

Pacific Journal of Mathematics

ON THE FRACTIONAL PARTS OF A SET OF POINTS. II

ROGER COOK

ON THE FRACTIONAL PARTS OF A SET OF POINTS II

R. J. COOK

Heilbronn proved that for any $\varepsilon > 0$ there exists a number $C(\varepsilon)$ such that for any real numbers θ and $N \geq 1$ there is an integer n such that

$$1 \leq n \leq N \quad \text{and} \quad \|n^2\theta\| < C(\varepsilon)N^{-1/2+\varepsilon}$$

where $\|\alpha\|$ denotes the difference between α and the nearest integer, taken positively. The method depends on Weyl's estimates for trigonometric sums. The result was generalized by Davenport who obtained analogous results for polynomials which have no constant term.

The object here is to obtain a result for simultaneous approximations to quadratic polynomials f_1, \dots, f_R having no constant term:

For any $\varepsilon > 0$ there is a number $C = C(\varepsilon, R)$ such that for any $N \geq 1$ there is an integer n such that

$$1 \leq n \leq N \quad \text{and} \quad \|f_i(n)\| < CN^{-1/g(R)+\varepsilon}$$

for $i = 1, \dots, R,$

where $g(1) = 3$ and $g(R) = 4g(R-1) + 4R + 2$ for $R \geq 2$.

1. Introduction. In 1948 Heilbronn [4] proved the result stated above on the distribution of the sequence $n^2\theta \pmod{1}$. This was generalized to polynomials which have no constant term by Davenport [2].

THEOREM. *Let $\varepsilon > 0$ and let R be a positive integer. Then there is a number $C = C(\varepsilon, R)$ such that for any quadratic polynomials f_1, \dots, f_R having no constant term, and for any $N \geq 1$, there is an integer n such that*

$$(1) \quad 1 \leq n \leq N \quad \text{and} \quad \|f_i(n)\| < CN^{-1/g(R)+\varepsilon}$$

for $i = 1, \dots, R,$

where

$$(2) \quad g(1) = 3 \quad \text{and} \quad g(R) = 4g(R-1) + 4R + 2 \quad \text{for} \quad R \geq 2,$$

the result being uniform in f_1, \dots, f_R .

It can be readily verified by induction that an explicit formula for $g(R)$ is

$$(3) \quad 18g(R) = 29 \cdot 4^R - 24R - 20, \quad \text{for} \quad R \geq 2.$$

2. **Preliminaries to the proof.** The case $R = 1$ was proved by Davenport [2]. The theorem will be proved by induction on R , so we suppose the theorem is true for $R - 1$. ε denotes a small positive number and $r(\varepsilon)$ denotes a multiple of ε depending only on R , note that $r(\varepsilon)$ differs in its various occurrences. We may suppose that $N > N_0(\varepsilon, R)$. $F \ll G$ means that $|F| < CG$ where C depends at most on ε and R . $e(z) = \exp(2\pi iz)$.

LEMMA 1 (Vinogradov). *Let Δ satisfy $0 < \Delta < 1/2$ and let a be a positive integer. Then there exists a function $\psi(z)$, periodic with period 1, which satisfies*

$$(4) \quad \psi(z) = 0 \quad \text{for } \|z\| > \Delta$$

and

$$\psi(z) = \sum_{m=-\infty}^{\infty} a_m e(mz)$$

where the a_m are real numbers, $a_0 = \Delta$, $a_m = a_{-m}$, $m = 1, 2, \dots$, and

$$(5) \quad |a_m| < A \min(\Delta, m^{-a-1} \Delta^{-a}), \quad m \neq 0,$$

where A depends only on a .

Proof. This is a particular case of Lemma 12 of Chapter 1 of Vinogradov [5].

LEMMA 2 (Weyl). *Let A and P be real numbers, $P \geq 1$. Let $\alpha = aq^{-1} + \beta$ where $(a, q) = 1$, $q \geq 1$ and $|\beta| \leq q^{-2}$. Then*

$$(6) \quad \left| \sum_{A \leq n \leq A+P} e(\alpha n^2 + \alpha_1 n) \right|^2 \ll P^2 (q^{-1}P + 1)(P + q \log q).$$

Proof. See, for example, Lemma 1 of Davenport [1].

