.

The main result of this paper is Theorem 7.1, which gives an infinite-dimensional version of the finite-dimensional handle straightening result of Kirby-Siebenmann [10]. This result is a crucial step in the author's recent proof that Whitehead torsion is a topological invariant [6]. Theorem 7.1 is also used to establish the following theorem. We will use the notation R^n for Euclidean n-space, and for any $r > 0$ we use $B_r^n = [-r, r]^n$ and $\text{Int}(B_r^n) = (-r, r)^n$ to denote the standard n-cell of side $2r$ and its interior.

THEOREM. *If X is a triangulable Q-manifold and $h: R^n \times Q \to X$ is an open embedding, for $n \geq 2$, then $X \backslash h(\text{Int}(B_1^n) \times Q)$ is triangulable.*

The above theorem is the main tool used in proving that every compact Q-manifold can be triangulated. We refer the reader to [5] for details of the proof of this result and for a discussion of its corollaries.

In broad outline the proof of Theorem 7.1 follows the proof of the corresponding finite-dimensional result. In particular, the idea of torus homeomorphisms is used. However to achieve the details we will have to apply a considerable amount of infinite-dimensional machinery.

A crucial step in the proof of Theorem 7.1 is Theorem 3.1, where we establish an infinite-dimensional surgery result. For the most part, the techniques used in proving Theorem 3.1 differ from those used in corresponding finite-dimensional situations. In particular we don't need Poincaré duality or transversality.

In §2 we give some definitions and notation which will be used throughout this paper. As mentioned earlier, §3 is devoted to surgery. In §§4, 5, and 6 we establish some technical results which will be used in §7, where we prove our version of infinite-dimensional handle straightening. Finally in Section 8 we prove the theorem cited earlier.

2. **Definitions and notation.** In this section we will describe some of the basic terminology that will be used throughout this paper. We will use the representation $Q = \prod_{i=1}^{\infty} I_i$, where each I_i is the closed interval $[-1, 1]$. If X is any space and k is any integer, then we use p_k to denote the projection of $X \times Q$ onto $X \times I^k$, where $I^k = I_1 \times \cdots I_k$. We also let $Q_{k+1} = I_{k+1} \times I_{k+2} \times \cdots$, for all $k > 0$. Let S^1 denote the standard 1-sphere in R^2 and let $e: R \to S^1$ be the covering projection defined by $e(x) = \exp(\pi i x/4)$. Let the n-torus be denoted by $T^n = S^1 \times \cdots \times S^1$ and let $e^n: R^n \to T^n$ be the product covering projection defined by $e^n = e \times \cdots \times e$.

For any space X and $A \subset X$ we use $\mathrm{Int}_X(A)$ and $\mathrm{Bd}_X(A)$ to denote the topological interior and boundary, respectively, of A in X. As usual, the subscript will be omitted when the meaning is clear. If M is an n-manifold (i.e. a finite-dimensional manifold), then ∂M will denote the combinatorial boundary of M. We will use definitions and notation from [8] concerning PL spaces and manifolds, regular neighborhood theory, etc.

A *map* is a continuous function and *embedding* $f: X \to Y$ is a homeomorphism of X onto $f(X) \subset Y$. When we say that $f: X \to Y$ is a homeomorphism, we mean that f is a homeomorphism of X *onto* Y. If $f, g: X \to Y$ are homotopic maps, then we write $f \cong g$. We will denote composition of maps by juxtaposition.

We will also need Anderson's notion of Property Z [1]. A closed subset A of a space X is said to be a Z-set *in* X provided that given any nonnull, homotopically trivial open subset U of X, $U \backslash A$ is also nonnull and homotopically trivial. Z-sets in Q-manifolds are important because of the following result [2]. Homeomorphism Extension Theorem. *Let K_1, K_2 be compact Z-sets in a Q-manifold X and let $h: K_1 \to K_2$ be a homeomorphism such that $h \cong \mathrm{id}$ (i.e. h is homotopic to the inclusion of K_1 into X). Then h can be extended to a homeomorphism of X onto itself.*

3. **Infinite-dimensional surgery.** Our main result is Theorem

3.1, where we establish an infinite-dimensional surgery result.   In Theorem 3.2 we apply a special case of Theorem 3.1 to prove a result which will be used directly in §7 for handle straightening.

Let P be a compact connected PL space such that P is 1-connected or $\pi_1(P) = Z$, M be a finite-dimensional PL manifold, and let $\phi: P \times Q \times R \to M \times Q$ be an open embedding.   We will use this notation throughout this section.   The proof of Theorem 3.1 will be made more readable by establishing Lemmas 3.1—3.5, each of which strengthens the preceding one.   In the argument that follows we have explicitly given the details for the case $\pi_1(P) = Z$.   The case in which P is 1-connected is quite similar, and easier.

LEMMA 3.1.   *There exists an integer $k > 0$ and a compact PL submanifold X of $M \times I^k$ such that*
(1)   *X is connected,*
(2)   *Bd (X) is a PL submanifold of $M \times I^k$ which is PL bicollared[1],*
(3)   *Bd (X) consists of exactly two components,*
(4)   *$\phi(P \times Q \times \{0\}) \subset \text{Int}(X) \times Q_{k+1} \subset X \times Q_{k+1} \subset \phi(P \times Q \times R)$.*

Proof.   Choose an integer $k_1 > 0$ large enough so that
$$p_{k_1}\phi(P \times Q \times \{0\}) \times Q_{k_1+1} \subset \phi(P \times Q \times R) \ .$$
Choose an open set $U \subset M \times I^{k_1}$ such that
$$p_{k_1}\phi(P \times Q \times \{0\}) \times Q_{k_1+1} \subset U \times Q_{k_1+1} \subset \phi(P \times Q \times R) \ .$$
Then let $X_1 \subset U$ be a regular neighborhood of any compact connected PL subspace of U which contains $p_{k_1}\phi(P \times Q \times \{0\})$, and such that X_1 meets ∂U regularly.   Clearly X_1 is a compact connected PL submanifold of $M \times I^{k_1}$ such that Bd (X_1) is a PL submanifold which is PL bicollared and
$$\phi(P \times Q \times \{0\}) \subset \text{Int}(X_1) \times Q_{k_1+1} \subset X_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R) \ .$$

Bd $(X_1) \times Q_{k_1+1}$ is compact and since P is connected it follows that each component of $\varphi(P \times Q \times R)\backslash(\text{Int}(X_1) \times Q_{k_1+1})$ meets Bd $(X_1) \times Q_{k_1+1}$.   Thus $\varphi(P \times Q \times R)\backslash(\text{Int}(X_1) \times Q_{k_1+1})$ has only finitely many components, and it is clear that exactly two of these are unbounded. Let $\{A_1, \cdots, A_p\}$ be the collection of bounded components of $\phi(P \times Q \times R)\backslash(\text{Int}(X_1) \times Q_{k_1+1})$.   Choose $k_2 \geq k_1$ large enough so that
$$p_{k_2}(A_i) \times Q_{k_2+1} \subset \phi(P \times Q \times R) \ ,$$
for $1 \leq i \leq p$.   Let $X_2' = X_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$ and note that $p_{k_2}(A_i)$

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[1] Bd (X) is PL bicollared means that there exists a PL open embedding h: Bd $(X) \times (-1, 1) \to M \times I^k$ such that $h(x, 0) = x$, for all $x \in$ Bd (X).

is a relatively open subset of $(M \times I^{k_2})\backslash\text{Int}(X_2')$, for $1 \leq i \leq p$. Put

$$X_2 = X_2' \cup (\bigcup_{i=1}^{p} p_{k_2}(A_i))$$

and note that X_2 is a compact connected PL submanifold of $M \times I^{k_2}$ such that $\text{Bd}(X_2)$ is a PL submanifold of $M \times I^{k_2}$, $\text{Bd}(X_2)$ PL bicollared, $\phi(P \times Q \times \{0\}) \subset \text{Int}(X_2) \times Q_{k_2+1} \subset X_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$, and $\phi(P \times Q \times R)\backslash(\text{Int}(X_2) \times Q_{k_2+1})$ has exactly two components.

Let A, B denote the components of $\phi(P \times Q \times R)\backslash(\text{Int}(X_2) \times Q_{k_2+1})$. We will show how to reduce the number of components of $(\text{Bd}(X_2) \times Q_{k_2+1}) \cap A$. The procedure for reducing the number of components of $(\text{Bd}(X_2) \times Q_{k_2+1}) \cap B$ will be similar. Thus choose distinct components C, D of $(\text{Bd}(X_2) \times Q_{k_2+1}) \cap A$. We can find a path σ in A from C to D (i.e. $\sigma \cap C \neq \phi$ and $\sigma \cap D \neq \phi$). Choose $k_3 \geq k_2$ large enough so that $p_{k_3}(\sigma) \times Q_{k_3+1} \subset A$. Let $X_3' = X_2 \times I_{k_2+1} \times \cdots \times I_{k_3}$ and note that $p_{k_3}(\sigma)$ is a path in $(M \times I^{k_3})\backslash\text{Int}(X_3')$ from $p_{k_3}(C)$ to $p_{k_3}(D)$. We can use $p_{k_3}(\sigma)$ to find a PL arc α in $(M \times I^{k_3})\backslash[\text{Int}(X_3') \cup \partial(M \times I^{k_3})]$ from $p_{k_3}(C)$ to $p_{k_3}(D)$ such that one endpoint is in $p_{k_3}(C)$, the other is in $p_{k_3}(D)$, and the remainder of the arc misses X_3'.

