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**SIMULTANEOUS APPROXIMATION AND INTERPOLATION IN  
 $L_1$  AND  $C(T)$**

JOSEPH MICHAEL LAMBERT

## SIMULTANEOUS APPROXIMATION AND INTERPOLATION IN $L_1$ AND $C(T)$

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Given a dense subspace of  $M$  of a Banach space  $X$ , an element  $x$  in  $X$  and a finite collection of linear functions in  $X^*$ , the problem of simultaneous approximation and interpolation is to interpolate  $x$  at the given functionals in  $X^*$  by an element  $m$  of  $M$ , with the restriction that the norms of  $x$  and  $m$  be equal and their difference in norm be arbitrarily small. A solution is given for the space  $L_1$  with dense subspace, the simple functions in  $L_1$ , and any collection of functions in  $L_\infty$ . In addition the problem is studied in the space  $C(T)$ , with any dense subalgebra and any finite collection of linear functionals in  $C(T)^*$ .

In [1] the concept of simultaneous approximation and interpolation which preserves the norm, (SAIN), was introduced.

DEFINITION [1]. Let  $X$  be a normed linear space,  $M$  a dense subset of  $X$ ,  $L$  a finite dimensional subspace of  $X^*$ . The triple  $(X, M, L)$  has property (SAIN) if for every  $x$  in  $X$  and  $\varepsilon > 0$  there exists  $y$  in  $M$  such that  $\|x - y\| < \varepsilon$ ,  $\|x\| = \|y\|$  and  $\lambda(x) = \lambda(y)$  for all  $\lambda$  in  $L$ .

Other papers concerned with this topic are [4], [5], and [6].

In [5] it was shown that if  $L$  is any finite dimensional subspace of  $l_\infty$  and if  $M$  is the subspace of  $l_1$  consisting of the elements having only finitely many nonzero components, then  $(l_1, M, L)$  had property (SAIN). In this paper, we let  $M$  be the subspace of simple functions in  $L_1$ . We show  $(L_1, M, T)$  has property (SAIN) for any finite dimensional subspace  $T$  in  $L_\infty$ .

In [1], the space  $C(T)$  is studied, where  $T$  is a compact Hausdorff space. One finds there

THEOREM 4.1. *Let  $A$  be a dense subalgebra of  $C(T)$  and  $t_1, \dots, t_n$  in  $T$ . Then  $(C(T), A, \{\delta_{t_1}, \dots, \delta_{t_n}\})$  has property (SAIN). ( $\delta_t$  is the linear functional on  $C(T)$  given by point evaluation at  $t$ .)*

When arbitrary linear functionals in  $C(T)$  are used, examples in [1] show that  $(C(T), A, \{\nu\})$  may or may not have property (SAIN) depending on  $\nu$ .

In this paper we wish to find sufficient conditions on  $f$  in  $C(T)$  and  $M$  dense in  $C(T)$  such that given  $\{\nu_1, \dots, \nu_n\}$  in  $C(T)^*$  and  $\varepsilon > 0$  there exists  $m$  in  $M$  such that  $\|f - m\| < \varepsilon$ ,  $\|f\| = \|m\|$  and

$$\int f d\nu_i = \int m d\nu_i, i = 1, \dots, n .$$

In particular one finds that if  $f$  attains its norm at most a finite number of times, then any dense subalgebra of  $C(T)$  will satisfy these conditions.

In this paper the following notation and terminology is used.  $X$  is to denote a real normed linear space.  $X^*$  is to denote the continuous dual of  $X$ ,  $U(X)$  and  $S(X)$ , the closed unit ball and its boundary in  $X$ . A set  $E$  contained in a set  $F$  is  $F$ -extremal if whenever  $tx + (1 - t)y$  is in  $E$ , with  $0 < t < 1$  and  $x, y$  in  $F$  then  $x, y$  are in  $E$ . A hyperplane  $H$  supports a set  $K$ , if it bounds  $K$  and intersects  $K$ . The real valued function  $\text{sgn}(\cdot): \text{Reals} \rightarrow \{-1, 0, 1\}$  is defined via  $\text{sgn}(0) = 0$  and  $\text{sgn}(x) = x/|x|, x \neq 0$ . Then convex hull of a set  $A$  is to be denoted by  $\text{co}(A)$ . All other notation will correspond to that of [3].

