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LOCAL CONTROL AND FACTORIZATION OF THE FOCAL SUBGROUP

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There are a number of results that yield information about the structure of a finite group G , based on information about its local subgroups; that is, the normalizers of its p -subgroups. Alperin and Gorenstein defined a notion of control of transfer, fusion, or strong fusion in a group G by a conjugacy functor W , or rather by $N(W(P))$, where P is a Sylow p -subgroup of G , and proved that if W controls transfer, fusion or strong fusion in every local subgroup of G , then W controls, respectively, transfer, fusion or strong fusion in the entire group G .

This paper gives a definition of control by two functors, W and V , or rather $N(W(P))$ and $C(V(P))$, and proves that when W contains the center, and when V is contained in the center and satisfies another condition, control in certain local subgroups implies control in the entire group G . In particular, the result on transfer says that $P \cap G'$, the focal subgroup of G , can be factorized whenever their focal subgroups can be properly factorized. From theorems of Glauberman, Goldschmidt, and Thompson which give conditions under which a group can be factorized, it is possible to obtain results which say that the focal subgroup of G can be factorized whenever certain local subgroups are p -solvable and satisfy the hypotheses of the factorization theorems.

1. Introduction. We use throughout the notation of [5]. In particular, H is a subgroup of G is denoted by $H \leq G$, with $H < G$ reserved for proper subgroups. We fix G a finite group, p a prime dividing the order of G , and P a Sylow p -subgroup of G .

DEFINITION 1.1. ([2], p. 242) Let \mathcal{H} be the set of all non-identity p -subgroups of G . A conjugacy functor W on \mathcal{H} is a mapping from \mathcal{H} to \mathcal{H} satisfying, for each $H \in \mathcal{H}$

- (a) $W(H) \leq H$, and
- (b) $W(H^x) = W(H)^x$ for all $x \in G$.

DEFINITION 1.2. A central functor V on \mathcal{H} is a conjugacy functor satisfying, for any $H, K \in \mathcal{H}$

- (a) $V(H) \leq Z(H)$
- (b) if $Z(H) \leq Z(K)$, then $V(H) \leq V(K)$.

REMARK 1.3. Examples of conjugacy functors include the identity

map $W(H) = H$, and the Thompson subgroup, $J(H)$, which is the subgroup of H generated by all Abelian subgroups of H for which the minimal number of generators is maximal. Examples of central functors include the center $Z(H)$ and the functor $V(H) = \mathcal{O}^n(Z(H))$, the subgroup of H generated by all p^n th powers of elements of $Z(H)$.

We now define a special notion of fusion.

DEFINITION 1.4. Let A and B be subsets of G , and K_1 and K subsets of G with $K_1 \subseteq K$, and $g \in \langle A, B \rangle$ such that $K_1^g \subseteq K$. We say g takes K_1 to K_1^g in K via A and B if we can write $g = g_1 g_2 \cdots g_n$ with $g_i \in A$ or B , and $K_1^{g_1 \cdots g_i} \subseteq K$ for all $i = 1, 2, \dots, n$. That is, we require that K_1 be conjugate to K_1^g by a series of conjugations so that each intermediate step lies in K .

REMARK 1.5. (a) If g takes K_1 to K_1^g in K via A and B , and if $K \subseteq L$; $A \subseteq C$, $B \subseteq D$ and if $K_2 \subseteq K_1$ then g takes K_2 to K_2^g in L via C and D .

(b) If g_1 takes K_1 to K_2 in K via A and B , and g_2 takes K_2 to K_3 in K via A and B then the product $g_1 g_2$ takes K_1 to K_3 in K via A and B .

Using this notion of fusion, we can now define local control, and state our main theorem.

DEFINITION 1.6. Let W be a conjugacy functor on \mathcal{H} , and V a central functor on \mathcal{H} , and L a subgroup of G with order divisible by p . We say, respectively, that W with V controls

- (1) *transfer in L*
- (2) *fusion in L*
- (3) *strong fusion in L*

provided there exists a Sylow p -subgroup Q of L such that

- (1) $Q \cap L' = (Q \cap N_L(W(Q)))' \cdot (Q \cap C_L(V(Q)))'$

(2) Whenever X and Y are subsets of Q conjugate in L , then they are conjugate by an element of $\langle N_L(W(Q)), C_L(V(Q)) \rangle$ which takes X to Y in Q .

(3) Whenever X and Y are subsets of Q , with $Y = X^a$ for some $a \in L$, then $a = cb$, with $c \in C_L(X)$, $b \in \langle N_L(W(Q)), C_L(V(Q)) \rangle$, and b takes X to Y in Q .

DEFINITION 1.7. A *local* subgroup of G is a subgroup of the form $L = N(H)$, for some $H \in \mathcal{H}$. Define \mathcal{L} as the set of those local subgroups satisfying $L = N(H)$, with $Z(P) \leq H \leq P$.

We now state our main theorem, relating local and global control by two functors, W and V . Note that we require $Z(H) \leq W(H)$ and $V(H) \leq Z(H)$.

THEOREM 1.8. *Let W be a conjugacy functor on \mathcal{H} satisfying $Z(H) \leq W(H)$ for all $H \in \mathcal{H}$, and let V be a central functor on \mathcal{H} . Assume that W with V controls*

- (1) *transfer,*
- (2) *fusion, or*
- (3) *strong fusion*

in all local subgroups $L \in \mathcal{L}$. Then W with V controls, respectively, transfer, fusion, or strong fusion in G .

We use Theorem 1.8 (1), on control of transfer, to obtain factorizations of the focal subgroup, $P \cap G'$, when the local subgroups in \mathcal{L} are all p -solvable and satisfy certain other conditions. This is discussed in §6.