In a standard way we can use α to attach a PL handle to X_3' in $(M \times I^{k_3})\backslash[\text{Int}(X_3') \cup \partial(M \times I^{k_3})]$ which connects the components $p_{k_3}(C)$ and $p_{k_3}(D)$ (i.e. take a regular neighborhood of α in $(M \times I^{k_3})\backslash\partial(M \times I^{k_3})$ and add it to X_3'). In this manner we obtain a compact connected PL submanifold X_3 of $M \times I^{k_3}$ such that $\text{Bd}(X_3)$ is a PL submanifold, $\text{Bd}(X_3)$ is PL bicollared, $\text{Bd}(X_3)$ has one less component than $\text{Bd}(X_3')$, $\phi(P \times Q \times \{0\}) \subset \text{Int}(X_3) \times Q_{k_3+1} \subset X_3 \times Q_{k_3+1} \subset \phi(P \times Q \times R)$ and $\phi(P \times Q \times R)\backslash(\text{Int}(X_3) \times Q_{k_3+1})$ has exactly two components. [For this last condition we need to assume that $k_3 \geq 3$.].

It is now clear that we can continue to eliminate boundary components in this manner to inductively arrive at an integer $k \geq k_3$ and a compact connected PL submanifold X of $M \times I^k$ which fulfills our requirements.

Our next task is to alter the X obtained in Lemma 3.1 so that, in addition to the properties listed there, if C is a component of $\text{Bd}(X)$, then the inclusion $C \times Q_{k+1} \subset \phi(P \times Q \times R)$ induces an isomorphism of $\pi_1(C \times Q_{k+1})$ onto $\pi_1(\phi(P \times Q \times R))$. In Lemma 3.2 below we achieve the surjectivity of this induced homomorphism and Lemmas 3.3, and 3.4 give an inductive procedure for eliminating the kernel of this homomorphism, while maintaining the surjectivity.

LEMMA 3.2. *There exists an integer $k > 0$ and a compact PL submanifold X of $M \times I^k$ such that*

(1) X is connected,

(2) Bd (X) is a PL submanifold of $M \times I^k$ which is PL bicollared,

(3) Bd (X) consists of exactly two components,

(4) $\phi(P \times Q \times \{0\}) \subset \text{Int}(X) \times Q_{k+1} \subset X \times Q_{k+1} \subset \phi(P \times Q \times R)$,

(5) if C is any component of Bd (X), then the inclusion $C \times Q_{k+1} \subset \phi(P \times Q \times R)$ induces a surjection of $\pi_1(C \times Q_{k+1})$ onto $\pi_1(\phi(P \times Q \times R))$.

Proof. Using Lemma 3.1 there exists an integer $k_1 > 0$ and a compact connected PL submanifold X_1 of $M \times I^{k_1}$ such that Bd (X_1) is a PL submanifold of $M \times I^{k_1}$, Bd (X_1) is PL bicollared and has exactly two components, and $\phi(P \times Q \times \{0\}) \subset \text{Int}(X_1) \times Q_{k_1+1} \subset X_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$. The procedure is now to "fatten up" $X_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$ in $M \times I^{k_2}$, for some sufficiently large integer k_2, so that (5) above is satisfied. Since $\pi_1(\phi(P \times Q \times R) = Z$, this amounts to "fattening up" $X_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$ so that a loop is introduced into each component of its boundary, where this loop is appropriately chosen to achieve (5) above. We will not give the details of the construction since they are similar to the "fattening up" process used in Lemma 3.5 later on.

LEMMA 3.3. *Let $k_1 > 0$ be an integer and let $X_1 \subset M \times I^{k_1}$ be a compact connected PL submanifold of $M \times I^{k_1}$ such that Bd (X_1) is a PL submanifold, Bd (X_1) is PL bicollared and has exactly two components, and $\phi(P \times Q \times \{0\}) \subset \text{Int}(X_1) \times Q_{k_1+1} \subset X_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$. Let C_1, D_1 be the components of Bd (X_1) and let i_1, j_1 be the inclusions $C_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$, $D_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$. If $\alpha \in \pi_1(C_1 \times Q_{k_1+1})$ and $\beta \in \pi_1(D_1 \times Q_{k_1+1})$ satisfy $(i_1)_*(\alpha) = 1$ and $(j_1)_*(\beta) = 1$, where $(i_1)_*$ and $(j_1)_*$ are the induced homomorphisms on the fundamental groups, then there exists an integer $k_2 \geqq k_1$ and a compact PL submanifold X_2 of $M \times I^{k_2}$ such that*

(1) Bd (X_2) is a PL submanifold of $M \times I^{k_2}$,

(2) Bd (X_2) is PL bicollared and has exactly two components,

(3) $\phi(P \times Q \times \{0\}) \subset \text{Int}(X_2) \times Q_{k_2+1} \subset X_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$,

(4) *if C_2, D_2 are the components of Bd (X_2) (with notation appropriately chosen) and i_2, j_2 are the inclusions $C_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$, $D_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$, then there exist surjective homomorphisms $u: \pi_1(C_1 \times Q_{k_1+1}) \rightarrow \pi_1(C_2 \times Q_{k_2+1})$ and $v: \pi_1(D_1 \times Q_{k_1+1}) \rightarrow \pi_1(D_2 \times Q_{k_2+1})$ such that $u(\alpha) = 1$, $v(\beta) = 1$, $(i_2)_*u = (i_1)_*$, and $(j_2)_*v = (j_1)_*$.*

Proof. Let A_1, B_1 be the components of $\phi(P \times Q \times R) \backslash (\text{Int}(X_1) \times Q_{k_1+1})$ and choose notation so that $C_1 \times Q_{k_1+1} \subset A_1 \subset \phi(P \times Q \times (0, \infty))$ and $D_1 \times Q_{k_1+1} \subset B_1 \subset \phi(P \times Q \times (-\infty, 0))$. Since $(i_1)_*(\alpha) = 1$ and $(j_1)_*(\beta) = 1$ we can find maps $f_1: B_1^2 \rightarrow \phi(P \times Q \times (0, \infty))$ and $g_1: B_1^2 \rightarrow \phi(P \times Q \times$

$(-\infty, 0))$ such that $f_1(\mathrm{Bd}\,(B_1^2)) \subset C_1 \times Q_{k_1+1}$, $[f_1 | \mathrm{Bd}\,(B_1^2)] = \alpha$, $g_1(\mathrm{Bd}\,(B_1^2)) \subset D_1 \times Q_{k_1+1}$, and $[g_1 | \mathrm{Bd}\,(B_1^2)] = \beta$ (here $[\ \ ]$ denotes homotopy class). Choose $k_2 \geq \max\,(k_1, 5)$ *large enough so that*

$$p_{k_2} f_1(B_1^2) \times Q_{k_2+1} \subset \phi(P \times Q \times (0, \infty))$$

and

$$p_{k_2} g_1(B_1^2) \times Q_{k_2+1} \subset \phi(P \times Q \times (-\infty, 0)) \,.$$

Since $k_2 \geq 5$ we can approximate f_1 and g_1 by *PL* embeddings
$$f_2 \colon B_1^2 \longrightarrow (M \times I^{k_2}) \backslash \partial (M \times I^{k_2})$$

and

$$g_2 \colon B_1^2 \longrightarrow (M \times I^{k_2}) \backslash \partial (M \times I^{k_2})$$

such that

(1) $f_2(\mathrm{Bd}\,(B_1^2)) \subset C_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$ and $g_2(\mathrm{Bd}\,(B_1^2)) \subset D_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$,

(2) $f_2(B_1^2) \times Q_{k_2+1} \subset \phi(P \times Q \times (0, \infty))$ and $g_2(B_1^2) \times Q_{k_2+1} \subset \phi(P \times Q \times (-\infty, 0))$,

(3) $[((f_2 | \mathrm{Bd}\,(B_1^2)), 0)] = \alpha$ and $[((g_2 | \mathrm{Bd}\,(B_1^2)), 0)] = \beta$, where $((f_2 | \mathrm{Bd}\,(B_1^2)), 0) \colon \mathrm{Bd}\,(B_1^2) \to C_1 \times Q_{k_1+1}$ and $((g_2 | \mathrm{Bd}\,(B_1^2)), 0) \colon \mathrm{Bd}\,(B_1^2) \to D_1 \times Q_{k_1+1}$ are defined by $((f_2 | \mathrm{Bd}\,(B_1^2)), 0)(x) = (f_2(x), 0)$ and $((g_2 | \mathrm{Bd}\,(B_1^2)), 0)(x) = (g_2(x), 0)$, where $0 = (0, 0, \cdots) \in Q_{k_2+1}$,

(4) $f_2(B_1^2)$ is in general position with respect to $C_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$,

(5) $g_2(B_1^2)$ is in general position with respect to $D_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$. Choose an open set $U \subset M \times I^{k_2}$ such that

$$(C_1 \times I_{k_1+1} \times \cdots \times I_{k_2}) \cup f_2(B_1^2) \subset U$$

and $U \times Q_{k_2+1} \subset \phi(P \times Q \times (0, \infty))$. Similarly we can choose an open set $V \subset M \times I^{k_2}$ such that

$$(D_1 \times I_{k_1+1} \times \cdots \times I_{k_2}) \cup g_2(B_1^2) \subset V$$

and $V \times Q_{k_2+1} \subset \phi(P \times Q \times (-\infty, 0))$. The procedure of Browder now applies here. That is, Lemma 3.1 of [3] implies that we can *exchange* discs to construct a compact connected *PL* submanifold X_2 of $M \times I^{k_2}$ such that

(1) $\mathrm{Bd}\,(X_2)$ is a *PL* submanifold of $M \times I^{k_2}$ which has exactly two components, one in U and the other in V,

(2) $\mathrm{Bd}\,(X_2)$ is *PL* bicollared,

(3) $\phi(P \times Q \times \{0\}) \subset \mathrm{Int}\,(X_2) \times Q_{k_2+1} \subset X_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$,

(4) if C_2 and D_2 are the components of $\mathrm{Bd}\,(X_2)$, where $C_2 \subset U$ and $D_2 \subset V$, then the homomorphisms u and v of the statement of the

lemma exist and satisfy the required properties.

We now use Lemma 3.3 to inductively construct an X which satisfies the requirements of Lemma 3.1 and which also satisfies the property that if C is a component of $\mathrm{Bd}\,(X)$, then the inclusion $C \times Q_{k+1} \subset \phi(P \times Q \times R)$ induces an isomorphism of $\pi_1(C \times Q_{k+1})$ onto $\pi_1(\phi(P \times Q \times R))$.