### 1. Minimal closed $U(X)$ extremal subsets.

**DEFINITION 1.1.**  $F(x)$  is to denote the minimal closed  $U(X)$ -extremal set containing  $x$ .  $Q(x)$  is the intersection of all  $U(X)$  supporting hyperplanes at  $x$ .

**THEOREM 1.1.** *Let  $X$  be a normed linear space,  $M$  a dense subspace of  $X$  and  $L = \text{span} \{\varphi_1, \dots, \varphi_k\}$  a finite dimensional subspace of  $X^*$ , and  $x$  in  $S(X)$ . If  $F(x) \cap M$  is dense in  $F(x)$  then given  $\varepsilon > 0$  there exists  $m$  in  $S(X)$  such that  $\varphi_i(x) = \varphi_i(m), i = 1, \dots, k$  and  $\|x - m\| < \varepsilon$ .*

*Proof.* Define the continuous function  $\varphi: F(x) \rightarrow R^k$  via  $\varphi(x) = (\varphi_1(x), \dots, \varphi_k(x))$ . Assume that  $F(x) \subset \varphi_i^{-1}(\varphi_i(x))$  for  $i = 0, 1, \dots, k$  and that this is the largest set of linearly independent elements of  $L$  for which this is true. If no such set exists,  $k = 0$ . In  $R^{n-k}$  we assert the existence of  $m_\alpha \in F(x) \cap M$  with  $\|x - m_\alpha\| < \varepsilon$  such that

$$(\varphi_{k+1}(x), \dots, \varphi_n(x)) \in \text{co}(\varphi_{k+1}(m_\alpha), \dots, \varphi_n(m_\alpha) | \alpha \in A) ,$$

A an arbitrary index set. If not, then in  $R^{n-k}$  there exists a linear functional  $\tau$ , a linear combination of the  $\varphi_i, i > k$  such that without loss of generality  $\tau(m) \leq \tau(x)$  for all  $m \in F(x) \cap M$  such that  $\|x - m\| < \varepsilon$ . But this implies  $\tau(m) \leq \tau(x)$  for all  $m \in F(x) \cap M$ , since if there exists  $m_0 \in F(x) \cap M$  with  $\|x - m_0\| > \varepsilon$  then the set

$$\{y \in F(x) | \tau(y) > \tau(x), \|y - x\| < \varepsilon\}$$

is  $F(x)$  relatively open and nonempty (choose a suitable combination

of  $x$  and  $m_0$ ) and hence contains  $m$  in  $F(x) \cap M$  contradicting  $\tau(m) \leq \tau(x)$  with  $\|m - x\| < \varepsilon$ . Since  $F(x) \cap M$  is dense in  $F(x)$  this implies  $\tau(y) \leq \tau(x) \forall y \in F(x)$ . Let  $K = \{y \in F(x) \mid \tau(y) = \tau(x)\}$ .  $K$  is convex closed and  $F(x)$ -extremal since  $tz + (1 - t)y$  in  $K$  implies  $t\tau(z) + (1 - t)\tau(y) = \tau(x)$  with  $\tau(z) \leq \tau(x), \tau(y) \leq \tau(x)$ . Hence  $\tau(z) = \tau(y) = \tau(x)$  and  $z, y \in K$ . Hence  $K$  is closed  $U(X)$ -extremal and  $K = F(x)$ . Thus  $F(x) \subset \tau^{-1}(\tau(x))$ . Since  $\tau$  is linearly independent of  $\varphi_i, i = 1, \dots, k$ , this contradicts the maximal choice of  $\varphi_i$  at the start of the proof. Therefore

$$(\varphi_{k+1}(x), \dots, \varphi_n(x)) \in \text{co}(\varphi_{k+1}(m_\alpha), \dots, \varphi_n(m_\alpha) \mid \alpha \in A)$$

with  $\|x - m_\alpha\| < \varepsilon$ . This yields the result by the convexity of  $M$  and  $\varphi(M)$ .