2. Assumed results. We need a number of concepts and results from Alperin [1] and Alperin and Gorenstein [2]. Throughout, as in the previous chapter, G is a finite group, and P is a Sylow p -subgroup of G . Also, W is a conjugacy functor on \mathcal{H} , as in Definition 1.1, and in addition, satisfies $W(H) \geq Z(H)$ for all $H \in \mathcal{H}$.

DEFINITION 2.1. ([1], p. 236) A *family* is a collection of pairs (H, T) where H is a subgroup of P and T is a subset of $N(H)$.

DEFINITION 2.2. ([1], p. 237) A family F is called a *conjugation family* provided that whenever A and B are subsets of P and $B = A^g$ for some $g \in G$, then there exist elements $(H_1, T_1), (H_2, T_2), \dots, (H_n, T_n)$ of F , and elements x_1, x_2, \dots, x_n , and y of G satisfying

- (a) $g = x_1 \cdots x_n y$
- (b) $x_i \in T_i$, for $i = 1, 2, \dots, n$, and $y \in N(P)$
- (c) $A \leq H_1, A^{x_1 \cdots x_i} \leq H_{i+1}$, for $i = 1, \dots, n + 1$.

We now define a particular conjugation family which we use for the remainder of the paper.

DEFINITION 2.3. ([1], p. 225) If Q and R are Sylow p -subgroups of G , then we say $Q \cap R$ is a *tame intersection* provided that $N_Q(Q \cap R)$ and $N_R(Q \cap R)$ are Sylow p -subgroups of $N(Q \cap R)$.

DEFINITION 2.4. ([2], p. 245) If H is a subgroup of P , we set $W_1(H) = H$, $M_1(H) = N_P(H)$, and $N_1(H) = N(H)$, and define recursively $W_{i+1}(H) = W(M_i(H))$, $M_{i+1}(H) = N_P(W_{i+1}(H))$, and $N_{i+1}(H) = N(W_{i+1}(H))$.

REMARK 2.5. As noted in [2], page 245, the sequence $M_i(H)$ is strictly increasing until it stabilizes at P . Also, if $M_i(H) = P$, then $W_{i+1}(H) = W(P)$, so that the sequence $W_i(H)$, though not necessarily

increasing, stabilizes at $W(P)$.

DEFINITION 2.6. ([2], p. 245) If H is a subgroup of P , we say that H is well-placed in P with respect to W if $M_i(H)$ is a Sylow p -subgroup $N_i(H)$, for all i .

PROPOSITION 2.7. ([2], p. 252) *If H is a subgroup of P , then some conjugate of H is well-placed in P .*

DEFINITION 2.8. ([2], p. 246) We define the family F to consist of all pairs (H, T) such that there exists a Sylow p -subgroup S of G so that $H = P \cap S$ is a well-placed tame intersection, and

- (a) $T = C(H)$ if $C_P(P \cap S) \not\leq P \cap S$
- (b) $T = N(H)$ if $C_P(P \cap S) \leq P \cap S$.

REMARK 2.9. Note that $(P, N(P)) \in F$. This fact eliminates the need for special mention of $y \in N(P)$ in Definition 2.2.

PROPOSITION 2.10. *F is a conjugation family.*

Proof. Section 2 of [2] is devoted to a proof of this fact.

In the next section, we use the following result, which applies to all conjugation families.

PROPOSITION 2.11. ([1], p. 238) *If P^* is a subgroup of P , and if $[P, N(P)] \leq P^* < P \cap G'$, then there is an element (H, T) of F such that $[H, T] \not\leq P^*$.*

3. Control of transfer. We now turn to the proof of Theorem 1.8 (1), on local control of transfer.

LEMMA 3.1. *Let P^* be a subgroup of G satisfying $P \cap (N(W(P)))' \leq P^* < P \cap G'$. Then there exists a non-identity subgroup K of P such that*

- (a) $Z(P) \leq K$
- (b) $P \cap L$ is a Sylow p -subgroup of $L = N(K)$
- (c) $P \cap L' \not\leq P^*$, and
- (d) $P \cap M' \leq P^*$ where $M = N(W(P \cap L))$.

Proof. By the definition of conjugacy functor, if $x \in N(P)$, then $W(P) = W(P^x) = W(P)^x$, so that $N(P) \leq N(W(P))$. Thus, $P \cap N(P)' \leq P \cap N(W(P))'$, and the hypotheses of this lemma imply that $P \cap N(P)' \leq P^*$. Thus all the hypotheses of Proposition 2.11 are fulfilled, and we conclude that there exists $(H, T) \in F$ such that $[H, T] \not\leq P^*$.

Now, if $C_P(H) \not\leq H$, then by the definition of F , Definition 2.8, we have $T = C(H)$, so $[H, T] \leq P^*$, contrary to our choice of (H, T) . Thus, we must have $C_P(H) \leq H$ and $T = N(H)$.

Thus, $Z(P) \leq H$, so (a) is satisfied for $K = H$. Furthermore, as $[H, N(H)] \leq N(H)'$, and $[H, N(H)] \leq P$, we have $[H, N(H)] \leq P \cap N(H)'$. Thus, $[H, N(H)] \not\leq P^*$ implies $P \cap N(H)' \not\leq P^*$, as required in (c). Also, H is, by choice, a tame intersection in P , so $N_P(H) = P \cap N(H)$ is a Sylow p -subgroup of $N(H)$, as required in (b). Thus, (a), (b), and (c) hold for $K = H$.

Define $W_i = W_i(H)$, so that $W_1 = H$. We shall find an m so that W_m satisfies (d) as well as (a), (b), and (c).

First, we show that $Z(P) \leq W_i$ for all i . For $i = 1$, we have seen that $Z(P) \leq W_1 = H$. For $i > 1$, as $W_{i-1} \leq P$, we have $Z(P) \leq N(W_{i-1}) \cap P$, so $Z(P) \leq Z(N(W_{i-1}) \cap P)$. By our choice of W , we have $Z(L) \leq W(L)$ for all $L \in \mathcal{H}$, so

$$Z(P) \leq Z(N(W_{i-1}) \cap P) \leq W(N(W_{i-1}) \cap P) = W_i.$$

Thus, $Z(P) \leq W_i$ for all i , so each W_i satisfies (a).