LEMMA 3.4.   *There exists an integer $k > 0$ and a compact PL submanifold X of $M \times I^k$ such that*

    (1) X *is connected,*

    (2) $\mathrm{Bd}\,(X)$ *is a PL submanifold of $M \times I^k$ which is PL bicollared,*

    (3) $\mathrm{Bd}\,(X)$ *consists of exactly two components,*

    (4) $\phi(P \times Q \times \{0\}) \subset \mathrm{Int}\,(X) \times Q_{k+1} \subset X \times Q_{k+1} \subset \phi(P \times Q \times R)$,

    (5)   *if C is a component of $\mathrm{Bd}\,(X)$, then the inclusion $C \times Q_{k+1} \subset \phi(P \times Q \times R)$ induces an isomorphism of $\pi_1(C \times Q_{k+1})$ onto $\pi_1(\phi(P \times Q \times R))$.*

Proof. Using Lemma 3.2 there exists an integer $k_1 > 0$ and a compact connected *PL* submanifold X_1 of $M \times I^{k_1}$ such that $\mathrm{Bd}\,(X_1)$ is a *PL* submanifold of $M \times I^{k_1}$, $\mathrm{Bd}\,(X_1)$ is *PL* bicollared and has exactly two components, and $\phi(P \times Q \times \{0\}) \subset \mathrm{Int}\,(X_1) \times Q_{k_1+1} \subset X_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$. Moreover if C_1 and D_1 are the components of $\mathrm{Bd}\,(X_1)$ and $i_1\colon C_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$, $j_1\colon D_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$ are inclusions, then $(i_1)_*\colon \pi_1(C_1 \times Q_{k_1}) \to \pi_1(\phi(P \times Q \times R))$ and $(j_1)_*\colon \pi_1(D_1 \times Q_{k_1+1}) \to \pi_1(\phi(P \times Q \times R))$ are surjections. Let α be a generator of $\pi_1(\phi(P \times Q \times R))$ and choose $\alpha_1 \in \pi_1(C_1 \times Q_{k_1+1})$, $\beta_1 \in \pi_1(D_1 \times Q_{k_1+1})$ such that $(i_1)_*(\alpha_1) = \alpha$ and $(j_1)_*(\beta_1) = \alpha$. Since $\pi_1(C_1 \times Q_{k_1+1})$ and $\pi_1(D_1 \times Q_{k_1+1})$ are finitely generated we can choose generators $\gamma_1, \cdots, \gamma_m$ of $\pi_1(C_1 \times Q_{k_1+1})$ and $\delta_1, \cdots, \delta_n$ of $\pi_1(C_1 \times Q_{k_1+1})$. We now apply Lemma 3.3 to inductively reduce the number of generators of $\pi_1(C_1 \times Q_{k_1+1})$ and $\pi_1(D_1 \times Q_{k_1+1})$.

Note that $(i_1)_*(\gamma_1) = \alpha^r$ and $(j_1)_*(\delta_1) = \alpha^s$, for some integers r and s. Thus $(i_1)_*(\gamma_1\alpha_1^{-r}) = 1$ and $(j_1)_*(\delta_1\beta_1^{-s}) = 1$. Using Lemma 3.3 we can choose an integer $k_2 \geqq k_1$ and a compact connected *PL* submanifold X_2 of $M \times I^{k_2}$ such that $\mathrm{Bd}\,(X_2)$ is a *PL* submanifold of $M \times I^{k_2}$ which is *PL* bicollared and has exactly two components, and $\phi\{P \times Q \times \{0\}) \subset \mathrm{Int}\,(X_2) \times Q_{k_2+1} \subset X_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$. Moreover if C_2, D_2 are the components (appropriately named) of $\mathrm{Bd}\,(X_2)$ and $i_2\colon C_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$, $j_2\colon D_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$ are inclusions, then there exist surjective homomorphisms $u\colon \pi_1(C_1 \times Q_{k_1+1}) \to \pi_1(C_2 \times Q_{k_2+1})$, $v\colon \pi_1(D_1 \times Q_{k_1+1}) \to \pi_1(D_2 \times Q_{k_2+1})$ such that $u(\gamma_1\alpha_1^{-r}) = 1$, $v(\delta_1\beta_1^{-s}) = 1$, $(i_2)_*u = (i_1)_*$, and $(j_2)_*v = (j_1)_*$. We have $\pi_1(C_2 \times Q_{k_2+1})$ generated by $u(\alpha_1)$, $u(\gamma_1)$, $\cdots$, $u(\gamma_m)$, where $u(\gamma_1) = (u(\alpha_1))^r$. Thus $\pi_1(C_2 \times Q_{k_2+1})$ is generated by $u(\alpha_1)$, $u(\gamma_2)$, $\cdots$, $u(\gamma_m)$. Also we have $(i_2)_*(u(\alpha_1)) = \alpha$. Similarly we have $\pi_1(D_2 \times Q_{k_2+1})$ generated by $v(\beta_1)$, $v(\delta_2)$, $\cdots$, $v(\delta_n)$ and $(j_2)_*(v(\beta_1)) = \alpha$.

Inductively continuing this procedure we will eventually arrive at an integer $k_3 \geqq k_2$ and an appropriate subspace X_3 of $M \times I^{k_3}$, with boundary components C_3 and D_3, and inclusions $i_3 \colon C_3 \times Q_{k_3+1} \subset \phi(P \times Q \times R)$ and $j_3 \colon D_3 \times Q_{k_3+1} \subset \phi(P \times Q \times R)$, such that $\pi_1(C_3 \times Q_{k_3+1})$ is generated by an element α_3 which satisfies $(i_3)_*(\alpha_3) = \alpha$ and $\pi_1(D_3 \times Q_{k_3+1})$ is generated by an element β_3 which satisfies $(j_3)_*(\beta_3) = \alpha$. This implies that $(i_3)_*$ and $(j_3)_*$ are isomorphisms.

We now prove that the X of Lemma 3.1 can be constructed so that the inclusion $X \times Q_{k+1} \subset \phi(P \times Q \times R)$ is a homotopy equivalence.

LEMMA 3.5. *There exists an integer $k > 0$ and a compact connected PL submanifold X of $M \times I^k$ such that*
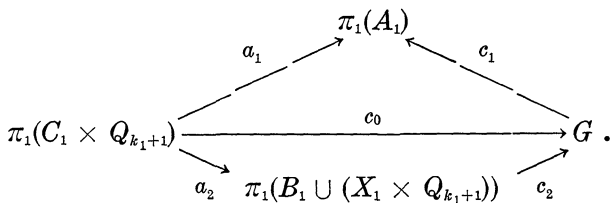   (1) $\mathrm{Bd}\,(X)$ *is a PL submanifold of $M \times I^k$ which has exactly two components,*
    (2) $\mathrm{Bd}\,(X)$ *is PL bicollared,*
    (3) $\phi(P \times Q \times \{0\}) \subset \mathrm{Int}\,(X) \times Q_{k+1} \subset X \times Q_{k+1} \subset \phi(P \times Q \times R)$,
    (4) *the inclusion $X \times Q_{k+1} \subset \phi(P \times Q \times R)$ is a homotopy equivalence.*

Proof. Using Lemma 3.4 we can find an integer $k_1 > 0$ and a compact connected PL submanifold X_1 of $M \times I^{k_1}$ such that $\mathrm{Bd}\,(X_1)$ is a PL submanifold of $M \times I^{k_1}$ which has exactly two components, $\mathrm{Bd}\,(X_1)$ is PL bicollared, $\phi(P \times Q \times \{0\}) \subset \mathrm{Int}\,(X_1) \times Q_{k_1+1} \subset X_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$, and if C_1, D_1 are the components of $\mathrm{Bd}\,(X_1)$, then the inclusions $C_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$ and $D_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$ induce isomorphisms on π_1. Let A_1, B_1 denote the components of $\phi(P \times Q \times R) \backslash (\mathrm{Int}\,(X_1) \times Q_{k_1+1})$, where notation is chosen so that $\mathrm{Bd}\,(A_1) = C_1 \times Q_{k_1+1}$ and $\mathrm{Bd}\,(B_1) = D_1 \times Q_{k_1+1}$.

Consider the following diagram, where all the homomorphisms are induced by inclusions.

$$
\begin{array}{ccc}
 & \pi_1(A_1) & \\
{\scriptstyle a_1}\nearrow & & \searrow{\scriptstyle b_1} \\
\pi_1(C_1 \times Q_{k_1+1}) & \xrightarrow{\ \ b_0\ \ } & \pi_1(\mathscr{P}(P \times Q \times R)) \ . \\
{\scriptstyle a_2}\searrow & & \nearrow{\scriptstyle b_2} \\
 & \pi_1(B_1 \cup (X_1 \times Q_{k_1+1})) &
\end{array}
$$
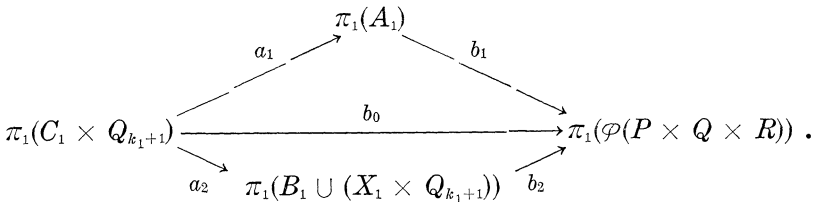
We are given that b_0 is an isomorphism, thus a_1, a_2 are one-to-one and b_1, b_2 are onto. If G denotes the amalgated free product of $\pi_1(A_1)$ and $\pi_1(B_1 \cup (X_1 \times Q_{k_1+1}))$, with the subgroup $a_1(\pi_1(C_1 \times Q_{k_1+1}))$ of $\pi_1(A_1)$ amalgated with subgroup $a_2(\pi_1(C_1 \times Q_{k_1+1}))$ of $\pi_1(B_1 \cup (X_1 \times Q_{k_1+1}))$, then there exist monomorphisms c_0, c_1, c_2 such that the two triangles commute in the following diagram (see [11], p. 32).

$$\begin{array}{ccc}
& \pi_1(A_1) & \\
a_1 \nearrow & & \nwarrow c_1 \\
\pi_1(C_1 \times Q_{k_1+1}) & \xrightarrow{\quad c_0 \quad} & G \; . \\
a_2 \searrow & \pi_1(B_1 \cup (X_1 \times Q_{k_1+1})) & \nearrow c_2
\end{array}$$

Using the Seifert-Van Kampem theorem there exists a homomorphism $d: \pi_1(\mathcal{P}(P \times Q \times R)) \to G$ such that $c_1 = db_1$ and $c_2 = db_2$ (see [12], p. 114). Thus b_1, b_2 must be isomorphisms, hence the inclusions $A_1 \subset \mathcal{P}(P \times Q \times R)$ and $B_1 \subset \mathcal{P}(P \times Q \times R)$ induce isomorphisms on π_1.