Let

$$(7) \quad f_i(n) = \theta_i n^2 + \phi_i n, \quad i = 1, \dots, R.$$

We choose a positive number δ so that there is no integer n with

$$(8) \quad 1 \leq n \leq N \quad \text{and} \quad \|f_i(n)\| \leq N^{-\delta}, \quad i = 1, \dots, R.$$

We may suppose that $\delta < 1/g(R)$. We take $\Delta = N^{-\delta}$ and $a = [2\varepsilon^{-1}] + 1$ in Lemma 1. Then

$$\sum_{n=1}^N \prod_{i=1}^R \psi(f_i(n)) = 0$$

so

$$N^{1-R\delta} + \sum^* a_{m_1} \dots a_{m_R} T(m) = 0$$

where Σ^* denotes a summation over $-\infty < m_1 < \infty, \dots, -\infty < m_R < \infty$, $\mathbf{m} = (m_1, \dots, m_R) \neq \mathbf{0}$,

$$(9) \quad T(\mathbf{m}) = \sum_{n=1}^N e(\mathbf{m} \cdot \boldsymbol{\theta} n^2 + \mathbf{m} \cdot \boldsymbol{\phi} n),$$

$$(10) \quad \mathbf{m} \cdot \boldsymbol{\theta} = \sum_{i=1}^R m_i \theta_i \quad \text{and} \quad \mathbf{m} \cdot \boldsymbol{\phi} = \sum_{i=1}^R m_i \phi_i.$$

Summing over terms in the region $|m_1| > N^{\delta+\varepsilon}$ we have

$$\begin{aligned} \sum |a_{m_1} \cdots a_{m_R} T(\mathbf{m})| &\ll N \sum N^{a\delta} m_1^{-a-1} \\ &\ll N^{1-a\varepsilon} \end{aligned}$$

by Lemma 1, and similarly for other regions $|m_i| > N^{\delta+\varepsilon}$. Thus

$$(11) \quad \begin{aligned} 1 &\ll N^{-1+R\delta} \Sigma' |a_{m_1} \cdots a_{m_R} T(\mathbf{m})| \\ &\ll N^{-1} \Sigma' |T(\mathbf{m})| \end{aligned}$$

where Σ' denotes a summation over $\max |m_i| \leq N^{\delta+\varepsilon}$, $\mathbf{m} \neq \mathbf{0}$. Taking the square of this inequality and applying Cauchy's inequality we have

$$(12) \quad 1 \ll N^{-2+R\delta+R\varepsilon} S$$

where

$$(13) \quad S = \Sigma' |T(\mathbf{m})|^2.$$

We now proceed to estimate S . Let $Q = N^A$, $T = N^B$ where A and B will be chosen later. By Dirichlet's theorem on Diophantine approximation, see Theorem 185 of Hardy and Wright [3], for each \mathbf{m} there exist integers a, b, q and t such that

$$(14) \quad \mathbf{m} \cdot \boldsymbol{\theta} = aq^{-1} + \alpha \quad \text{with} \quad (a, q) = 1, \quad 1 \leq q \leq Q, \quad q|\alpha| \leq Q^{-1}$$

$$(15) \quad \mathbf{m} \cdot \boldsymbol{\phi} = bt^{-1} + \beta \quad \text{with} \quad (b, t) = 1, \quad 1 \leq t \leq T, \quad t|\beta| \leq T^{-1}.$$

3. The induction step. For any \mathbf{m} in the sum for S we have

$$(16) \quad \max |m_i| \leq N^{\delta+\varepsilon}.$$

Since $\mathbf{m} \neq \mathbf{0}$ and $|T(-\mathbf{m})| = |T(\mathbf{m})|$ we may suppose that $m_R > 0$.