As H is well-placed, each $M_i(H)$ is a Sylow p -subgroup of $N_i(H)$. That is, $N_P(W_i(H)) = P \cap (N(W_i(H)))$ is a Sylow p -subgroup of $N(W_i(H))$. Thus, (b) is also satisfied for each W_i .

Now, as noted in Remark 2.5, $W_i = W(P)$ for all i sufficiently large, and, as we have $P \cap N(W(P))' \leq P^*$, (c) does not hold for W_i with i sufficiently large. However, (c) does hold for $i = 1$. Thus, we can choose m so that (c) holds W_m , but not for W_{m+1} . But $W_{m+1} = W(P \cap N(W_m))$, so $P \cap N(W_{m+1})' \leq P^*$, as (c) fails for W_{m+1} , and so $P \cap N(W(P \cap N(W_m)))' \leq P^*$. But this last inequality states that condition (d) holds for W_m .

Thus, setting $K = W_m$ completes the proof of the lemma.

We now complete the proof of the theorem on control of transfer.

Proof of Theorem 1.8 (1). Assume that the conclusion of the theorem is false. Then we have

$$P \cap N(W(P))' \cdot P \cap C(V(P))' < P \cap G'.$$

Call the group on the left P^* . Then, as $P^* < P \cap G'$, we may apply Lemma 3.1, and find subgroups K , L , and M as defined in the lemma.

As L is a local subgroup, the hypotheses of our theorem imply that W with V controls transfer in L , so there exists Q , a Sylow p -subgroup of L , satisfying

$$Q \cap L' = [Q \cap C_L(V(Q))'] \cdot [Q \cap N_L(W(Q))'].$$

By conjugacy, this statement holds for all Sylow p -subgroups of

L , so we may take $Q = P \cap L$, which is a Sylow p -subgroup of L by (b) of the lemma. Thus, using the fact that $M = N(W(P \cap L))$, the above equation becomes

$$(P \cap L) \cap L' = [(P \cap L) \cap C_L(V(P \cap L))'] \cdot [(P \cap L) \cap (M \cap L)']$$

which yields

$$(3.2) \quad P \cap L' = [P \cap C_L(V(P \cap L))'] [P \cap (M \cap L)'] .$$

From (d) of the lemma, $P \cap M' \leq P^*$, so certainly $P \cap (M \cap L)' \leq P^*$. Also, from (a) of the lemma $Z(P) \leq K$, and $K \leq L$, so $Z(P) \leq P \cap L \leq P$, which implies that $Z(P) \leq Z(P \cap L)$, and so $C(V(P \cap L)) \leq C(V(P))$. Thus, $P \cap C_L(V(P \cap L))' \leq P \cap C(V(P))' \leq P^*$ by the definition of P^* . Hence, from (3.2), we have

$$P \cap L' = [P \cap C_L(V(P \cap L))'] \cdot [P \cap (M \cap L)'] \leq P^* ,$$

contrary to (c) of the lemma.

This contradiction completes the proof of Theorem 1.8 (1).

4. Control of fusion. Before proving Theorem 1.8 (2), we need first the following lemma. Recall that, if $g \in \langle A, B \rangle$ takes X to Y in P , then Definition 1.4 requires that the intermediate steps lie in P .

LEMMA 4.1. *Assume the hypotheses of Theorem 1.8 (2). Assume further that we have Q , a Sylow p -subgroup of G , so that $P \cap Q$ is a well-placed, tame intersection and $C_P(P \cap Q) \leq P \cap Q$, and that we have X and Y subsets of $P \cap Q$, satisfying $X^g = Y$ for some $g \in N(P \cap Q)$. Then X and Y are conjugate by an element of $\langle N(W(P)), C(V(P)) \rangle$ which takes X to Y in P .*

Proof. Let $W_i = W_i(P \cap Q)$, $M_i = M_i(P \cap Q)$, and $N_i = N_i(P \cap Q)$, as defined in Definition 2.4, and let $C_i = \langle N_i, C(V(P)) \rangle$. We show by induction on i that there exists $c_i \in C_i$ satisfying $Y = X^{c_i}$ and c_i takes X to Y in M_i via N_i and $C(V(P))$. Since, by Remark 2.5, $N_i = N(W(P))$ and $M_i = P$ for i sufficiently large, this will suffice.

For $i = 1$, we have $N_1 = N(P \cap Q)$, and $N_1 \leq C_1$. Thus, $g \in C_1$, and $X^g = Y$ and g takes X to Y in M_1 via N_1 and $C(V(P))$, as $g \in N_1$ and $X, Y \leq P \cap Q \leq M_1$. Thus, $c_1 = g$.

Assume, by induction, that $X^{c_i} = Y$ for some $c_i \in C_i$, and c_i takes X to Y in M_i . That is, we have $c_i = a_1 a_2 \cdots a_n$, with $a_k \in N_i$ or $a_k \in C(V(P))$, for $k = 1, 2, \dots, n$, and, if we call $X_k = X^{a_1 \cdots a_k}$, then we have $X = X_0$, $Y = X_n$, and $X_k \leq M_i$ for $k = 1, \dots, n$.

It suffices to show that, for each $k = 1, \dots, n$, we have X_{k-1} and X_k conjugate by an element $b_k \in C_{i+1}$ which takes X_{k-1} to X_k in M_{i+1} via N_{i+1} and $C(V(P))$. If we show this, then the product $b_1 \dots b_n$ takes X to Y in M_{i+1} , as required.