It follows from [14] (Complement 6.6(b), p. 48) that A_1 and B_1 are dominated by finite complexes. It follows from [16] that A_1 and B_1 have finite homotopy type if and only if the Wall obstructions $\sigma(A_1)$ and $\sigma(B_1)$ vanish. But since $\pi_1(A_1) \cong \pi_1(B_1) \cong Z$ it follows that $\sigma(A_1)$ and $\sigma(B_1)$ vanish (see [16], p. 67, for references). Thus A_1 and B_1 have finite homotopy type.

Our strategy will be to use the fact that A_1 and B_1 have finite homotopy type to "fatten up" X_1 to get our desired X. This is the first time that we really have to exploit the fact that we are dealing with infinite-dimensional spaces. Instead of inductively killing homology groups (as is done in the finite-dimensional situation), we make the jump from X_1 to X in one step.

Choose a finite complex K and maps $f: A_1 \to |K|$, $g: |K| \to A_1$ which are homotopy inverses. Let $i: C_1 \to A_1$ be defined by $i(c) = (c, 0)$. Since C_1 is a compact PL space we can find a PL map $\alpha: C_1 \to |K|$ which is homotopic to $fi: C_1 \to |K|$. Let Y be the simplicial mapping cylinder of α, for appropriate subdivisions of C_1 and K (see [7] for details). Then C_1 and $|K|$ have natural identifications in Y. Let $r: Y \to |K|$ be the retraction which collapses the fiber over each point of $|K|$ in Y to that point. With this identification we obviously must have $r \,|\, C_1 = \alpha$. Then r is a deformation retraction. Thus $gr: Y \to A_1$ is a homotopy equivalence. Using results from [2] we can find an embedding $h: Y \to A_1$ such that $h(Y)$ is a Z-set and $h \cong gr$. Let $j: C_1 \subset Y$ be inclusion and note that

$$hj \cong grj = g\alpha \cong gfi \cong (\mathrm{id})i = i \; ,$$

where id denotes the identity. Thus $h \,|\, C_1: C_1 \to A_1$ is an embedding such that $h \,|\, C_1 \cong i$ and $h(C_1)$ is a Z-set. Using the Homeomorphism Extension Theorem of §2 we can adjust h so that the condition $h \,|\, C_1 = i$ is additionally satisfied.

Now choose $k_2 \geq k_1$ large enough so that $p_{k_2}h(Y) \times Q_{k_2+1} \subset A_1$ and let $\beta = p_{k_2}h$. If k_2 is chosen large enough, then we can replace β by a PL embedding $\gamma: Y \to M \times I^{k_2}$ such that $\gamma \,|\, C_1 = (\mathrm{id}, 0): C_1 \to M \times$

I^{k_2} and γ is as close to β as we want. Thus γ can be chosen so that $\gamma(Y) \times Q_{k_2+1} \subset A_1$ and $\gamma \cong \beta$. By making one more adjustment to γ we can additionally require that

$$\gamma(Y \backslash C_1) \cap (C_1 \times I_{k_1+1} \times \cdots \times I_{k_2}) = \varnothing .$$

Note that $(\gamma, 0) \colon Y \to A_1$ is a homotopy equivalence. If $C_2 = C_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$, then there is a deformation retraction of $C_2 \cup \gamma(Y)$ onto $(C_1 \times \{0\}) \cup \gamma(Y) = \gamma(Y)$. Thus the inclusion $(C_2 \cup \gamma(Y)) \times Q_{k_2+1} \subset A_1$ is a homotopy equivalence. Let

$$N = (C_2 \times I_{k_2+1}) \cup (\gamma(Y) \times \{1\}) \subset M \times I^{k_2+1}$$

and observe that there is a strong deformation retraction of N onto $(C_2 \cup \gamma(Y)) \times \{1\}$, thus the inclusion $N \times Q_{k_2+1} \subset A_1$ is a homotopy equivalence. Also we have $N \subset \partial((M \times I^{k_2+1}) \backslash \mathrm{Int}\,(X_3))$, where $X_3 = X_1 \times I_{k_1+1} \times \cdots \times I_{k_2+1}$. Choose a regular neighborhood N^* of N in $\partial((M \times I^{k_2+1}) \backslash \mathrm{Int}\,(X_3))$ such that $N^* \times Q_{k_2+2} \subset A_1$. Clearly the inclusion $N^* \times Q_{k_2+2} \subset A_1$ is a homotopy equivalence. Put $k_3 = k_2 + 1$.

Since $(M \times I^{k_3}) \backslash \mathrm{Int}\,(X_3)$ has a boundary collar we can find a PL embedding $\theta \colon N^* \times [0, 1] \to (M \times I^{k_3}) \backslash \mathrm{Int}\,(X_3)$ such that $\theta(n, 0) = n$, for all $n \in N^*$, and $\theta(N^* \times [0, 1]) \cap \partial((M \times I^{k_3}) \backslash \mathrm{Int}\,(X_3)) = N^*$. By choosing $\theta(N^* \times [0, 1])$ in a neighborhood of N^* which is sufficiently close to N^*, we can additionally assume that $\theta(N^* \times [0, 1]) \times Q_{k_3+1} \subset A_1$. Let $\mathrm{Bd}\,(N^*)$ denote the boundary of N^* in $\partial((M \times I^{k_3}) \backslash \mathrm{Int}\,(X_3))$ and let $\psi \colon \mathrm{Bd}\,(N^*) \times [0, 1] \to N^*$ be a boundary collar of N^*, where $\psi(n, 0) = n$, for all $n \in \mathrm{Bd}\,(N^*)$. Let us additionally assume that $\psi(\mathrm{Bd}\,(N^*) \times [0, 1]) \cap N = \varnothing$. Let $\tau \colon N^* \backslash \psi(\mathrm{Bd}\,(N^*) \times [0, 1/2)) \to [0, 1]$ be a PL map such that $\tau(N) = \{1\}$ and $\tau^{-1}(0) = \psi(\mathrm{Bd}\,(N^*) \times \{1/2\})$. Put $X_3' = X_3 \cup G$, where

$$G = \bigcup \{\theta(n, t) \mid n \in N^* \backslash \psi(N^* \times [0, 1/2)) \text{ and } 0 \leqq t \leqq \tau(n)/2\} .$$

It is now clear that X_3' is a compact connected PL submanifold of $M \times I^{k_3}$ such that $\mathrm{Bd}\,(X_3')$ is a PL submanifold which is PL bicollared, $\mathrm{Bd}\,(X_3')$ has exactly two components, and $X_1 \times Q_{k_1+1} \subset X_3' \times Q_{k_3+1} \subset \phi(P \times Q \times R)$. Let A_3', B_3' be the components of $\phi(P \times Q \times R) \backslash (\mathrm{Int}\,(X_3') \times Q_{k_3+1})$ and let C_3', D_3' be the components of $\mathrm{Bd}\,(X_3')$, where notation is chosen so that $\mathrm{Bd}\,(A_3') = C_3' \times Q_{k_3+1}$ and $\mathrm{Bd}\,(B_3') = D_3' \times Q_{k_3+1}$. We can also choose notation so that $B_3' = B_1$ and $D_3' = D_1 \times I_{k_1+1} \times \cdots \times I_{k_3}$. Then it follows that the inclusion $C_3' \times Q_{k_3+1} \subset A_3'$ is a homotopy equivalence. Thus we have successfully operated on one side of X_1.

It is clear that the above procedure can be repeated to find an integer $k_4 > 0$ and a compact connected PL submanifold X_4 of $M \times I^{k_4}$ such that $\mathrm{Bd}\,(X_4)$ is a PL submanifold of $M \times I^{k_4}$ which is PL bicollared, $\mathrm{Bd}\,(X_4)$ has exactly two components, say C_4 and D_4, $\phi(P \times$

$Q \times \{0\}) \subset \text{Int} \, (X_4) \times Q_{k_4+1} \subset X_4 \times Q_{k_4+1} \subset \phi(P \times Q \times R)$, and if A_4, B_4 are the components of $\phi(P \times Q \times R) \backslash (\text{Int} \, (X_4) \times Q_{k_4+1})$ such that $\text{Bd} \, (A_4) = C_4 \times Q_{k_4+1}$ and $\text{Bd} \, (B_4) = D_4 \times Q_{k_4+1}$, then the inclusions $C_4 \times Q_{k_4+1} \subset A_4$ and $D_4 \times Q_{k_4+1} \subset B_4$ are homotopy equivalences. A well-known property of ANR's implies that $C_4 \times Q_{k_4+1}$ is a strong deformation retract of A_4 and $D_4 \times Q_{k_4+1}$ is a strong deformation retract of B_4 (for example see [15], p. 31). This implies that the inclusion $X_4 \times Q_{k_4+1} \subset \mathscr{P}(P \times Q \times R)$ is a homotopy equivalence. Then put $X = X_4$ and $k = k_4$ to fulfill our requirements.

