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**COMMUTATIVE ENDOMORPHISM RINGS**

SAMUEL HARRY COX, JR.

## COMMUTATIVE ENDOMORPHISM RINGS

S. H. COX, JR.

Two problems of W. V. Vasconcelos are partially solved:  
 (1) The total quotient ring of a commutative noetherian ring  $R$  is quasi-frobenius if and only if  $\text{End}_R(A)$  is commutative for each ideal  $A$  of  $R$ . (2) Let  $R$  be a commutative quasi-regular ring and  $E$  a finitely presented  $R$ -module. If  $E$  is faithful and  $\text{End}_R(E)$  is semi-prime, then  $E$  is isomorphic to an ideal of  $R$ . Only commutative rings with unit and unital modules are considered.

1. In [3] Vasconcelos considers problems concerning commutative endomorphism rings. Toward the end he asks for a characterization of rings  $R$  for which  $\text{End}(A)$  is commutative for each ideal  $A \subset R$ . He conjectures the following answer for noetherian rings.

**THEOREM 1.1.** *Let  $R$  be a noetherian ring with total quotient ring  $T$ . If  $\text{End}_R(A)$  is commutative for each ideal  $A$  of  $R$  then  $T$  is quasi-frobenius.*

*Proof.* It is sufficient to show that for each maximal ideal  $p$  of  $T$  the local ring  $S = T_p$  has Krull dimension zero and  $\text{Ann}_S(q)$  is a one dimensional  $S/q$ -vector space where  $q = pS$  [2, Theorem 221]. Each ideal  $I$  of  $S$  has a commutative endomorphism ring since we can select  $J \subset R$  such that  $J \otimes S = I$  and observe that the natural map  $\text{End}_R(J) \otimes_R S \rightarrow \text{End}_S(J \otimes_R S)$  is a ring isomorphism [1, p. 39, Proposition 11]. Also because  $R$  is noetherian  $q = \text{Ann}_S(a)$  for some  $a \in S$  [2, Theorem 82]. Since  $\text{End}_{S/q}(\text{Ann}_S(q)) = \text{End}_S(\text{Ann}_S(q))$  is commutative and  $\text{Ann}_S(q) \neq 0$ , then  $\text{Ann}_S(q)$  is a one dimensional  $S/q$ -vector space. It remains to show  $S$  is zero dimensional, i.e.,  $q$  is nilpotent. Since  $a \neq 0$ , then there is an integer  $n$  such that  $a \notin q^n$  [2, Theorem 79]. Since  $\text{Ann}_S(q)$  is simple we must have  $Sa = \text{Ann}_S(q)$  and  $q^n \cap Sa = 0$ . Now we show  $q^n = 0$ . Suppose not, choose  $b \in q^n, b \neq 0$ . Then  $\text{Ann}_S(b) \subset q = \text{Ann}_S(a)$ . Thus the correspondence  $xb \rightarrow xa$  defines an  $S$ -homomorphism  $f: Sb \rightarrow Sa$ . Let  $J = Sa + Sb$ . This sum is direct since  $Sa \cap Sb \subset Sa \cap q^n = 0$ . Let  $u, w \in \text{End}_S(J)$  be the following composites:

$$u: J \longrightarrow Sa \subset J$$

$$w: J \longrightarrow Sb \xrightarrow{f} Sa \subset J.$$

Then  $uw(b) = a$  and  $wu(b) = 0$  contradicting the commutativity of  $\text{End}_S(J)$ . Thus  $q^n = 0$ , and  $S$  is zero dimensional.

The converse to 1.1 is true also. This is because  $T$  is an injective  $T$ -module if  $T$  is quasi-frobenius. Indeed if  $R$  is any commutative ring whose total quotient ring  $T$  is  $T$ -injective then for  $A \subset R$ ,  $\text{End}_R(A)$  can be viewed as a subring of  $\text{End}_T(AT)$  which is a homomorphic image of  $T$  and therefore commutative.

The next proposition gives a sufficient condition on  $A \subset R$  for  $\text{End}(A)$  to be commutative.

**PROPOSITION 1.2.** *Let  $R$  be a commutative ring and  $A$  an ideal of  $R$ . If  $A \cap \text{Ann}(A) = 0$ , then  $\text{End}_R(A)$  is commutative.*

*Proof.* Let  $f, g \in \text{End}_R(A)$  and  $c = fg - gf$ . For  $a, b \in A$  we have  $ac(b) = c(ab) = f(g(ab)) - g(f(ab)) = f(ag(b)) - g(bf(a)) = g(b)f(a) - f(a)g(b) = 0$ . Hence  $Ac(A) = 0$  implies  $c(A) \subset A \cap \text{Ann}(A) = 0$ . Therefore  $\text{End}(A)$  is commutative.

An  $R$ -algebra will be called semi-prime if it has no non-zero nilpotent elements.