If $a_k \in C(V(P))$, then X_{k-1} and X_k are conjugate by an element of C_{i+1} , namely a_k itself, and, as $M_i \leq M_{i+1}$, we see that a_k takes X_{k-1} to X_k in M_{i+1} as required. In this case, let $b_k = a_k$.

So we may assume $a_k \in N_i$. We now claim that W with V controls fusion in N_i . To establish this, we need only prove that $Z(P) \leq W_i$, since $N_i = N(W_i)$ and we are assuming that W with V controls fusion in every local subgroup of the form $L = N(H)$ with $Z(P) \leq H \leq P$.

Since $C_P(P \cap Q) \leq P \cap Q$, by hypothesis of the lemma, we have $Z(P) \leq P \cap Q = W_1$. Assume now, by induction, that $Z(P) \leq W_j$. Then $Z(M_j) \leq W(M_j) = W_{j+1}$, by hypothesis on W . But $Z(P) \leq W_j \leq M_j \leq P$, so $Z(P) \leq Z(M_j)$ and $Z(P) \leq W_{j+1}$. Thus, for all i , $Z(P) \leq W_i$, and W with V controls fusion in N_i as claimed.

Thus, as we have X_{k-1} and X_k , subsets of M_i , conjugate by $a_k \in N_i$ and M_i is a Sylow p -subgroup of N_i , as H is well-placed, we conclude that X_{k-1} and X_k are conjugate by some element

$$b_k \in \langle N(W(M_i)), C(V(M_i)) \rangle$$

which takes X_{k-1} to X_k in M_i . Also, $Z(P) \leq P \cap N_i = M_i$, so $Z(P) \leq Z(M_i)$ and $V(P) \leq V(M_i)$. Thus $C(V(M_i)) \leq C(V(P))$, and

$$b_k \in \langle N(W(M_i)), C(V(P)) \rangle = C_{i+1},$$

and since $M_i \leq M_{i+1}$, it follows that b_k takes X_{k-1} to X_k in M_{i+1} , as required. This completes the induction on i and the proof of the lemma.

We now complete the proof of Theorem 1.8 (2).

Proof of Theorem 1.8 (2). Let X and Y be subsets of P conjugate in G . We wish to show that X and Y are conjugate by an element of $\langle N(W(P)), C(V(P)) \rangle$ which takes X to Y in P via $N(W(P))$ and $C(V(P))$.

Using the conjugation family F of Definition 2.8, we know there exist X_0, X_1, \dots, X_n , subsets of P , and $(H_i, T_i) \in F$ such that $X = X_0$, $X_n = Y$, and X_{i-1} and X_i are contained in H_i and conjugate by an element of T_i .

To prove our theorem, it suffices to prove that for each $i = 1, \dots, n$, X_{i-1} and X_i are conjugate by an element $g_i \in \langle N(W(P)), C(V(P)) \rangle$ taking X_{i-1} to X_i in P , for then their product $g_1 g_2 \dots g_n \in \langle N(W(P)),$

$C(V(P))$ takes X to Y in P via $N(W(P))$ and $C(V(P))$.

If $C_P(H_i) \not\leq H_i$, then $T_i = C(H_i)$, by definition of F , so $X_{i-1} = X_i$, and the assertion holds. In this case, set $g_i = 1$.

If, on the other hand, $C_P(H_i) \leq H_i$, we may use Lemma 4.1, for H_i is a well-placed, tame intersection of P with some other Sylow p -subgroup of G . Thus, we can apply Lemma 4.1 to X_{i-1} and X_i , and conclude that X_{i-1} and X_i are conjugate by an element

$$g_i \in \langle N(W(P)), C(V(P)) \rangle$$

taking X_{i-1} to X_i in P .

Thus, $g_1 g_2 \cdots g_n$ takes X to Y in P as required, and the proof of Theorem 1.8 (2) is complete.

5. Control of strong fusion. To prove Theorem 1.8 (3), on control of strong fusion, we need first the following theorem.

THEOREM 5.1. *Assume the hypotheses of Theorem 1.8 (3). Let X be a subgroup of P , and $g \in N(X)$. Then we can write $g = ab$, where $a \in C(X)$, $b \in \langle N(W(P)), C(V(P)) \rangle$ and b takes X to X in P .*

REMARK 5.2. Proving Theorem 5.1 will suffice to prove Theorem 1.8 (3). For if X and Y are subsets of P , with $X^g = Y$ for some $g \in G$, then, by Theorem 1.8 (2), there exists $h \in \langle N(W(P)), C(V(P)) \rangle$ such that $X^h = Y$ and h takes X to Y in P . Thus, $gh^{-1} \in N(X)$. We may apply the theorem above to the subgroup $\langle X \rangle$, and as $gh^{-1} \in N(\langle X \rangle)$, we have $gh^{-1} = ab$ for some $a \in C(\langle X \rangle) = C(X)$ and $b \in \langle N(W(P)), C(V(P)) \rangle$ and b takes X to Y in P . Then $g = abh$, where $a \in C(X)$, $bh \in \langle N(W(P)), C(V(P)) \rangle$ and bh takes X to Y in P , as required to prove the theorem on control of strong fusion.

To prove Theorem 5.1, we need the following lemmas. For the remainder of this section assume the hypotheses of Theorem 1.8 (3), namely that W with V controls strong fusion in all local subgroups of the form $L = N(H)$ with $Z(P) \leq H \leq P$.

LEMMA 5.3. *If H is well-placed in P , and $Z(P) \leq H$, and if $g \in N(H)$, then $g \in \langle N(W(P)), C(V(P)) \rangle$, and g takes H to H in P .*

Proof. Let $W_i = W_i(H)$, $M_i = M_i$, $N_i = N_i(H)$, and

$$C_i = \langle N_i, C(V(P)) \rangle,$$

as before. For $g \in N(H)$, we shall show by induction on i that $g \in C_i$ and that g takes H to H in M_i , via N_i and $C(V(P))$. As before, this will suffice to prove the lemma.