We are now ready to state and prove our main surgery result. What we are going to do is modify the boundary of the X which was obtained in Lemma 3.5. For the reader who is comparing this process with finite-dimensional surgery, this is the step in which we exchange handles.

THEOREM 3.1. *There exists an integer $k > 0$ and a compact PL submanifold X of $M \times I^k$ such that*
 (1)   Bd (X) *is a PL submanifold of $M \times I^k$ which is PL bicollared,*
 (2)   Bd (X) *has exactly two components,*
 (3) $\phi(P \times Q \times \{0\}) \subset \text{Int} \, (X) \times Q_{k+1} \subset X \times Q_{k+1} \subset \phi(P \times Q \times R)$,
 (4)   *the inclusion $X \times Q_{k+1} \subset \phi(P \times Q \times R)$ is a homotopy equivalence,*
 (5)   *for each component C of* Bd (X), *the inclusion $C \times Q_{k+1} \subset \phi(P \times Q \times R)$ is a homotopy equivalence.*

Proof. Using Lemma 3.5 we can find an integer $k_1 > 0$ and a compact *PL* submanifold X_1 of $M \times I^{k_1}$ such that
 (1)   Bd (X_1) is a PL submanifold of $M \times I^{k_1}$ which is *PL* bicollared,
 (2)   Bd (X_1) has exactly two components,
 (3) $\phi(P \times Q \times \{0\}) \subset \text{Int} \, (X) \times Q_{k_1+1} \subset X \times Q_{k+1} \subset \phi(P \times Q \times R)$,
 (4)   the inclusion $X_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$ is a homotopy equivalence. Let A_1, B_1 be the components of $\phi(P \times Q \times R) \backslash (\text{Int} \, (X_1) \times Q_{k_1+1})$ and let C_1, D_1 be the components of Bd (X_1) so that Bd $(A_1) = C_1 \times Q_{k_1+1}$ and Bd $(B_1) = D_1 \times Q_{k_1+1}$. It is possible that the inclusions $C_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$ and $D_1 \times Q_{k_1+1} \subset \phi(P \times Q \times R)$ are not homotopy equivalences. To remedy this defect we now dig back into X_1 to produce our required X. This amounts to "fattening up" A_1 and B_1. We will only give the details for operating on A_1. A similar procedure will work on B_1. Choose notation so that we have $A_1 \subset \phi(P \times Q \times (0, \infty))$ and $B_1 \subset \phi(P \times Q \times (- \infty, 0))$.

Choose $r > 0$ so that $\phi(P \times Q \times [- r, r]) \subset \text{Int} \, (X_1) \times Q_{k_1+1}$. Then the inclusion $\phi(P \times \{0\} \times \{r\}) \subset X_1 \times Q_{k_1+1}$ is a homotopy equivalence, where $0 = (0, 0, \cdots) \in Q$. Put $P_0 = \phi(P \times \{0\} \times \{r\})$, let $f: P_0 \to X_1 \times Q_{k_1+1}$ be inclusion, and let $g: X_1 \times Q_{k_1+1} \to P_0$ be a map such that f and

g are homotopy inverses. Let $i: C_1 \rightarrow X_1 \times Q_{k_1+1}$ be the map defined by $i(c) = (c, 0)$ and let $\alpha: C_1 \rightarrow P_0$ be a PL approximation to gi (i.e., $\alpha \cong gi$). [We are regarding P_0 as having a fixed *PL* structure which is inherited from P.] Let Y be the simplicial mapping cylinder of α, where C_1 and P_0 have natural identifications in Y, and let $r: Y \rightarrow P_0$ be the map which collapses the fiber over each point of P_0 to that point. Then $fr: Y \rightarrow X_1 \times Q_{k_1+1}$ is a homotopy equivalence and $fr \,|\, C_1: C_1 \rightarrow X_1 \times Q_{k_1+1}$ is homotopic to $i: C_1 \rightarrow X_1 \times Q_{k_1+1}$ (in $X_1 \times Q_{k_1+1}$). But $\phi(P \times Q \times [r, \infty))$ is a retract of $\phi(P \times Q \times R)$. Thus the homotopy joining i to $fr \,|\, C_1$ can be realized in $(X_1 \times Q_{k_1+1}) \cap \phi(P \times Q \times [r, \infty))$. Therefore the main idea of Lemma 3.5 can be used to dig back into $X_1 \times Q_{k_1+1}$ to produce an integer $k_2 \geqq k_1$ and a compact *PL* submanifold X_2 of $M \times I^{k_2}$ such that

(1) $\phi(P \times Q \times \{0\}) \subset \text{Int}\,(X_2) \times Q_{k_2+1} \subset X_2 \times Q_{k_2+1} \subset X_1 \times Q_{k_1+1}$,

(2) Bd (X_2) is a PL submanifold of $M \times I^{k_2}$ which is *PL* bicollared,

(3) Bd (X_2) has exactly two components,

(4) one component of Bd (X_2) is $D_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$,

(5) if C_2 is the other component of Bd (X_2), then $C_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$ is a homotopy equivalence,

(6) the inclusion $X_2 \times Q_{k_2+1} \subset \phi(P \times Q \times R)$ is a homotopy equivalence. Thus we have successfully operated on A_1. Using a similar procedure on B_1 we can produce our desired X and k.

In Theorem 3.2 below we establish a special case of Theorem 3.1 which will also be used in the proof of Theorem 7.1.

THEOREM 3.2.   *Let M be a PL manifold and let $h: R^n \times Q \rightarrow M \times Q$ be an open embedding, where $n \geqq 2$. Then there exists an integer $k > 0$ and a compact contractible PL submanifold X of $M \times I^k$ such that*

(1) *Bd (X) is a PL submanifold of $M \times I^k$,*

(2) *Bd (X) is PL bicollared,*

(3) $h(B_1^n \times Q) \subset \text{Int}\,(X) \times Q_{k+1} \subset X \times Q_{k+1} \subset h(B_2^n \times Q)$,

(4) *the inclusion* Bd $(X) \times Q_{k+1} \subset h((B_2^n \backslash \text{Int}\,(B_1^n)) \times Q)$ *is a homotopy equivalence.*

Proof.   Let $\phi = h \,|\, (\text{Int}\,(B_{2.5}^n) \backslash B_{.5}^n) \times Q$, which may be viewed as an open embedding of Bd $(B_1^n) \times Q \times R$ into $M \times Q$. Using Theorem 3.1 we can find an integer $k > 0$ and a compact *PL* submanifold Y of $M \times I^k$ such that Y is *PL* bicollared, $Y \times Q_{k+1} \subset h((\text{Int}\,(B_2^n) \backslash B_1^n) \times Q)$, this inclusion is a homotopy equivalence, and $Y \times Q_{k+1}$ separates $h((\text{Int}\,(B_{2.5}^n) \backslash B_{.5}^n) \times Q)$ into two components, one containing $h(\text{Bd}\,(B_1^n) \times Q)$ and the other containing $h(\text{Bd}\,(B_2^n) \times Q)$.

Clearly $h^{-1}(Y \times Q_{k+1})$ separates $R^n \times Q$ into two components, one component is bounded and contains $B_1^n \times Q$. Let C be the closure

of this component and note that C is contractible. Then $X = p_k h(C)$ fulfills our requirements.

4. **Simple homotopy type.** In this section we develop the machinery on simple homotopy type which will be needed in the proof of Theorem 7.1. The first result strengthens a theorem of West [17].

LEMMA 4.1. *Let K and L be finite connected complexes and let $f: |K| \to |L|$ be a simple homotopy equivalence. Then $f \times$ id: $|K| \times Q \to |L| \times Q$ is homotopic to a homeomorphism.*

Proof. In [17] West proved that $|K| \times Q$ is homeomorphic to $|L| \times Q$. We will modify West's argument to give our desired conclusion. Actually no new ideas are involved and our proof is just West's proof with a little more attention paid to appropriate details. Using Whitehead [18] there exist pairs $(K_0, L_0), (K_1, L_1), \cdots, (K_n, L_n)$ of finite complexes such that $K = K_0$, $L_n = L$, each $|L_{i-1}|$ is PL homeomorphic to $|K_i|$, and each L_i is a formal deformation of K_i (we use the language of [13]). For each i let us define a map $f_i: |K_i| \to |L_i|$ as follows: f_i is inclusion if $K_i \to L_i$ is a formal expansion and f_i is a retraction which is homotopic to the identity if $K_i \to L_i$ is a formal contraction. For each i let $g_{i-1}: |L_{i-1}| \to |K_i|$ be the given PL homeomorphism. Then $f: |K| \to |L|$ is homotopic to $f_n g_{n-1} f_{n-1} g_{n-2} \cdots f_1 g_0 f_0$. But in [17] West proved that each $f_i \times$ id: $|K_i| \times Q \to |L_i| \times Q$ is homotopic to a homeomorphism. This clearly gives us our desired result.

We now use this result to obtain the main result of this section: It is used in the proof of Theorem 7.1.

THEOREM 4.1. *Let K and L be finite connected complexes such that the Whitehead group $\mathrm{Wh}(\pi_1(|K|)) = 0$, let X be a compact metric space, and let $i: X \to |K| \times Q$, $j: X \to |L| \times Q$ be embeddings such that i and j are homotopy equivalences and $i(X), j(X)$ are Z-sets. Then there exists a homeomorphism $f: |K| \times Q \to |L| \times Q$ such that $fi = j$.*

Proof. Let $g: |K| \times Q \to X$ be a map such that i and g are homotopy inverses and let $h: |K| \to |K| \times Q$ be defined by $h(x) = (x, 0)$, where $0 = (0, 0, \cdots) \in Q$. If $p: |L| \times Q \to |L|$ is the projection map, then clearly $pjgh: |K| \to |L|$ is a homotopy equivalence. Since $\mathrm{Wh}(\pi_1(|K|)) = 0$ it follows then that $pjgh$ is a simple homotopy equivalence ([18], p. 43). Using Lemma 4.1 it follows that $pjgh \times$ id: $|K| \times Q \to |L| \times Q$ is homotopic to a homeomorphism $\phi: |K| \times Q \to |L| \times Q$. Then $\phi i: X \to |L| \times Q$ is an embedding such that $\phi i(X)$ is a Z-set. It is easy to see that $\phi i \cong j$. Using the Homeomorphism Extension Theorem of §2 we can therefore adjust ϕ to get our desired

homeomorphism $f: |K| \times Q \to |L| \times Q$.

