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In this paper a criterion for divisoriality of a scheme is given. The corresponding relative notion is defined and a criterion is given in this case as well.

1. In [2], Borelli proves that any coherent sheaf on a divisorial scheme is the quotient of a locally free sheaf. An examination of his proof reveals that he actually proves the stronger result that any quasi-coherent sheaf of finite type on a divisorial scheme is the quotient of a direct sum of invertible sheaves. Furthermore, these invertible sheaves are tensor powers of the inverses of the sheaves giving the scheme as divisorial.

The purpose of this note is to prove the (weaker) converse of this stronger result. In § 3, we define the notion of divisorial morphism and prove an analogous characterization theorem. The author would like to express his gratitude to Steven Kleiman who made several helpful suggestions.

Recall that X is divisorial if and only if, for every $x \in X$ there exists an invertible sheaf $\mathscr{L}$ on X, an element $s \in \Gamma(X, \mathscr{L}^{\otimes m})$ such that X_s is affine and $x \in X_s$. If $\mathscr{L}$ is any invertible sheaf on X, we write

$$X_{\mathscr{L}} = \bigcup \{X_s \mid X_s \text{ affine}, s \in \Gamma(X, \mathscr{L}^{\otimes m}), m = 1, 2, 3, \cdots\} .$$

With this notation, X is divisorial if and only if there exists a finite number of invertible sheaves $\mathscr{L}_1, \cdots, \mathscr{L}_n$ such that $X = \bigcup \mathscr{L}_i$. In this case, we will say that X is divisorial for $\mathscr{L}_1, \cdots, \mathscr{L}_n$.

2. THEOREM 1. *Let X be a scheme and let $\mathscr{L}_1, \cdots, \mathscr{L}_n$ be invertible sheaves on X. Suppose that for any quasi-coherent sheaf $\mathscr{F}$ of finite type, there exists a surjection $\mathscr{E} \to \mathscr{F}$ where $\mathscr{E}$ is a direct sum of tensor powers of the inverses of the $\mathscr{L}_i$. Then X is divisorial for $\mathscr{L}_1, \cdots, \mathscr{L}_n$.*

Proof. By Corollary 3.1 of [1], it suffices to show that the family $\{X_s \mid s \in \Gamma(X, \mathscr{L}_i^{\otimes m}); i = 1, \cdots, n; m = 1, 2, 3, \cdots\}$ forms a base for the topology of X. To see this, let U be an open subset of X and let $x \in U$. If $Y = X - U$ and $\mathscr{J}$ is the Ideal of $\mathscr{O}$ corresponding to Y, then $\mathscr{J}$ is quasi-coherent. Since X is a scheme, $\mathscr{J}$ is the direct limit of its quasi-coherent sub-Ideals $\mathscr{J}_\lambda$ of finite type ([3] I, 9.4.9). Each $\mathscr{J}_\lambda$ corresponds to a closed subscheme $Y_\lambda \supset Y$. If x were contained in each Y_λ, then x would be an element of Y. But $x \in U$ so there exists some $Y_\lambda \supset Y$ such that $x \notin Y_\lambda$ and the associated Ideal is of finite type. Thus, we may assume that I itself is of finite type.

By hypothesis, there exists a surjection $\mathscr{E} \to \mathscr{I}$ where $\mathscr{E}$ is a direct sum of tensor powers of the inverses of the $\mathscr{L}_i$. For convenience, let us write $\mathscr{E}$ as $\mathscr{L}'_1 \oplus \cdots \oplus \mathscr{L}'_t$. The surjection $\mathscr{E} \to \mathscr{I}$ induces a surjection $\gamma\colon \mathscr{E}_x \to \mathscr{I}_x$ of stalks at x and maps $\gamma_j\colon \mathscr{L}'_{j,x} \to \mathscr{I}_x = \mathscr{O}_x$ (since $x \notin Y$). We claim that at least one of the γ_j is surjective since, if not, the images of all the γ_j would be contained in the maximal ideal of $\mathscr{O}_x$ and this would imply that the image of γ itself is contained in the maximal ideal. Let α denote the composite map of sheaves $\mathscr{L}'_j \to \mathscr{I} \to \mathscr{O}$ where the second map is the canonical inclusion and j is chosen so that γ_j is surjective. Taking the tensor product of α with $\mathscr{L}'^{-1}_j$, we obtain $\alpha'\colon \mathscr{O} \to \mathscr{L}'^{-1}_j$ which is therefore defined by some $s \in \Gamma(X, \mathscr{L}'^{-1}_j)$. Since α' is surjective on the stalk at x, it follows that $s(x) \neq 0$. On the other hand, for all $y \in Y$, the image of α_y is contained in $\mathscr{I}_y$ and hence in the maximal ideal of $\mathscr{O}_y$, so that $s(y) = 0$. Thus $x \in X_s \subset U$. Realizing that $\mathscr{L}'^{-1}_j$ is a tensor power of one of the $\mathscr{L}_i$ completes the proof.

3. The notion of divisoriality has a relativization entirely analogous to the notion of relative ampleness given in [3] II, 4. 6. 1.

DEFINITION: Let $f\colon X \to Y$ be a morphism of schemes and let $\mathscr{L}_1, \cdots, \mathscr{L}_n$ be invertible sheaves on X. We say that f is divisorial for $\mathscr{L}_1, \cdots, \mathscr{L}_n$ or that X is divisorial over Y for $\mathscr{L}_1, \cdots, \mathscr{L}_n$ provided that there exists an open affine cover of Y such that $V = f^{-1}(U)$ is divisorial for $\mathscr{L}_1 | V, \cdots, \mathscr{L}_n | V$ for every U in the cover.