**COROLLARY 1.3.** *If  $R$  is semi-prime then  $\text{End}(A)$  is commutative and semi-prime for each ideal  $A \subset R$ .*

*Proof.*  $A \cap \text{Ann}(A)$  consists of nilpotents so  $\text{End}(A)$  is commutative by 1.2. If  $f \in \text{End}(A)$  is nilpotent, say  $f^n = 0$ , then for  $x \in A$   $0 = f^n(x^n) = (f(x))^n$ . Since  $R$  is semi-prime  $f(x) = 0$  for  $x \in A$ . Thus  $f = 0$ .

If  $R$  is an integral domain, we can characterize the ideals of  $R$  as those torsionless  $R$ -modules  $E$  having  $\text{End}(E)$  commutative. For if  $x \in E$   $x \neq 0$  there is  $f: E \rightarrow R$  with  $f(x) \neq 0$ . Let  $y \in E$ . The two homomorphisms  $z \rightarrow f(z)x$  and  $z \rightarrow f(z)y$  commute. Hence  $f(y)x = f(x)y$ , so  $f(y) = 0$  implies  $y = 0$ . Thus  $f$  is injective.

The next section is concerned with how well the property  $\text{End}(A)$  is semi-prime distinguishes the ideals of a semi-prime ring  $R$  from other  $R$ -modules.

2. In [3] Vasconcelos proves that when  $R$  is noetherian and semi-prime a finitely generated faithful  $R$ -module  $E$  with  $\text{End}_R(E)$  commutative and semi-prime is isomorphic to an ideal of  $R$ . He conjectures that the result may remain valid for a finitely presented  $E$  even if  $R$  is not noetherian. I could not resolve this but generalize his result to include those rings having an absolutely flat total quotient ring (called quasi-regular rings). The methods make no use of the commutativity of  $\text{End}(E)$ . Thus in the situation considered (in 2.2 below) semi-prime implies commutativity. Although we are con-

sidering only commutative rings here, our generalization, unlike the original version of the theorem, can at least be conjectured for non-commutative rings.

This is the first step in the proof:

**THEOREM 2.1.** *Let  $R$  be a ring and  $E$  a finitely present  $R$ -module. If  $x \in E$  is nonzero, then there exists  $f \in \text{End}(E)$  nonzero such that  $f(E) \subset Rx$ .*

*Proof.* First suppose  $R$  is noetherian. Let  $p$  be a prime minimal over  $\text{Ann}(x)$ . Then there exists  $y \in Rx$  such that  $p = \text{Ann}(y)$  [2, Theorem 86]. Localize at  $p$ . Let  $K = R_p/p_p$ . Since  $E_p \neq 0$ ,  $E_p \otimes K$  is at least one dimensional by the Nakayama lemma [2, Theorem 78]. Thus there is a surjection  $h: E_p \otimes K \rightarrow K$ . As an  $R_p$ -module,  $K = (Ry)_p$ . Let  $g$  be the composite  $E_p \rightarrow E_p \otimes K \xrightarrow{h} (Ry)_p \subset (Rx)_p$ . Since  $E$  is finitely presented, we have  $\text{Hom}_R(E, Rx)_p \cong \text{Hom}_{R_p}(E_p, Rx_p)$  [1, p. 39, Proposition 11]. Hence  $g = f/s$  for some  $f: E \rightarrow Rx$  and  $s \in R \setminus p$ . Clearly,  $f$  has the required properties. Thus the result holds when  $R$  is noetherian. Since  $E$  is finitely presented we can use the following well known technique to reduce to the noetherian case: Let  $R^m \xrightarrow{A} R^n \xrightarrow{B} E \rightarrow 0$  be a presentation of  $E$ , select bases  $f_i, e_j$ : let  $A(f_i) = \sum a_{ij}e_j$ ,  $B(e_j) = m_j$ ,  $x = \sum x_j m_j$  with  $a_{ij}, x_j \in R$ . Let  $S$  be the subring generated by 1 and all the  $x$ 's and  $a$ 's.  $S$  is noetherian by the Hilbert Basis Theorem. Let  $F$  be the  $S$ -submodule generated by the  $m$ 's. Then  $F \otimes_S R = E$  and  $x \in F$ . Since  $S$  is noetherian there is nonzero  $g: F \rightarrow Sx$ . Tensoring with  $S$  yields a commutative diagram:

$$\begin{array}{ccccc}
 E & \longrightarrow & Sx \otimes R & \longrightarrow & Rx \\
 \cup & & \uparrow & & \parallel \\
 F & \xrightarrow{g} & Sx & \subset & Rx
 \end{array}$$

Hence we can take  $f$  to be the composite of the maps on the upper row.

For an ideal  $I$  of  $R$  let  $\text{Min}(I)$  denote the primes of  $R$  minimal over  $I$ . For an  $R$ -module  $E$  let  $\text{Ass}(E)$  denote the Bourbaki associated primes of  $E$ . Thus  $\text{Ass}(E)$  is the union over  $x \in E$  of the sets  $\text{Min}(\text{Ann}(x))$ .