5. **Immersions.** If X and Y are Q-manifolds, then we define an *immersion* to be a map $\alpha: X \to Y$ which is locally an open embedding. In Lemma 5.1 below we show that if Y is triangulated, then "large" compacta in X lie in triangulated submanifolds of X. Of course if X is compact, then this implies that X is triangulated; however our use of this result is for noncompact X. Using an argument along the lines of Theorem 8 of [4], one could actually prove that noncompact X are triangulated. We settle for the weaker version, which has a more direct proof.

LEMMA 5.1. *Let X be a Q-manifold, M be a finite-dimensional PL manifold, and let $\alpha: X \to M \times Q$ be an immersion. If $K_0 \subset G \subset K \subset X$, where K_0 and K are compact, G is open, and $\alpha \,|\, G$ is one-to-one, then there exists a finite-dimensional PL manifold N, a compact PL submanifold N_0 of N, and an embedding $\phi: N \times Q \to X$ such that*

(1) $K_0 \subset \mathrm{Int}\,(\phi(N_0 \times Q)) \subset \phi(N_0 \times Q) \subset G \subset K \subset \phi(N \times Q)$,

(2) *there exists an integer $n > 0$ and a PL submanifold Y of $M \times I^n$ such that $\alpha\phi(N_0 \times Q) = Y \times Q_{n+1}$,*

(3) $\alpha\phi \,|\, N_0 \times \{0\}: N_0 \times \{0\} \to Y \times \{0\}$ *is a PL homeomorphism,*

(4) *for any $(n, (q_i)) \in N_0 \times Q$ we have $\alpha\phi(n, (q_i)) = (y, (q_i))$, where y is chosen so that $\alpha\phi(n, 0) = (y, 0)$.*

Proof. Note that $\alpha(G)$ is open in $M \times Q$ and $\alpha(K_0)$ is a compact set in $\alpha(G)$. Thus we can find an integer $m > 0$ and compact PL submanifolds M_0', M_1' of $M \times I^m$ such that $M_0' \subset \mathrm{Int}\,(M_1')$, $M_1' \times Q_{m+1} \subset \alpha(G)$, $\alpha(K_0) \subset M_0' \times Q_{m+1}$. Since α is an immersion we can choose an open cover $\mathcal{U}$ of $X_0 = X \backslash (\alpha \,|\, G)^{-1}(M_0' \times Q_{m+1})$ such that if $U^* \in \mathrm{St}\,(\mathcal{U})$, then $\alpha \,|\, U^*$ is one-to-one, and if $U \in \mathcal{U}$ and $U \cap (\alpha \,|\, G)^{-1}(M_1' \times Q_{m+1}) \neq \varnothing$, then $U \subset G$. [We use $\mathrm{St}\,(\mathcal{U})$ to denote the set whose elements are $\cup \{V \,|\, V \in \mathcal{U},\ V \cap U \neq \varnothing\}$, where $U \in \mathcal{U}$.] We say that an open subset W of Q is a *basic open set* provided that there exists an integer $k > 0$ and an open subset W' of I^k such that $W = W' \times Q_{k+1}$ and such that W' is a product of intervals. For each $x \in X_0$ we can choose an open subset A_x of X such that

$$x \in A_x \subset U_x \in \mathcal{U}\,,$$

for some $U_x \in \mathcal{U}$, and such that $\alpha(A_x)$ is the product of an open neighborhood in M with a basic open set in Q. Choose a finite subset $\{A_i\}_{i=1}^p$ of $\{A_x\}_{x \in X_0}$ which covers $K \backslash (\alpha \,|\, G)^{-1}(\mathrm{Int}\,(M_1') \times Q_{m+1})$. Then we can find an integer $n \geqq m$ large enough so that for each i we have $\alpha(A_i) = B_i \times Q_{n+1}$, for some open subset B_i of $M \times I^n$. Put $M_1 = M_1' \times I_{m+1} \times \cdots \times I_n$ and let

$$N = \left(\overset{p}{\underset{i=1}{\cup}} (\alpha \mid A_i)^{-1}(B_i \times \{0\}) \right) \cup (\alpha \mid G)^{-1}(M_1 \times \{0\}) \,,$$

$$N_0 = (\alpha \mid G)^{-1}(M_1 \times \{0\}) \,,$$

$$Y = M_1 \,.$$

It is clear that the map $p_n \alpha \mid N : N \to M \times I^n$, which is locally an open embedding, induces a PL structure on N. Thus N is a PL manifold and N_0 is a PL submanifold of N. Let

$$Z = \left(\overset{p}{\underset{i=1}{\cup}} A_i \right) \cup (\alpha \mid G)^{-1}(M_1 \times Q_{n+1})$$

and define $A_0 = (\alpha \mid G)^{-1}(M_1 \times Q_{n+1})$. Choose $y \in A_i$, $0 \le i \le p$, and define

$$\psi(y) = ((\alpha \mid A_i)^{-1}(b, 0), q) \in N \times Q_{n+1} \,,$$

where $\alpha(y) = (b, q) \in (M \times I^n) \times Q_{n+1}$. It routinely follows that ψ is a homeomorphism of Z onto $N \times Q_{n+1}$. To check this we have to use the properties of the cover $\mathscr{U}$ which are listed at the beginning of this proof. Then $\varphi = \psi^{-1}$ clearly fulfills our requirements (if we identify Q_{n+1} with Q).

Using the covering projection $e^n : R^n \to T^n$ of §2 let $D^n = e^n([2, 3]^n)$ and $E^n = e^n(B_1^n)$. We will find the following result useful in the proof of Theorem 7.1.

LEMMA 5.2. *There exists an immersion $\alpha : (T^n \backslash D^n) \times Q \to \text{Int}(B_2^n) \times Q$ such that $\alpha(e^n \times \text{id}) \mid B_1^n \times Q : B_1^n \times Q \to B_1^n \times Q$ is the identity.*

Proof. On page 48 of [9] it is shown that there exists a map $\beta : T^n \backslash D^n \to \text{Int}(B_2^n)$ such that β is locally an open embedding and $\beta e^n \mid B_1^n : B_1^n \to B_1^n$ is the identity. Then $\alpha = \beta \times \text{id}$ fulfills our requirements.

6. Bounded homeomorphisms. A homeomorphism $h : R^n \times Q \to R^n \times Q$ is said to be *bounded* provided that the set

$$\{ \| x - y \| \mid h(x, q_1) = (y, q_2), \text{ for some } (x, q_1) \in R^n \times Q \}$$

is bounded above. In this section we describe the apparatus concerning bounded homeomorphisms which will be needed in the proof of Theorem 7.1. Our first result is just an observation due to Kirby.

LEMMA 6.1. *Let $h : T^n \times Q \to T^n \times Q$ be a homeomorphism and let $\tilde{h} : R^n \times Q \to R^n \times Q$ be a covering homeomorphism of h which fixes lattice points, i.e. $\tilde{h}((a_i), 0) = ((a_i), 0)$, for all points $(a_i) \in R^n$ such that $a_i \equiv 4 \bmod 8$. Then $\tilde{h}$ is a bounded homeomorphism.*

Proof. We just take the argument given on page 44 of [9] and multiply everything by Q.

The next simple result gives us conditions which imply the existence of bounded homeomorphisms.

LEMMA 6.2. *Using the notation $E^n = e^n(B_1^n)$ (as introduced in §5), let $h: T^n \times Q \to T^n \times Q$ be a homeomorphism such that $h \mid (T^n \backslash E^n) \times \{0\} = \mathrm{id}$, where $n \geq 2$. Then h can be covered by a bounded homeomorphism $\tilde{h}: R^n \times Q \to R^n \times Q$.*

Proof. Let $X = (T^n \backslash E^n) \times \{0\}$ and note that $\tilde{X} = (e^n \times \mathrm{id})^{-1}(X)$ is connected, since $n \geq 2$. Let $\tilde{h}: R^n \times Q \to R^n \times Q$ be the covering homeomorphism of h which satisfies $h((4, 4, \cdots, 4), 0) = ((4, 4, \cdots, 4), 0)$. The existence of $\tilde{h}$ follows from standard covering space theory. Since $h \mid X = \mathrm{id}$ it follows that $\tilde{h} \mid \tilde{X}: \tilde{X} \to \tilde{X}$ is a covering transformation which satisfies $\tilde{h}((4, 4, \cdots, 4), 0) = ((4, 4, \cdots, 4), 0)$. Thus $\tilde{h} \mid \tilde{X} = \mathrm{id}$, which implies that $\tilde{h}$ fixes lattice points. It now follows from Lemma 6.1 that $\tilde{h}$ is bounded.

We now establish the main result of the section. It is one of the key technical results which will be needed in the proof of Theorem 7.1.