For $i = 1$, we have $g \in N(H) = N_1(H)$, so that $g \in C_1$. Further-

more, $H \leq N_P(H) = M_i(H)$, so that g takes H to H in M_i as required.

Assume now, by induction, that $g \in C_i$ and that g takes H to H in M_i . Thus, $g = b_1 \cdots b_n$, $b_k \in N_i$ or $b_k \in C(V(P))$, and $H^{b_1 \cdots b_k} \leq M_i$ for $k = 1, 2, \dots, n$. Define $H_0 = H$, and $H_k = H^{b_1 \cdots b_k}$, for $k = 1, \dots, n$, so that $H_n = H$.

We now show, by induction on k , that b_k takes H_{k-1} to H_k in M_{i+1} via N_{i+1} and $C(V(P))$. This will imply that $g = b_1 \cdots b_k$ takes $H = H_0$ to $H = H_n$ in M_{i+1} via N_{i+1} and $C(V(P))$, which will complete the induction on i and the proof of the lemma.

Assume $k = 1$. If $b_1 \in C(V(P))$, then $b_1 \in C_i$. As $M_i \leq M_{i+1}$, we have b_1 taking H_0 to H_1 in M_{i+1} via N_i and $C(V(P))$ as required.

Thus, we may assume that $b_1 \in N_i$.

We now show that W with V controls strong fusion in N_i . By the hypotheses of the lemma, we need only show that $Z(P) \leq W_i$, as $N_i = N(W_i)$. By hypothesis, $Z(P) \leq H = W_1$. Assume now, by induction, that $Z(P) \leq W_j$. Since $Z(M_j) \leq W(M_j)$ by hypothesis on W , and $W_{j+1} = W(M_j)$, we have $Z(M_j) \leq W_{j+1}$. As $Z(P) \leq M_j \leq P$, we have $Z(P) \leq Z(M_j)$, so $Z(P) \leq W_{j+1}$, completing the induction on j . Thus, W with V controls strong fusion in N_i for all i .

Now $H_0^{b_1} = H_1$, where H_0 and H_1 are subsets of M_i , a Sylow p -subgroup of N_i , and so by control of strong fusion, we conclude that $b_1 = c_1 d_1$, where $c_1 \in C(H)$, $d_1 \in \langle N(W(M_i)), C(V(M_i)) \rangle$, and d_1 takes H_0 to H_1 in M_i . Now since $V(P) \leq Z(P) \leq H$ by hypothesis, $C(H) \leq C(Z(P)) \leq C(V(P))$, so $c_1 \in C(V(P))$. Also, as we have seen, $Z(P) \leq M_i \leq P$, so $Z(P) \leq Z(M_i)$, and $C(V(M_i)) \leq C(V(P))$, which implies that $d_1 \in \langle N(W(M_i)), C(V(P)) \rangle = C_{i+1}$. Furthermore, we have $M_i \leq M_{i+1}$, and so $b_1 = c_1 d_1 \in C_{i+1}$, and takes H_0 to H_1 in M_{i+1} via N_{i+1} and $C(V(P))$, as required, completing the case $k = 1$.

We now consider the case $k > 1$. We assume, by induction, that for all $r < k$, the result is true for b_r ; that is that b_r takes H_{r-1} to H_r in M_{i+1} via N_{i+1} and $C(V(P))$.

If $b_k \in C(V(P))$, we are done, as in the case $k = 1$, for then $b_k \in C_{i+1}$ immediately.

So we may assume that $b_k \in N_i$. As we have seen above, W with V controls strong fusion in N_i , and so $b_k = c_k d_k$ with $c_k \in C(H_{k-1})$, $d_k \in C_{i+1}$, and d_k takes H_{k-1} to H_k in M_{i+1} via N_{i+1} and $C(V(P))$. Now $c_k \in C(H_{k-1})$, and $C(H_{k-1}) = C(H^{b_1 b_2 \cdots b_{k-1}}) = C(H)^{b_1 b_2 \cdots b_{k-1}}$. Thus, we have, for some $c \in C(H)$

$$(5.4) \quad c_k = b_{k-1}^{-1} \cdots b_1^{-1} c b_1 \cdots b_{k-1}.$$

By our induction hypothesis on k , each b_r , for $r = 1, \dots, k-1$ takes H_{r-1} to H_r in M_{i+1} . Thus, each b_r^{-1} takes H_r to H_{r-1} in M_{i+1} via N_{i+1} and $C(V(P))$. Furthermore, as $V(P) \leq Z(P) \leq H$, we have $C(H) \leq C(V(P))$, so $c \in C(H)$ also takes H to H in M_{i+1} via N_{i+1} and

$C(V(P))$.

We can now calculate the effect of c_k on H_{k-1} by examining the series of elements on the right-hand side of equation 5.4, and we find that c_k takes H_{k-1} to H_{k-1} in M_{i+1} via N_{i+1} and $C(V(P))$.

Thus, $b_k = c_k d_k$ takes H_{k-1} to H_k in M_{i+1} via N_{i+1} and $C(V(P))$, completing the induction on k and the proof of the lemma.

LEMMA 5.5. *Let $P \cap Q$ be a well-placed, tame intersection, with $C_P(P \cap Q) \leq P \cap Q$, for Q some Sylow p -subgroup of G . Let H be a subgroup of $P \cap Q$, let $x \in N(P \cap Q)$, and suppose that for any $g \in N(H)$, we can write $g = ab$, where $a \in C(H)$, $b \in \langle N(W(P)), C(V(P)) \rangle$, and b takes H to H in P . Then for any $h \in N(H^x)$, we can write $h = cd$, with $c \in C(H^x)$ and $d \in \langle N(W(P)), C(V(P)) \rangle$ and d takes H^x to H^x in P .*

REMARK 5.6. This lemma shows that if Theorem 5.1 holds for such a subgroup H , then it holds for any conjugate of H by an element of $N(P \cap Q)$.