We take

$$(17) \quad \sigma = 2g(R-1)\delta + 4g(R-1)\varepsilon,$$

$$(18) \quad A = \frac{3}{2} + (2g(R-1) + 1)\delta + (4g(R-1) + 3)\varepsilon,$$

and

$$(19) \quad B = \frac{1}{2} + 2\varepsilon .$$

Applying the case $R - 1$ of the theorem to the polynomials

$$(20) \quad f_i^*(n) = m_R q^2 t^2 \theta_i n^2 + q t \phi_i n, \quad i = 1, \dots, R - 1 ,$$

we see that there is an integer x such that

$$(21) \quad 1 \leq x \leq N^\sigma \quad \text{and} \quad \|f_i^*(x)\| \ll N^{-\sigma/g(R-1)+\varepsilon} , \\ i = 1, \dots, R - 1 .$$

Suppose that $q < N^{1/2-\sigma-\delta-4\varepsilon}$. Taking $y = m_R q t x$ we have $1 \leq y \leq N$ and for $i = 1, \dots, R - 1$

$$(22) \quad \|f_i(y)\| = \|m_R^2 q^2 t^2 \theta_i x^2 + m_R q t \phi_i x\| \\ \leq |m_R| \|f_i^*(x)\| \leq N^{-\delta} ,$$

by (16), (17), and (21). Also

$$(23) \quad \|f_R(y)\| = \|m_R^2 q^2 t^2 \theta_R x^2 + m_R q t \phi_R x\| \\ \leq \|m_R q^2 t^2 x^2 \mathbf{m} \cdot \boldsymbol{\theta}\| + \|\sum m_i (m_R q^2 t^2 \theta_i x^2 + q t \phi_i x)\| \\ + \|\sum m_i q t \phi_i x + m_R q t \phi_R x\| \\ \leq |m_R q t^2 x^2| \|q \mathbf{m} \cdot \boldsymbol{\theta}\| + \sum |m_i| \|f_i^*(x)\| \\ + |x q| \|t \mathbf{m} \cdot \boldsymbol{\phi}\| \\ \leq N^{-\delta} ,$$

by (14) – (21), where the summations are over $i = 1, \dots, R - 1$.

This contradicts the assumption that there were no integer solutions of (8). Therefore $q \geq N^{1/2-\sigma-\delta-4\varepsilon}$.

4. Completion of the proof of the theorem. From (6) we have

$$(24) \quad |T(\mathbf{m})|^2 \ll q^{-1} N^{2+\varepsilon} + q N^\varepsilon + N^{1+\varepsilon} .$$

For $N^{1/2-\sigma-\delta-4\varepsilon} \leq q \leq N$ we have

$$(25) \quad |T(\mathbf{m})|^2 \ll q^{-1} N^{2+\varepsilon} \ll N^{1+1/2+(2g(R-1)+1)\delta+r(\varepsilon)} .$$

Summing over $O(N^{R(\delta+\varepsilon)})$ such \mathbf{m} we have a contribution S_1 to S where

$$(26) \quad S_1 \ll N^{1+1/2+(2g(R-1)+R+1)\delta+r(\varepsilon)} .$$

For $N \leq q \leq M = N^4$ we have

$$(27) \quad |T(\mathbf{m})|^2 \ll q N^\varepsilon \ll N^{1+1/2+(2g(R-1)+1)\delta+r(\varepsilon)} .$$

Summing over $O(N^{R(\delta+\varepsilon)})$ such \mathbf{m} we have a contribution S_2 to S where

$$(28) \quad S_2 \ll N^{1+1/2+(2g(R-1)+R+1)\delta+r(\varepsilon)} .$$

Therefore, from (12), we have

$$(29) \quad 1 \ll N^{-1/2+(2g(R-1)+2R+1)\delta+r(\varepsilon)} .$$

Hence

$$-\varepsilon < -\frac{1}{2} + (2g(R-1) + 2R + 1)\delta + r(\varepsilon)$$

so

$$(30) \quad \delta > 1/g(R) - r(\varepsilon)$$

and the theorem is proved.