In a recent paper of Deutsch and Lindahl [2], they showed that in certain spaces that the set  $Q(x)$ , the intersection of all  $U(X)$  supporting hyperplanes at  $x$ , is equal to the closure of the minimal extremal subset containing  $x$ . Thus  $Q(x)$  is equal to the minimal closed extremal subset containing  $x$ . This occurs, in particular [2, Theorem 4.2], if  $(T, \Sigma, \nu)$  is a  $\sigma$ -finite measure space, in  $L_1(T, \Sigma, \nu)$ . Also, this occurs [2, Theorem 3.3] in the space  $C_0(T)$ , the space of continuous functions vanishing at infinity,  $T$  locally compact.

**THEOREM 2.1.** *Let  $(T, \Sigma, \nu)$  be a  $\sigma$ -finite measure space with  $L_1^*(T, \Sigma, \nu) = L_\infty(T, \Sigma, \nu)$ . Let  $M$  be the dense subspace of  $L_1$  consisting of the simple functions. Then  $(L_1, M, H)$  has property (SAIN) for any finite dimensional subspace  $H$  contained in  $L_\infty$ .*

*Proof.* Given  $x$  in  $S(L_1)$ . By [2, Theorem 4.2],  $Q(x) = \{z \in S(L_1) \mid \int z \text{sgn}(x) = 1\}$  and  $Q(x) = F(x)$ .  $M$  is dense in  $Q(x)$  and by Theorem 1.1 the result follows.

**THEOREM 2.2.** *Let  $T$  be a compact Hausdorff space,  $C(T)$  the space of real valued continuous functions on  $T$ . Let  $f$  in  $S(C(T))$  be such that  $Q(f) = \bigcap_{i=1}^r \varphi_i^{-1}(\|f\|)$  with  $\varphi_i$  in  $\text{rca}(T)$ . If*

$$(C(T), M, \{\varphi_i \mid i = 1, \dots, n\})$$

*has property SAIN then given any finite collection  $\mu_i$  in  $\text{rca}(T)$ ,  $\varepsilon > 0$  there exists  $m$  in  $M$  such that  $\|f - m\| < \varepsilon, \|f\| = \|m\|$  and  $\int f d\mu_i = \int m d\mu_i$ .*

*Proof.* By [2, Theorem 3.3]  $Q(f) = \{x \in C(T) \mid x(t) = f(t) \text{ for } t \in T \text{ such that } |f(t)| = 1\}$  and  $Q(f) = F(f)$ .  $(C(T), M, \{\varphi_i \mid i = 1, \dots, n\})$  having property (SAIN) implies  $F(f) \cap M$  is dense in  $F(f)$ . By Theorem 1.1 the result follows.

**COROLLARY 2.1.** *Let  $f$  be in  $S(C(T))$   $\varepsilon > 0$ . If  $|f|$  attains its norm finitely often then given  $\mu_i, i = 1, \dots, n$  in  $rca(T)$  there exists  $p$  in  $A$  (any dense subalgebra of  $C(T)$ ) such that  $\|p - f\| < \varepsilon \|p\| = \|f\|$  and  $\int pd\mu_i = \int fd\mu_i$ .*

*Proof.* By [1, Theorem 4.1] quoted in the introduction of this article  $((C(T), A, \{\delta_t | f(t) | = 1\})$  has property (SAIN). But  $Q(f) = \cap \{\delta_t^{-1}(f(t) | |f(t) | = 1\}$ . Hence apply Theorem 2.2.

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