SIMULTANEOUS APPROXIMATION AND INTERPOLATION IN $L_1$ AND $C(T)$

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Given a dense subspace of $M$ of a Banach space $X$, an element $x$ in $X$ and a finite collection of linear functions in $X^*$, the problem of simultaneous approximation and interpolation is to interpolate $x$ at the given functionals in $X^*$ by an element $m$ of $M$, with the restriction that the norms of $x$ and $m$ be equal and their difference in norm be arbitrarily small. A solution is given for the space $L_1$ with dense subspace, the simple functions in $L_1$, and any collection of functions in $L_1$. In addition the problem is studied in the space $C(T)$, with any dense subalgebra and any finite collection of linear functionals in $C(T)^*$.

In [1] the concept of simultaneous approximation and interpolation which preserves the norm, (SAIN), was introduced.

DEFINITION [1]. Let $X$ be a normed linear space, $M$ a dense subset of $X$, $L$ a finite dimensional subspace of $X^*$. The triple $(X, M, L)$ has property (SAIN) if for every $x$ in $X$ and $\varepsilon > 0$ there exists $y$ in $M$ such that $\|x - y\| < \varepsilon$, $\|x\| = \|y\|$ and $\lambda(x) = \lambda(y)$ for all $\lambda$ in $L$.

Other papers concerned with this topic are [4], [5], and [6].

In [5] it was shown that if $L$ is any finite dimensional subspace of $l_\infty$ and if $M$ is the subspace of $l_1$ consisting of the elements having only finitely many nonzero components, then $(l_1, M, L)$ had property (SAIN). In this paper, we let $M$ be the subspace of simple functions in $L_1$. We show $(L_1, M, T)$ has property (SAIN) for any finite dimensional subspace $T$ in $L_\infty$.

In [1], the space $C(T)$ is studied, where $T$ is a compact Hausdorff space. One finds there

THEOREM 4.1. Let $A$ be a dense subalgebra of $C(T)$ and $t_1, \ldots, t_n$ in $T$. Then $(C(T), A, \{\delta_{t_1}, \ldots, \delta_{t_n}\})$ has property (SAIN). ($\delta_t$ is the linear functional on $C(T)$ given by point evaluation at $t$.)

When arbitrary linear functionals in $C(T)$ are used, examples in [1] show that $(C(T), A, \nu)$ may or may not have property (SAIN) depending on $\nu$.

In this paper we wish to find sufficient conditions on $f$ in $C(T)$ and $M$ dense in $C(T)$ such that given $\nu_1, \ldots, \nu_n$ in $C(T)^*$ and $\varepsilon > 0$ there exists $m$ in $M$ such that $\|f - m\| < \varepsilon$, $\|f\| = \|m\|$ and
In particular one finds that if $f$ attains its norm at most a finite number of times, then any dense subalgebra of $C(T)$ will satisfy these conditions.

In this paper the following notation and terminology is used. $X$ is to denote a real normed linear space. $X^*$ is to denote the continuous dual of $X$, $U(X)$ and $S(X)$, the closed unit ball and its boundary in $X$. A set $E$ contained in a set $F$ is $F$-extremal if whenever $tx + (1 - t)y$ is in $E$, with $0 < t < 1$ and $x, y$ in $F$ then $x, y$ are in $E$. A hyperplane $H$ supports a set $K$, if it bounds $K$ and intersects $K$.

The real valued function $\text{sgn}(\cdot) : \text{Reals} \rightarrow \{-1, 0, 1\}$ is defined via $\text{sgn}(0) = 0$ and $\text{sgn}(x) = x/|x|, x \neq 0$. Then convex hull of a set $A$ is to be denoted by $\text{co}(A)$. All other notation will correspond to that of [3].

1. Minimal closed $U(X)$ extremal subsets.

**Definition 1.1.** $F(x)$ is to denote the minimal closed $U(X)$-extremal set containing $x$. $Q(x)$ is the intersection of all $U(X)$ supporting hyperplanes at $x$.

**Theorem 1.1.** Let $X$ be a normed linear space, $M$ a dense subspace of $X$ and $L = \text{span} \{\varphi_1, \ldots, \varphi\}$ a finite dimensional subspace of $X^*$, and $x$ in $S(X)$. If $F(x) \cap M$ is dense in $F(x)$ then given $\varepsilon > 0$ there exists $m$ in $S(X)$ such that $\|x - m\| < \varepsilon$.

**Proof.** Define the continuous function $\varphi : F(x) \rightarrow \mathbb{R}^n$ via $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$. Assume that $F(x) \subset \varphi_i^{-1}(\varphi_i(x))$ for $i = 0, 1, \ldots, k$ and that this is the largest set of linearly independent elements of $L$ for which this is true. If no such set exists, $k = 0$. In $R^{n-k}$ we assert the existence of $m_\alpha \in F(x) \cap M$ with $\|x - m_\alpha\| < \varepsilon$ such that

$$(\varphi_{k+1}(x), \ldots, \varphi_n(x)) \in \text{co} (\varphi_{k+1}(m_\omega), \ldots, \varphi_n(m_\alpha)|\alpha \in A),$$