REFERENCES

1. H. Davenport, *Analytic Methods for Diophantine Equations and Diophantine Inequalities*, Ann Arbor, Michigan, 1962.
2. ———, *On a theorem of Heilbronn*, Quart. J. Math. Oxford, (2), **18** (1967), 339-344.
3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford, 1965.
4. H. Heilbronn, *On the distribution of the sequence $n^2\theta \pmod{1}$* , Quart. J. Math. Oxford, **19** (1948), 249-256.
5. I. M. Vinogradov, *The Method of Trigonometrical Sums in the Theory of Numbers*, (Translated by K. F. Roth and Anne Davenport), Interscience Publishers, 1954.

Received December 14, 1971.

UNIVERSITY COLLEGE, CARDIFF

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

William George Bade, <i>Complementation problems for the Baire classes</i>	1
Ian Douglas Brown, <i>Representation of finitely generated nilpotent groups</i>	13
Hans-Heinrich Brungs, <i>Left Euclidean rings</i>	27
Victor P. Camillo and John Cozzens, <i>A theorem on Noetherian hereditary rings</i>	35
James Cecil Cantrell, <i>Codimension one embeddings of manifolds with locally flat triangulations</i>	43
L. Carlitz, <i>Enumeration of up-down permutations by number of rises</i>	49
Thomas Ashland Chapman, <i>Surgery and handle straightening in Hilbert cube manifolds</i>	59
Roger Cook, <i>On the fractional parts of a set of points. II</i>	81
Samuel Harry Cox, Jr., <i>Commutative endomorphism rings</i>	87
Michael A. Engber, <i>A criterion for divisoriality</i>	93
Carl Clifton Faith, <i>When are proper cyclics injective</i>	97
David Finkel, <i>Local control and factorization of the focal subgroup</i>	113
Theodore William Gamelin and John Brady Garnett, <i>Bounded approximation by rational functions</i>	129
Kazimierz Goebel, <i>On the minimal displacement of points under Lipschitzian mappings</i>	151
Frederick Paul Greenleaf and Martin Allen Moskowitz, <i>Cyclic vectors for representations associated with positive definite measures: nonseparable groups</i>	165
Thomas Guy Hallam and Nelson Onuchic, <i>Asymptotic relations between perturbed linear systems of ordinary differential equations</i>	187
David Kent Harrison and Hoyt D. Warner, <i>Infinite primes of fields and completions</i>	201
James Michael Hornell, <i>Divisorial complete intersections</i>	217
Jan W. Jaworowski, <i>Equivariant extensions of maps</i>	229
John Jobe, <i>Dendrites, dimension, and the inverse arc function</i>	245
Gerald William Johnson and David Lee Skoug, <i>Feynman integrals of non-factorable finite-dimensional functionals</i>	257
Dong S. Kim, <i>A boundary for the algebras of bounded holomorphic functions</i>	269
Abel Klein, <i>Renormalized products of the generalized free field and its derivatives</i> ...	275
Joseph Michael Lambert, <i>Simultaneous approximation and interpolation in L_1 and $C(T)$</i>	293
Kelly Denis McKennon, <i>Multipliers of type (p, p) and multipliers of the group L_p-algebras</i>	297
William Charles Nemitz and Thomas Paul Whaley, <i>Varieties of implicative semi-lattices. II</i>	303
Donald Steven Passman, <i>Some isolated subsets of infinite solvable groups</i>	313
Norma Mary Piacun and Li Pi Su, <i>Wallman compactifications on E-completely regular spaces</i>	321
Jack Ray Porter and Charles I. Votaw, <i>$S(\alpha)$ spaces and regular Hausdorff extensions</i>	327
Gary Sampson, <i>Two-sided L_p estimates of convolution transforms</i>	347
Ralph Edwin Showalter, <i>Equations with operators forming a right angle</i>	357
Raymond Earl Smithson, <i>Fixed points in partially ordered sets</i>	363
Victor Snaith and John James Ucci, <i>Three remarks on symmetric products and symmetric maps</i>	369
Thomas Rolf Turner, <i>Double commutants of weighted shifts</i>	379
George Kenneth Williams, <i>Mappings and decompositions</i>	387