THEOREM 6.1. *Let $h: T^n \times Q \to T^n \times Q$ be a homeomorphism and let $\tilde{h}: R^n \times Q \to R^n \times Q$ be a covering homeomorphism of h which is bounded. Assume that $A \subset \mathrm{Int}(E^n) \times Q$ is compact, connected, and $h(A) \subset \mathrm{Int}(E^n) \times Q$. Then there exists a homeomorphism $\hat{h}: E^n \times Q \to E^n \times Q$ such that $\hat{h} \mid \mathrm{Bd}(E^n) \times Q = \mathrm{id}$ and $\hat{h} \mid A = h \mid A$.*

Proof. Let $\tilde{A} = ((e^n \times \mathrm{id}) \mid B_1^n \times Q)^{-1}(A)$, which is a subset of $\mathrm{Int}(B_1^n) \times Q$. We will now show that there exists a covering transformation $\phi: R^n \times Q \to R^n \times Q$ such that $\phi \tilde{h}(\tilde{A}) \subset \mathrm{Int}(B_1^n) \times Q$. To see this note that

$$\alpha = ((e^n \times \mathrm{id}) \mid B_1^n \times Q)^{-1} h \mid A: A \to \mathrm{Int}(B_1^n) \times Q$$

gives a lifting of $h \mid A: A \to \mathrm{Int}(E^n) \times Q$. Choose any $a_0 \in A$ and note that $\alpha(a_0) = ((e^n \times \mathrm{id}) \mid B_1^n \times Q)^{-1} h(a_0) = \tilde{a}_0 \in \mathrm{Int}(B_1^n) \times Q$. Now

$$\beta = \tilde{h}((e^n \times \mathrm{id}) \mid B^n \times Q)^{-1} \mid A: A \to R^n \times Q$$

also gives a lifting of $h \mid A: A \to \mathrm{Int}(E^n) \times Q$. Let $\phi: R^n \times Q \to R^n \times Q$ be a covering transformation which satisfies $\phi \beta(a_0) = \tilde{a}_0$. Then we must have $\phi \beta = \alpha$, since A is connected. Thus $\phi \tilde{h}(\tilde{A}) \subset \mathrm{Int}(B_1^n) \times Q$. We also note that $\phi \tilde{h}$ is bounded. So without loss of generality we man assume that $\tilde{h}$ satisfies $\tilde{h}(\tilde{A}) \subset \mathrm{Int}(B_1^n) \times Q$. We will now construct

a homeomorphism $f: B_1^n \times Q \to B_1^n \times Q$ such that $f \mid \mathrm{Bd}\,(B_1^n) \times Q = \mathrm{id}$ and $f \mid \tilde{A} = \tilde{h} \mid \tilde{A}$. Once this is done we can get our required $\hat{h}$ by defining

$$\hat{h} = (e^n \times \mathrm{id}) f((e^n \times \mathrm{id}) \mid B_1^n \times Q)^{-1}.$$

Since $\tilde{A}$ is compact we can find $t, 0 < t < 1$, such that

$$\tilde{A} \cup \tilde{h}(\tilde{A}) \subset (-t, t)^n \times Q.$$

Let $\gamma: \mathrm{Int}\,(B_1^n) \to R^n$ be a radial expansion which satisfies $\gamma \mid B_t^n = \mathrm{id}$. Let $C(B_1^n \times Q) = (\mathrm{Int}\,(B_1^n) \times Q) \cup (\mathrm{Bd}\,(B_1^n) \times \{0\})$ and let $p: B_1^n \times Q \to C(B_1^n \times Q)$ be defined by $p \mid \mathrm{Int}\,(B_1^n) \times Q = \mathrm{id}$ and $p(x, q) = (x, 0)$ for all $(x, q) \in \mathrm{Bd}\,(B_1^n) \times Q$. Give $C(B_1^n \times Q)$ the identification topology determined by p. Since Q is known to be homeomorphic to its own cone, it follows that $C(B_1^n \times Q) \cong Q$. Define a homeomorphism $g: C(B_1^n \times Q) \to C(B_1^n \times Q)$ by

$$g = \begin{cases} \mathrm{id} & \text{on } \mathrm{Bd}\,(B_1^n) \times \{0\} \\ (\gamma \times \mathrm{id})^{-1} \tilde{h} (\gamma \times \mathrm{id}) & \text{on } \mathrm{Int}\,(B_1^n) \times Q. \end{cases}$$

It is clear that g is a homeomorphism because $\tilde{h}$ is bounded. Note that $g \mid \tilde{A} = \tilde{h} \mid \tilde{A}$. Choose a homeomorphism $\delta: B_1^n \times Q \to C(B_1^n \times Q)$ which satisfies $\delta \mid B_t^n \times Q = \mathrm{id}$ and δ is close enough to p so that $\delta(\mathrm{Bd}\,(B_1^n) \times Q) \cap (B_t^n \times Q) = \varnothing$. Then δ induces a homeomorphism $\hat{g}: B_1^n \times Q \to B_1^n \times Q$ which makes the following diagram commute.

$$
\begin{array}{ccc}
B_1^n \times Q & \xrightarrow{\hat{g}} & B_1^n \times Q \\
{\scriptstyle \delta} \downarrow & & \downarrow {\scriptstyle \delta} \\
C(B_1^n \times Q) & \xrightarrow{g} & C(B_1^n \times Q)
\end{array}
\quad .
$$

Note that $\hat{g} \mid \tilde{A} = g \mid \tilde{A} = h \mid \tilde{A}$. If δ is chosen sufficiently close to p then we have $\hat{g} \mid \mathrm{Bd}\,(B_1^n) \times Q \cong \mathrm{id}$ (in $(B_1^n \backslash B_t^n) \times Q$). Using the fact that $\mathrm{Bd}\,(B_1^n) \times Q$ is a Z-set in $B_1^n \times Q$, the Homeomorphism Extension Theorem of §2 applies to give us a homeomorphism $\theta: B_1^n \times Q \to B_1^n \times Q$ such that $\theta \mid B_t^n \times Q = \mathrm{id}$ and $\theta \mid \mathrm{Bd}\,(B_1^n) \times Q = \hat{g} \mid \mathrm{Bd}\,(B_1^n) \times Q$. Then $f = \theta^{-1} \hat{g}$ clearly fulfills our requirements.

7. **Handle straightening.** In this section we obtain our infinite-dimensional version of the handle-straightening results of Kirby-Siebenmann. The statement of the result is given in Theorem 7.1.

Let M be a finite-dimensional *PL* manifold and let $h: B_2^n \times Q \to M \times Q$ be an embedding such that $\mathrm{Bd}\,(h(B_2^n \times Q)) = h(\mathrm{Bd}\,(B_2^n) \times Q)$. Then we

say that h can be *straightened* if there exists a homeomorphism $f: B_2^n \times Q \to B_2^n \times Q$ and an integer $k > 0$ such that $f \mid \mathrm{Bd}\,(B_2^n) \times Q = \mathrm{id}$ and $hf(B_{1/2}^n \times Q) = N \times Q_{k+1}$, where N is a *PL* submanifold of $M \times I^k$ such that $\mathrm{Bd}\,(N)$ is a *PL* submanifold of $M \times I^k$ which is *PL* bicollared.

THEOREM 7.1. *Let M be a finite-dimensional PL manifold and let $h: B_2^n \times Q \to M \times Q$ be an embedding such that $\mathrm{Bd}\,(h(B_2^n \times Q)) = h(\mathrm{Bd}\,(B_2^n) \times Q)$. If $n \geqq 2$, then h can be straightened.*

Proof. We use the notation D^n and E^n of §5. Using Lemma 5.2 let $\alpha: (T^n \backslash D^n) \times Q \to \mathrm{Int}\,(B_2^n) \times Q$ be an immersion such that $\alpha(e^n \times \mathrm{id}) \mid B_1^n \times Q = \mathrm{id}$. Then it follows that $\alpha \mid E^n \times Q$ is one-to-one. Also $h\alpha: (T^n \backslash D^n) \times Q \to M \times Q$ is an immersion. Let

$$K_0 = e^n(B_{3/4}^n) \times Q \, ,$$
$$G = e^n(\mathrm{Int}\,(B_1^n)) \times Q \, ,$$
$$K = (T^n \backslash e^n((1.5, 3.5)^n)) \times Q \, .$$

Note that $K_0 \subset G \subset K$, where K_0 and K are compact and G is open. Moreover we see that $h\alpha \mid G$ is one-to-one. Using Lemma 5.1 there exists a *PL* manifold N, a compact *PL* submanifold N_0 of N, and an embedding $\phi: N \times Q \to (T^n \backslash D^n) \times Q$ such that

(1) $K_0 \subset \mathrm{Int}\,(\phi(N_0 \times Q)) \subset \phi(N_0 \times Q) \subset G \subset K \subset \phi(N \times Q)$,

(2)   there exists an integer $m > 0$ and a *PL* submanifold M_0 of $M \times I^m$ such that $h\alpha\phi(N_0 \times Q) = M_0 \times Q_{m+1}$,

(3) $h\alpha\phi \mid N_0 \times \{0\}: N_0 \times \{0\} \to M_0 \times \{0\}$ is a *PL* homeomorphism,

(4)   for any $(n, (q_i)) \in N_0 \times Q$ we have $h\alpha\phi(n, (q_i)) = (m, (q_i))$, where m is chosen so that $h\alpha\phi(n, 0) = (m, 0)$.

Note that $\phi^{-1} \mid \mathrm{Int}\,(K_0): \mathrm{Int}\,(K_0) \to \mathrm{Int}\,(N_0) \times Q$ is an open embedding. Using Theorem 3.2 there exists an integer $k_1 > 0$ and a compact contractible PL submanifold Y_1 of $\mathrm{Int}\,(N_0) \times I^{k_1}$ such that

(1) $\mathrm{Bd}\,(Y_1)$ is a *PL* submanifold of $\mathrm{Int}\,(N_0) \times I^{k_1}$ which is *PL* bicollared,

(2) $\phi^{-1}(e^n(B_{1/2}^n) \times Q) \subset \mathrm{Int}\,(Y_1) \times Q_{k_1+1} \subset Y_1 \times Q_{k_1+1}$
$$\subset \phi^{-1}(e^n(\mathrm{Int}\,(B_{2/3}^n)) \times Q) \, ,$$

(3)   the inclusion $\mathrm{Bd}\,(Y_1) \times Q_{k_1+1} \subset \phi^{-1}(e^n(B_{2/3}^n \backslash \mathrm{Int}\,(B_{1/2}^n)) \times Q)$ is a homotopy equivalence.