$A$ an arbitrary index set. If not, then in $R^{n-k}$ there exists a linear functional $\tau$, a linear combination of the $\varphi_i, i > k$ such that without loss of generality $\tau(m) \leq \tau(x)$ for all $m \in F(x) \cap M$ such that $\|x - m\| < \varepsilon$. But this implies $\tau(m) \leq \tau(x)$ for all $m \in F(x) \cap M$, since if there exists $m_\alpha \in F(x) \cap M$ with $\|x - m_\alpha\| > \varepsilon$ then the set

$$\{y \in F(x) : \tau(y) > \tau(x), \|y - x\| < \varepsilon\}$$

is $F(x)$ relatively open and nonempty (choose a suitable combination
of $x$ and $m_0$) and hence contains $m$ in $F(x) \cap M$ contradicting $\tau(m) \leq \tau(x)$ with $||m - x|| < \varepsilon$. Since $F(x) \cap M$ is dense in $F(x)$ this implies $\tau(y) \leq \tau(x)$ for all $y \in F(x)$. Let $K = \{y \in F(x) | \tau(y) = \tau(x)\}$. $K$ is convex closed and $F(x)$-extremal since $tz + (1 - t)y$ in $K$ implies $t\tau(z) + (1 - t)\tau(y) = \tau(x)$ with $\tau(z) \leq \tau(x), \tau(y) \leq \tau(x)$. Hence $\tau(z) = \tau(y) = \tau(x)$ and $z, y \in K$. Hence $K$ is closed $U(X)$-extremal and $K = F(x)$. Thus $F(x) \subset \tau^{-1}(\tau(x))$. Since $\tau$ is linearly independent of $\varphi_i, i = 1, \cdots, k$, this contradicts the maximal choice of $\varphi_i$ at the start of the proof. Therefore

$$(\varphi_{k+1}(x), \cdots, \varphi_n(x)) \in \text{co} (\varphi_{k+1}(m_a), \cdots, \varphi_n(m_a) \mid \alpha \in A)$$

with $||x - m_a|| < \varepsilon$. This yields the result by the convexity of $M$ and $\varphi(M)$.

In a recent paper of Deutsch and Lindahl [2], they showed that in certain spaces that the set $Q(x)$, the intersection of all $U(X)$ supporting hyperplanes at $x$, is equal to the closure of the minimal extremal subset containing $x$. Thus $Q(x)$ is equal to the minimal closed extremal subset containing $x$. This occurs, in particular [2, Theorem 4.2], if $(T, \Sigma, \nu)$ is a $\sigma$-finite measure space, in $L_1(T, \Sigma, \nu)$. Also, this occurs [2, Theorem 3.3] in the space $C_0(T)$, the space of continuous functions vanishing at infinity, $T$ locally compact.

**Theorem 2.1.** Let $(T, \Sigma, \nu)$ be a $\sigma$-finite measure space with $L^\ast_1(T, \Sigma, \nu) = L_\infty(T, \Sigma, \nu)$. Let $M$ be the dense subspace of $L_1$ consisting of the simple functions. Then $(L_1, M, H)$ has property (SAIN) for any finite dimensional subspace $H$ contained in $L_\infty$.

**Proof.** Given $x$ in $S(L_1)$. By [2, Theorem 4.2], $Q(x) = \{z \in S(L_1) | \int z \text{sgn}(x) = 1\}$ and $Q(x) = F(x)$. $M$ is dense in $Q(x)$ and by Theorem 1.1 the result follows.

**Theorem 2.2.** Let $T$ be a compact Hausdorff space, $C(T)$ the space of real valued continuous functions on $T$. Let $f$ in $S(C(T))$ be such that $Q(f) = \bigcap_{i=1}^n \varphi_i^{-1}(||f||)$ with $\varphi_i$ in rca $(T)$. If

$$(C(T), M, \{\varphi_i | i = 1, \cdots, n\})$$

has property SAIN then given any finite collection $\mu_i$ in rca $(T)$, $\varepsilon > 0$ there exists $m$ in $M$ such that $||f - m|| < \varepsilon, ||f|| = ||m||$ and

$$\int fd\mu_i = \int md\mu_i.$$ 

**Proof.** By [2, Theorem 3.3] $Q(f) = \{x \in C(T) | x(t) = f(t) \text{ for } t \in T \text{ such that } |f(t)| = 1\}$ and $Q(f) = F(f)$. $(C(T), M, \{\varphi_i | i = 1, \cdots, n\})$ having property (SAIN) implies $F(f) \cap M$ is dense in $F(f)$. By Theorem 1.1 the result follows.
Corollary 2.1. Let $f$ be in $S(C(T))$ $\varepsilon > 0$. If $|f|$ attains its norm finitely often then given $\mu_i$, $i = 1, \ldots, n$ in $\text{rca}(T)$ there exists $p$ in $A$ (any dense subalgebra of $C(T)$) such that $\|p - f\| < \varepsilon$ $\|p\| = \|f\|$ and $\int pd\mu_i = \int f d\mu_i$.

Proof. By [1, Theorem 4.1] quoted in the introduction of this article $((C(T), A, \{\delta_t | f(t) | = 1\})$ has property (SAIN). But $Q(f) = \cap \{\delta_t^{-1}(f(t)) | |f(t)| = 1\}$. Hence apply Theorem 2.2.

Acknowledgment. The author is indebted to the referee for his criticism of the initially submitted article and for his suggestions for revision.

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Received November 21, 1971 and in revised form April 17, 1972.

The Pennsylvania State University
Pacific Journal of Mathematics  
Vol. 45, No. 1  
September, 1973

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