Note that every such f will automatically be separated and quasi-compact.

THEOREM 2. *Let $f\colon X \to Y$ be a separated, quasi-compact morphism of schemes and let $\mathscr{L}_1, \cdots, \mathscr{L}_n$ be invertible sheaves on X. Then f is divisorial for $\mathscr{L}_1, \cdots, \mathscr{L}_n$ if and only if, for every quasi-coherent $\mathscr{O}_X$-Module $\mathscr{F}$ of finite type, there exists a surjection $\mathscr{E} \to \mathscr{F}$ where $\mathscr{E}$ is a direct sum of sheaves of the form $\mathscr{L}_j^{\otimes -m} \otimes f^*(\mathscr{G})$ and each $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-Module of finite type.*

Proof. Suppose f is divisorial. Let U be an element of the open affine cover of Y which exhibits the divisoriality of f and let $V = f^{-1}(U)$. By definition, V can be covered by finitely many sets of the form $V_{\overline{\mathscr{L}}}$ where $\overline{\mathscr{L}} = \mathscr{L} \mid V$. By Lemma 3.2 of [2], for a sufficiently large integer m, $\mathscr{F} \otimes \mathscr{L}^{\otimes m} \mid V_{\overline{\mathscr{L}}}$ is generated by finitely many elements of $\Gamma(V, \mathscr{F} \otimes \mathscr{L}^{\otimes m})$. But then, the canonical morphism $(f \mid V)^*(f \mid V)_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m} \mid V) \to \mathscr{F} \otimes \mathscr{L}^{\otimes m} \mid V$ is surjective on $V_{\overline{\mathscr{L}}}$

by [3] 0_I, 5.1.2. Now the first member is just $f^*f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m}) \mid V$ so that we can say that the global morphism $f^*f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m}) \to \mathscr{F} \otimes \mathscr{L}^{\otimes m}$ is surjective on $V_{\mathscr{L}}$. Since f is quasi-coherent and separated, $f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m})$ is quasi-coherent on Y([3] I, 9.2.2) whence it is the direct limit of its quasi-coherent sub-$\mathscr{O}_Y$-Modules $\mathscr{G}_\lambda$ of finite type ([3] I, 9.4.9). Since inverse image commutes with direct limit, we have $f^*f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m}) = \text{dir. lim. } f^*(\mathscr{G}_\lambda)$. Since X is quasi-compact, $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ of finite type and the map $f^*f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m}) \to \mathscr{F} \otimes \mathscr{L}^{\otimes m}$ surjective on $V_{\mathscr{L}}$ it follows that there exists an index λ such that $f^*(\mathscr{G}_\lambda) \to \mathscr{F} \otimes \mathscr{L}^{\otimes m}$ is surjective on $V_{\mathscr{L}}$([3] 0_I, 5. 2. 3). But then $\mathscr{L}^{\otimes -m} \otimes f^*(\mathscr{G}_\lambda) \to \mathscr{F}$ is surjective on $V_{\mathscr{L}}$ and since X is covered by finitely many sets of the form $V_{\mathscr{L}}$, the result follows.

To prove sufficiency, let U be an arbitrary open affine subset of Y and let $V = f^{-1}(U)$. We will show that V is divisorial for $\mathscr{L}_1 \mid V, \cdots, \mathscr{L}_n \mid V$ by the criterion of Theorem 1. To this end, let $\mathscr{F}$ be any quasi-coherent Module of finite type on V. By [3] I, 9.4.8, there exists a quasi-coherent Module $\mathscr{F}'$ of finite type on X such that $\mathscr{F}' \mid V = \mathscr{F}$.

By hypothesis, there exists a surjection

$$\mathscr{L}'_1 \otimes f^*(\mathscr{G}_1) \oplus \cdots \oplus \mathscr{L}'_t \otimes f^*(\mathscr{G}_t) \longrightarrow \mathscr{F}'$$

where we have written $\mathscr{L}'_j$ to represent negative tensor powers of the $\mathscr{L}_i$ as we did in the proof of Theorem 1. The above map is, of course, still a surjection when restricted to V and we will write $\overline{\mathscr{L}'} = \mathscr{L}' \mid V$. Since the $\mathscr{G}_j$ are quasi-coherent and U is affine, $\mathscr{G}_j \mid U$ is generated by its global sections whence we have surjections $\mathscr{O}_U^{r_j} \to \mathscr{G}_j \mid U$ and, taking inverse images by $f \mid V$, $\mathscr{O}_V^{r_j} \to (f \mid V)^*(\mathscr{G}_j \mid U) = f^*(\mathscr{G}_j) \mid V$. In this way, we obtain surjections

$$(\overline{\mathscr{L}}'_j)^{r_j} = \overline{\mathscr{L}}'_j \otimes \mathscr{O}_V^{r_j} \longrightarrow \overline{\mathscr{L}}'_j \otimes f^*(\mathscr{G}_j) \mid V = (\mathscr{L}'_j \otimes f^*(\mathscr{G}_j)) \mid V\,.$$

Finally, by taking direct sums over j and composing with the restriction of the given surjection to V, we obtain

$$(\overline{\mathscr{L}}'_1)^{r_1} \oplus \cdots \oplus (\overline{\mathscr{L}}'_t)^{r_t} \longrightarrow \mathscr{F} \longrightarrow 0 \text{ is exact on } V\,.$$

Since $\mathscr{F}$ was arbitrary, this implies that V is divisorial for $\overline{\mathscr{L}}_1, \cdots, \overline{\mathscr{L}}_n$. Since U can be taken as an element of an affine cover of Y, the result follows.

Note. The above results remain valid if the hypothesis of separation is relaxed to quasi-separation in view of the results of [3] IV, 1.7.

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