Proof of lemma. Let $h \in N(H^x)$. Then $h^{x^{-1}} \in N(H)$, and so $h^{x^{-1}} = ab$, where $a \in C(H)$, $b \in \langle N(W(P)), C(V(P)) \rangle$ and b takes H to H in P .

Since $C_P(P \cap Q) \leq P \cap Q$, we have $Z(P) \leq P \cap Q$. Thus, the hypotheses of Lemma 5.3 are satisfied, with $P \cap Q$ in the place of H , and so, as $x \in N(P \cap Q)$, we have $x \in \langle N(W(P)), C(V(P)) \rangle$ and x takes $P \cap Q$ to $P \cap Q$ in P . Since H and H^x are subsets of $P \cap Q$, the subset property noted in Remark 1.5 shows that x takes H to H^x in P . Similarly, x^{-1} takes H^x to H in P .

Now $h^{x^{-1}} = ab$, so $h = (a^x)(b^x)$. As $a \in C(H)$, $a^x \in C(H^x)$, as required. Furthermore, $b^x \in \langle N(W(P)), C(V(P)) \rangle$, and as we have seen that x^{-1} takes H^x to H in P , that b takes H to H in P , and that x takes H to H^x in P , we can see that b^x takes H^x to H^x in P , by the transitivity property of Remark 1.5.

Thus, $h = (a^x)(b^x)$ is the required factorization of h , and the proof is complete.

LEMMA 5.7. *Let Y be a well-placed subgroup of P with $Z(P) \not\leq Y$, and assume that Theorem 5.1 holds for every subgroup V of P such that $Y < V$. Then Theorem 5.1 holds for Y .*

Proof. Since $Z(P) \not\leq Y$, it follows that $C_P(Y) \not\leq Y$. Hence, $Y < Y \cdot C_P(Y)$. Call $Y^* = Y \cdot C_P(Y)$, and note that, by hypothesis, the theorem holds for Y^* .

Now Y is well-placed in P , so $N_P(Y)$ is a Sylow p -subgroup of $N(Y)$. Also, $Y \cdot C(Y)$ is normal in $N(Y)$, so $N_P(Y) \cap Y \cdot C(Y)$ is a Sylow p -subgroup of $Y \cdot C(Y)$. But $N_P(Y) \cap Y \cdot C(Y) = N(Y) \cap P \cap$

$Y \cdot C(Y) = P \cap Y \cdot C(Y)$, as $Y \cdot C(Y) \leq N(Y)$. Furthermore, $P \cap Y \cdot C(Y) = Y \cdot C_P(Y) = Y^*$ by the Dedekind identity. Thus, Y^* is a Sylow p -subgroup of $Y \cdot C(Y)$, which is a normal subgroup of $N(Y)$. So we may apply the Frattini argument, and conclude $N(Y) \leq Y \cdot C(Y) \cdot N(Y^*)$.

But, as $Y \leq Y^*$, certainly $Y \leq N(Y^*)$, so the above relation may be rewritten as $N(Y) \leq C(Y) \cdot N(Y^*)$.

Thus, if $g \in N(Y)$, we can write $g = c \cdot n$, for some $c \in C(Y)$, $n \in N(Y^*)$. Now Theorem 5.1 holds for Y^* , so $n = ab$, for some $a \in C(Y^*)$, $b \in \langle N(W(P)), C(V(P)) \rangle$, and b takes Y^* to Y^* in P .

Now $Z(P) \leq Y^*$, since $C_P(Y) \leq Y^*$, so $C(Y^*) \leq C(Z(P)) \leq C(V(P))$, and so $a \in \langle N(W(P)), C(V(P)) \rangle$, and takes Y^* to Y^* in P . But $n \in N(Y)$, and thus the subset property of Remark 1.5 shows that n actually takes Y to Y in P .

Thus, $g = c \cdot n$, with $c \in C(Y)$, and $n \in \langle N(W(P)), C(V(P)) \rangle$, with n taking Y to Y in P . Thus, the theorem holds for Y , and the lemma is proved.

Proof of Theorem 5.1. We complete the proof of this theorem by induction on $|P: X|$. If $X = P$, then $N(X) \leq N(W(P))$, by the definition of conjugacy functor, and so Theorem 5.1 is true immediately.

Thus, we may assume that $X < P$, and, by induction, that the theorem is true for all subgroups Y of P such that $|P: Y| < |P: X|$.

We show first that there is some conjugate Y of X , where Y is a subgroup of P , such that the theorem holds for Y . By Proposition 2.7, some conjugate Y of X is well-placed in P . If $Z(P) \leq Y$, then Lemma 5.3 shows that the theorem holds for Y . If $Z(P) \not\leq Y$, then our induction hypothesis and Lemma 5.7 show that the theorem holds in Y . In any case, there is a conjugate Y of X such that the theorem holds in Y .

Since the family F , described in Definition 2.8 is a conjugation family, and since $(P, N(P)) \in F$, we have the following, using Definition 2.2:

There exist subgroups X_1, \dots, X_n of P , with $Y = X_1$, and $X = X_n$, and Sylow p -subgroups Q_1, \dots, Q_n of G , satisfying

- (a) $P \cap Q_i$ is a well-placed, tame intersection
- (b) X_i and X_{i+1} are contained in $P \cap Q_i$,
- (c) if $C_P(P \cap Q_i) \not\leq P \cap Q_i$, then X_i and X_{i+1} are conjugate in $C(P \cap Q_i)$, and
- (d) if $C_P(P \cap Q_i) \leq P \cap Q_i$, then X_i and X_{i+1} are conjugate in $N(P \cap Q_i)$.

We show by induction on j that the theorem holds for all X_j . As $X = X_n$, this suffices to prove the theorem.