It follows from the construction of ϕ that $h\alpha\phi(Y_1 \times \{0\})$ is a *PL* subspace of $M \times I^{m+k_1} \times \{0\} \subset M \times Q$ (where $Y_1 \times \{0\} \subset (N_0 \times I^{k_1}) \times Q_{k_1+1}$). We will construct a homeomorphism $f: B_2^n \times Q \to B_2^n \times Q$ such that $f \mid \mathrm{Bd}\,(B_2^n) \times Q = \mathrm{id}$ and $f(B_{1/2}^n \times Q) = \alpha\phi(Y_1 \times Q_{k_1+1})$. This will fulfill our requirements. In order to construct f we will first construct a homeomorphism $\tilde{f}: E^n \times Q \to E^n \times Q$ such that $\tilde{f} \mid \mathrm{Bd}\,(E^n) \times Q = \mathrm{id}$ and $\tilde{f}(e^n(B_{1/2}^n) \times Q) = \phi(Y_1 \times Q_{k_1+1})$. Once we have $\tilde{f}$ we can define

f by

$$f = \begin{cases} \mathrm{id} & \text{on } (B_2^n \backslash B_1^n) \times Q \\ \alpha \tilde{f}(\alpha \mid E^n)^{-1} & \text{on } B_1^n \times Q \,. \end{cases}$$

It is clear that $\phi^{-1} \mid e^n((1.3, 3.7)^n \backslash [1.4, 3.6]^n) \times Q$ gives an open embedding of $e^n((1.3, 3.7)^n \backslash [1.4, 3.6]^n) \times Q$ into $N \times Q$. Using Theorem 3.1 we can find an integer $k_2 \geqq k_1$ and a compact PL submanifold Y_2 of $N \times I^{k_2}$ such that

(1) Y_2 is PL bicollared,
(2) $Y_2 \times Q_{k_2+1} \subset \phi^{-1}(e^n((1.3, 3.7)^n \backslash [1.4, 3.6]^n) \times Q)$,
(3) the above inclusion is a homotopy equivalence,
(4) $Y_2 \times Q_{k_2+1}$ separates $\phi^{-1}(e^n([1.3, 3.7]^n \backslash (1.4, 3.6)^n) \times Q)$ into two components, one containing $\phi^{-1}(e^n(\mathrm{Bd}\,([1.3, 3.7]^n)) \times Q)$ and the other containing $\phi^{-1}(e^n(\mathrm{Bd}\,([1.4, 3.6]^n)) \times Q)$.

It is easily seen that $\varphi(Y_2 \times Q_{k_2+1})$ separates $T^n \times Q$ into two components A and B, where $\mathrm{Cl}\,(A)$ (the closure of A) contains $e^n([1.4, 3.6]^n) \times Q$ and $\mathrm{Cl}\,(B)$ contains $(T^n \times Q) \backslash (e^n((1.3, 3.7)^n) \times Q)$. Let A', B' denote the closures of the intersections of A, B with $e^n([1.3, 3.7]^n \backslash (1.4, 3.6)^n) \times Q$, respectively. Then $\varphi^{-1}(A')$ and $\varphi^{-1}(B')$ are the closures of the components of (4) above. Recalling the comments concerning ANR's made at the end of the proof of Lemma 3.5, there exists a strong deformation retraction of $A' \cup B'$ onto $\varphi(Y_2 \times Q_{k_2+1})$; let $D_t \colon A' \cup B' \to A' \cup B'$, $0 \leqq t \leqq 1$, be a homotopy which gives such a strong deformation retraction, where $D_0 = \mathrm{id}$ and D_1 is a retraction of $A' \cup B'$ onto $\varphi(Y_2 \times Q_{k_2+1})$. Let $r \colon A' \cup B' \to B'$ be the retraction defined by $r \mid B' = \mathrm{id}$ and $r \mid A' = D_1 \mid A'$. Then define a homotopy $D_t' \colon B' \to B'$ by $D_t' = rD_t \mid B'$, for $0 \leqq t \leqq 1$. It is clear that D_t' defines a strong deformation retraction of B' onto $\varphi(Y_2 \times Q_{k_2+1})$. Using D_t' it follows that $\mathrm{Cl}\,(A)$ is contractible, and it is clearly a Q-manifold. But it is known that all compact, contractible Q-manifolds are homeomorphic to Q [4]. Thus $\mathrm{Cl}\,(A) \cong Q$. [One is tempted to apply Lemma 4.1 here, but it will not work because it is not known that $\mathrm{Cl}\,(A)$ is triangulated.] We can easily use similar tricks to show that the inclusion

$$(T^n \times Q) \backslash (e^n((1.3, 3.7)^n) \times Q) \subset \mathrm{Cl}\,(B)$$

is a homotopy equivalence.

Let $Y_1' = Y_1 \times I_{k_1+1} \times \cdots \times I_{k_2}$ and let $Y_2' = p_{k_2} \phi^{-1}(\mathrm{Cl}\,(A))$, where $p_{k_2} \colon N \times Q \to N \times I^{k_2}$ is the projection mapping. Put $R = (N \times I^{k_2}) \backslash (\mathrm{Int}\,(Y_1') \cup \mathrm{Int}\,(Y_2'))$. Then R is a compact PL submanifold of $N \times I^{k_2}$ and $\mathrm{Bd}\,(R) = \mathrm{Bd}\,(Y_2') \cup \mathrm{Bd}\,(Y_2')$. It follows (as above) that if

$$C = [T^n\backslash e^n((1.3, 3.7)^n \cup \text{Int}(B_{2/3}^n))] \times \{0\} \subset \phi(R \times Q_{k_2+1}) \ ,$$

then the inclusion $i\colon C \subset \phi(R \times Q_{k_2+1})$ is a homotopy equivalence. Let

$$j\colon C \subset [T^n\backslash(\text{Int}(D^n) \cup e^n(\text{Int}(B_{1/2}^n)))] \times Q$$

denote inclusion, which is also a homotopy equivalence.

If $n \geq 3$, then $\pi_1(C) = Z \oplus Z \oplus \cdots \oplus Z(n$ copies of $Z)$. If $n = 2$, then it easily follows that $\pi_1(C) = Z*Z*Z$ (where $*$ denotes free product). In any case we have $\text{Wh}(\pi_1(C)) = 0$ (see [13], page 373, for references). Using Theorem 4.1 we can find a homeomorphism

$$\gamma'\colon [T^n\backslash(\text{Int}(D^n) \cup e^n(\text{Int}(B_{1/2}^n)))] \times Q \to \phi(R \times Q_{k_2+1})$$

such that $\gamma' \mid C = \text{id}$. Using the Homeomorphism Extension Theorem of §2 we can adjust γ' to get a homeomorphism

$$\gamma\colon [T^n\backslash(\text{Int}(D^n) \cup e^n(\text{Int}(B_{1/2}^n)))] \times Q \to \phi(R \times Q_{k_2+1})$$

which satisfies

$$\gamma \mid C = \text{id}, \gamma(\text{Bd}(D^n) \times Q) = \phi(Y_2 \times Q_{k_2+1}) \ ,$$

and

$$\gamma(\text{Bd}(e^n(B_{1/2}^n)) \times Q) = \phi(\text{Bd}(Y_1') \times Q_{k_2+1}) \ .$$

Note that $T^n \times Q = \phi(R \times Q_{k_2+1}) \cup \text{Cl}(A) \cup \phi(Y_1' \times Q_{k_2+1})$, where $\text{Cl}(A)$ has already been observed to be a Hilbert cube and $\varphi(Y_1' \times Q_{k_2+1})$ is a Hilbert cube by Lemma 4.1. By applying the Homeomorphism Extension Theorem again to

$$\gamma \mid \text{Bd}(D^n) \times Q\colon \text{Bd}(D^n) \times Q \to \text{Bd}(\text{Cl}(A))$$

and

$$\gamma \mid \text{Bd}(e^n(B_{1/2}^n)) \times Q\colon \text{Bd}(e^n(B_{1/2}^n)) \times Q \to \phi(\text{Bd}(Y_1') \times Q_{k_2+1}) \ ,$$

it easily follows that γ can be extended to a homeomorphism

$$\delta\colon T^n \times Q \to T^n \times Q \ .$$

Using Lemma 6.2 it follows that δ can be covered by a bounded homeomorphism $\tilde{\delta}\colon R^n \times Q \to R^n \times Q$. Using Theorem 6.1 the existence of our required $\tilde{f}$ now follows.

8. *Proof of the theorem.* We are given a triangulated Q-manifold X and an open embedding $h\colon R^n \times Q \to X$, where $n \geq 2$. We want to prove that $X\backslash h(\text{Int}(B_1^n) \times Q)$ is a triangulated Q-manifold.

Let $\phi\colon X \to |K| \times Q$ be a homeomorphism, for some countable

locally-finite simplicial complex K. Then there exists a finite sub-complex K_0 of K such that $\phi h(B_2^n \times Q) \subset \text{Int}(|K_0| \times Q)$ and $\text{Bd}(|K_0|)$ is a PL subspace of $|K|$ which is PL bicollared. Let us regard $|K_0|$ as a PL subspace of some I^n (where $I^n \cap |K| = |K_0|$), with $\text{Bd}(|K_0|) \subset \text{Bd}(I^n)$ and $\text{Int}(|K_0|) \subset \text{Int}(I^n)$. Choose a regular neighborhood M of $|K_0|$ in I^n which meets $\text{Bd}(I^n)$ regularly. Then we get a new com-plex $|L| = M \cup (|K| \backslash |K_0|)$, which is a countable locally-finite simplicial complex containing M as a PL subspace.

It follows from [17] that there exists a homeomorphism $\psi \colon |L| \times Q \to |K| \times Q$ such that ψ is the identity on $(|L| \backslash M) \times Q$. Then $\psi^{-1}\phi \colon X \to |L| \times Q$ is a homeomorphism which satisfies

$$\psi^{-1}\phi h(B_2^n \times Q) \subset \text{Int}(M) \times Q .$$

Since M is a PL manifold our result now follows from Theorem 7.1.

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