**THEOREM 2.2.** *Let  $R$  be a semi-prime ring,  $E$  a finitely presented  $R$ -module. If  $\text{End}(E)$  is semi-prime, then  $\text{End}(E)$  is commutative and  $\text{Ass}(E) = \text{Min}(\text{Ann}(E))$ .*

*Proof.* For any finitely presented  $R$ -module  $E$   $\text{Min}(\text{Ann}_R(E)) \subset \text{Ass}(E)$  and the mapping  $\text{End}(E) \rightarrow \prod_{p \in \text{Ass}(E)} \text{End}_{R_p}(E_p)$  induced by

$\text{End}(E) \rightarrow \text{End}_{R_p}(E_p)$  is an injective ring homomorphism. Thus it is sufficient to establish that if  $p \in \text{Min}(\text{Ann}(x))$  and  $x \in E$ , then  $E_p = (Rx)_p$ . For then  $[\text{End}_R(E)]_p \cong \text{End}_{R_p}(E_p)$  is commutative and  $\text{Ann}_{R_p}(Rx_p) = \text{Ann}_{R_p}(E_p) = [\text{Ann}_R(E)]_p$ . Thus by relationship between primes of  $R_p$  and primes of  $R$  contained in  $p$  we get  $p \in \text{Min}(\text{Ann}(E))$ . So let  $p \in \text{Min}(\text{Ann}(x))$ ,  $x \in E$ . Put  $T = R_p$ ,  $q = pT$ ,  $F = E_p$ ,  $y = x/1 \in F$ .  $T$  is quasi-local semi-prime with maximal ideal  $q = \sqrt{\text{Ann}_T(y)}$ . By 2.1 there is nonzero  $f: F \rightarrow F$  with  $f(F) \subset Ty$ . Let  $f(y) = ay$ . Then  $a \notin q$  else  $f$  is nilpotent and consequently zero since  $\text{End}_T(F)$  is semi-prime. Let  $b \in q$  and define  $h = ba^{-1}f$ . Then  $h$  is nilpotent since  $b \in \sqrt{\text{Ann}(y)}$  and  $h(F) \subset Ty$ . Thus  $h = 0$ . Hence  $0 = h(y) = by$ . Therefore  $q = \text{Ann}_T(y)$ . Let  $M = F/Ty$  and suppose  $M \neq qM$ . Then  $M/qM$  is at least one dimensional over  $T/q$  so there is a surjection  $g: M \rightarrow T/q = Ty$ . Let  $k$  be the composite of the natural map  $F \rightarrow M$  followed by  $g$ . Then  $k \in \text{End}(F)$  and  $k^2 = 0$ . Since  $\text{End}(F)$  is semi-prime we get the contradiction  $k = 0$  and  $Ty = 0$ . Therefore  $M = qM$ ; and thus  $F = Ty$  by Nakayama. Hence  $E_p = (Rx)_p$  as required.

**THEOREM 2.3.** *Let  $R$  be quasi-regular ring and  $E$  a finitely presented  $R$ -module. If  $\text{End}(E)$  is semi-prime and if there is an ideal  $I \subset R$  such that  $\text{Ann}(I) = \text{Ann}(E)$ , then  $E$  is isomorphic to an ideal of  $R$ .*

*Proof.* By 2.2  $\text{Ass}(E) = \text{Min}(\text{Ann}(E)) = \text{Min}(\text{Ann}(I))$ . Thus each associated prime of  $E$  consists of zero divisors of  $R$  [2, Theorem 84]. Therefore the natural map  $E \rightarrow E \otimes T$  is injective where  $T$  denotes the total quotient ring of  $R$ . Let  $F = E \otimes T$ .  $\text{End}_T(E) \cong \text{End}_R(E) \otimes T$  is semi-prime. Since  $T$  is absolutely flat, then  $F$  is a direct sum  $Te_1 \oplus \cdots \oplus Te_n$  of ideals of  $T$  each of which is generated by an idempotent  $e_i$  of  $T$  [1, Exercise 18, p. 64]. Let  $i \neq j$ ,  $h \in \text{Hom}_T(Te_i, Te_j)$ . Define  $f: F \rightarrow F$  by  $f(e_k) = 0$  for  $k \neq i$  and  $f(e_i) = h(e_i)$ . Then  $f^2 = 0$  and thus  $h = 0$ . Hence the idempotents  $e_i$  are mutually orthogonal and therefore  $F = Te_1 + \cdots + Te_n$  is an ideal of  $T$ . Now multiplication by a suitable regular element will move the image of  $E$  in  $F$  inside  $R$ .

The hypothesis on  $\text{Ann}_R(E)$  in 2.3 is satisfied when  $E$  is faithful.

There is some evidence that 2.3 may be valid for noncommutative rings. For example if  $R$  is an absolutely flat semi-prime ring and  $E$  a finitely presented right  $R$ -module (or more generally a projective right  $R$ -module) and if  $\text{End}(E)$  is semi-prime then  $E$  is isomorphic to an ideal.

*Added March 12, 1973.* S. Alamelu has independently obtained Theorem 1.1. Her results will appear in the Proceedings of the American Mathematical Society.

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