For $j = 1$, we have $X_1 = Y$, and we have chosen Y so that the

theorem holds in Y . Now suppose that the theorem holds in X_j . If (c) is true for the index j , then X_j and X_{j+1} are conjugate in $C(P \cap Q_i)$, and, as they lie in $P \cap Q_i$, they are equal, and the theorem holds for X_{j+1} .

If, on the other hand, (d) holds for the index j , then Lemma 5.5 shows that the theorem holds for X_{j+1} , as a conjugate of X_j by an element of $N(P \cap Q_i)$.

This induction completes the proof of Theorem 5.1, and so, of our result on control of strong fusion.

6. Factorizations of the focal subgroup. We first need the following result, which says that a factorization of a p -solvable group yields a factorization of its focal subgroup.

LEMMA 6.1. *Let G be a p -solvable group, P a Sylow p -subgroup of G , and suppose $G = \langle H, K, L \rangle$, where $P \leq H$, and $P \leq K$, and L is a normal p' -subgroup of G . Then $P \cap G' = (P \cap H') \cdot (P \cap K')$.*

Before we turn to the proof of Lemma 6.1, we prove a preliminary lemma.

LEMMA 6.2. *If P is a Sylow p -subgroup of the group G , and M is a subgroup of G with $P \leq M$, and N is a normal p' -subgroup of G , then $PN \cap M'N = (P \cap M')N$.*

Proof. Call $E = PN \cap M'N$. Then $E \leq PN$, and $N \leq E$, so, by the Dedekind identity, $E = (P \cap E)N$. Also, $P \cap E = P \cap PN \cap M'N = P \cap M'N$, so $E = (P \cap M'N)N$. Since $P \cap M'N$ is a p -group, and N is a p' -group, this factorization of E shows that $P \cap M'N$ is a Sylow p -subgroup of E . Also, since M' is a normal subgroup of M and $P \leq M$, we have that $P \cap M'$ is a Sylow p -subgroup of M' , and hence also of $M'N$. Since $P \cap M' \leq E$, and $E \leq M'N$, we have $P \cap M'$ a Sylow p -subgroup of E as well. But $P \cap M' \leq P \cap M'N$, and each of these is a Sylow p -subgroup of E . Hence, $P \cap M' = P \cap M'N$, and so $E = (P \cap M')N$, as required.

Proof of Lemma. 6.1. We proceed by induction on the order of G . Thus, we assume that the lemma is true of any group whose order is less than $|G|$.

Let N be a minimal normal subgroup of G . Define $\bar{G} = G/N$, $\bar{P} = PN/N$, $\bar{H} = HN/N$, $\bar{K} = KN/N$ and $\bar{L} = LN/N$. We claim that these homomorphic images satisfy the hypotheses of the lemma. By elementary properties of homomorphic images, \bar{G} is a p -solvable group, $\bar{G} = \langle \bar{H}, \bar{K}, \bar{L} \rangle$, and all the hypotheses of our lemma are satisfied. As

$|\bar{G}| < |G|$, our induction hypothesis shows that the conclusion of the lemma holds for \bar{G} ; that is, that $\bar{P} \cap \bar{G}' = (\bar{P} \cap \bar{H}') \cdot (\bar{P} \cap \bar{K}')$.

If we consider the corresponding equation for the pre-images in G of these subgroups of \bar{G} , using the fact that $\bar{G}' = (G/N)' = G'N/N$, we get

$$(6.3) \quad PN \cap G'N = (PN \cap H'N) \cdot (PN \cap K'N) .$$

Define $F = PN \cap G'N$. Now N is a minimal normal subgroup of the p -solvable group G , so N is either a p -group or a p' -group. We consider each of these cases separately.

Case 1. N is a p' -group. Using, in turn, G , then H , and then K for M in Lemma 6.2, we have $PN \cap G'N = (P \cap G')N$, and $PN \cap H'N = (P \cap H')N$, and $PN \cap K'N = (P \cap K')N$. Using these equations to rewrite (6.3), we get

$$F = (P \cap G')N \text{ and } F = (P \cap H') \cdot (P \cap K')N .$$

The first of these equations shows that $P \cap G'$ is a Sylow p -subgroup of F , and the second that $(P \cap H') \cdot (P \cap K')$ is a Sylow p -subgroup of F . But as $(P \cap H') \cdot (P \cap K') \leq P \cap G'$, we must have $P \cap G' = (P \cap H') \cdot (P \cap K')$ as required.

Case 2. N is a p -group. As N is a normal p -subgroup of G , we have $N \leq P$, so $PN = P$. Thus, (6.3) becomes

$$F = P \cap G'N = (P \cap H'N) \cdot (P \cap K'N) .$$

Now, we have $P \cap G'N \leq G'N$ and $N \leq P \cap G'N$, so by the Dedekind identity, $P \cap G'N = (P \cap G'N \cap G')N$. But $P \cap G'N \cap G' = P \cap G'$, so $P \cap G'N = (P \cap G')N$. Similarly, $P \cap H'N = (P \cap H')N$ and $P \cap K'N = (P \cap K')N$. Thus, the equation above becomes

$$F = (P \cap G')N = (P \cap H') \cdot (P \cap K')N .$$

Now N is either an eccentric or a central factor of G ; that is either $[N, G] > 1$ or $[N, G] = 1$.

If $[N, G] > 1$ then, as N is a normal subgroup of G , $[N, G]$ is a normal subgroup of G , and $[N, G] \leq N$, and so, by the minimality of N , we have $[N, G] = N$. We claim that in this case $N \leq (P \cap H') \cdot (P \cap K')$.

Let $n \in N$. Then, as $N = [N, G]$, there exist $m \in N$ and $g \in G$ satisfying $n = [m, g]$. As $G = \langle H, K, L \rangle$, we may write $g = g_1 g_2 \cdots g_k$, with each $g_i \in H, K$, or L .

We show by induction that any such commutator $[m, g_1 \cdots g_k] \in (P \cap H')(P \cap K')$, where each $g_i \in H, K$, or L . This will suffice to

establish the claim above.

For $i = 1$, we have $[m, g_1]$. If $g_1 \in H$ or K , then $[m, g_1] \in N$, since N is normal, and $[m, g_1] \in H'$, or K' since $m \in N \leq P \leq H \cap K$, so $[m, g_1] \in N \cap K' \leq P \cap K'$. If $g_1 \in L$, then $[m, g_1] \in N \cap L$, as each of these subgroups is normal; but they are of relatively prime order, so $[m, g_1] \in N \cap L = 1$. Thus in any case $[m, g_1] \in (P \cap H')(P \cap K')$.

For $i = k$, assume the result holds for $i = k - 1$. Call $h = g_1 \cdots g_{k-1}$. Then $[m, g_1 \cdots g_k] = [m, hg_k] = [m, g_k][m, h][[m, h], g_k]$. As in the case above $[m, g_k] \in (P \cap H')(P \cap K')$, and, since $[m, h] \in N$, we have

$$[[m, h], g_k] \in (P \cap H')(P \cap K')$$

for the same reasons. From our induction hypothesis,

$$[m, h] \in (P \cap H')(P \cap K'), \text{ so } [m, g_1, \dots, g_k] \in (P \cap H')(P \cap K').$$

This completes the induction, and shows that when $[N, G] > 1$, we have $N \leq (P \cap H')(P \cap K')$. Since $N \leq P \cap G'$ as well, (6.3) becomes $P \cap G' = (P \cap H')(P \cap K')$, as required.

The remaining case is $[N, G] = 1$; that is, $N \leq Z(G)$. From (6.3), we can calculate

$$|P \cap G'| = \frac{|F| \cdot |P \cap G' \cap N|}{|N|}$$

and

$$|(P \cap H') \cdot (P \cap K')| = \frac{|F| \cdot |(P \cap H') \cdot (P \cap K') \cap N|}{|N|}$$

Now, to show that $P \cap G' = (P \cap H') \cdot (P \cap K')$, it suffices to show that their orders are equal, and for this, it is clear from the equations above, it suffices to show that, $P \cap G' \cap N = (P \cap H') \cdot (P \cap K') \cap N$. Certainly, $(P \cap H') \cdot (P \cap K') \cap N \leq P \cap G' \cap N$.

A transfer theorem ([5], page 416) states that $P \cap G' \cap Z(G) \leq P'$. But $N \leq Z(G)$, so $P \cap G' \cap N \leq P'$, and $P \leq H$; we have $P' \leq P \cap H'$. Thus, $P \cap G' \cap N \leq P \cap H'$, and so $P \cap G' \cap N = (P \cap H')(P \cap K') \cap N$, as required. This completes the proof of Lemma 6.1.

We can immediately apply this result to obtain a factorization of the focal subgroup from Theorem 1.8.

THEOREM 6.4. *Let G be a group, P a Sylow p -subgroup of G , and W a conjugacy functor satisfying $Z(H) \leq W(H)$ for all $H \in \mathcal{H}$, and V a central functor. Assume that for every local subgroup $L \in \mathcal{L}$, L is p -solvable, and there is a Sylow p -subgroup Q of L such that $L = \langle N(W(Q)), C(V(Q)), O_{p'}(L) \rangle$. Then*

$$P \cap G' = (P \cap N(W(Q)))'(P \cap C(V(Q)))' .$$

Proof. The theorem follows directly from Lemma 6.1 and Theorem 1.8 (1), our result on local control of transfer.

We now quote several theorems which give conditions under which a group factorizes.

DEFINITION 6.5. A group H is *involved* in a group G if there are subgroups K and L of G , with K a normal subgroup of L , such that H is isomorphic to L/K ; that is, if H is a homomorphic image of some subgroup of G .

THEOREM 6.6. (Thompson [6]). *Let G be a p -solvable group, and P a Sylow p -subgroup of G , and suppose one of the following holds:*

- (a) $p \geq 5$
- (b) $p = 3$ and $SL(2, 3)$ is not involved in G .
- (c) $p = 2$ and $SL(2, 2)$ is not involved in G .

Then $G = N_G(J(P)) \cdot C_G(Z(P)) \cdot O_p(G)$.

THEOREM 6.7. (Glauberman [3], p. 19). *Let T be a Sylow 2-subgroup of G , and $C(O_2(G)) \subseteq O_2(G)$, and either*

- (a) $|N(H)/C(H)| < 6$ for every four-group H in $Z(O_2(G))$, or
- (b) the symmetric group on four letters is not involved in G .

Then $G = \langle N(J(T)), C(Z(T)) \rangle$.

THEOREM 6.8. (Goldschmidt [4]). *Suppose P is a Sylow p -subgroup of G , and $(\mathcal{O}^1 Z(P)) \subseteq O_p(G)$. Then $G = N(J(P)) \cdot C(\mathcal{O}^1(Z(P)))$.*

We can now combine any of these three results with Theorem 6.4, and obtain conditions on the local subgroups which imply that the focal subgroup factorizes.

THEOREM 6.9. *Assume that every local subgroup $L \in \mathcal{L}$ is p -solvable, and satisfies the hypotheses of*

- (a) Theorem 6.6.
- (b) Theorem 6.7, for $p = 2$.
- (c) Theorem 6.8.

Then, respectively,

- (a) and (b) $P \cap G' = (P \cap N(J(P)))'(P \cap C(Z(P)))'$, or
- (c) $P \cap G' = (P \cap N(J(P)))'(P \cap C(\mathcal{O}^1(Z(P))))'$.

Proof. Apply Theorem 6.4, with $W(H) = J(H)$, and, in the first two cases, $V(H) = Z(H)$, and $V(H) = \mathcal{O}^1(Z(H))$ in the third case.

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