MULTIPLIERS OF TYPE $(p, p)$ AND MULTIPLIERS OF THE GROUP $L_p$-ALGEBRAS

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Let \(G\) be a locally compact group with left Haar measure \(\lambda\) and suppose \(1 \leq p < \infty\). The purpose of this paper is to exhibit an isometric isomorphism \(\omega\) of the Banach algebra \(M_p\) of all right multipliers on \(L_p = L_p(G, \lambda)\) into the normed algebra \(m_p\) of all right multipliers on the group \(L_p\)-algebra \(L_p\). When \(G\) is either commutative or compact, \(\omega\) is surjective.

A function \(f \in L_p\) is said to be \(p\)-temperate if

1. \(h*f(x) = \int_x f(t)h(t^{-1}x)d\lambda(t)\) exists for \(\lambda\)-almost all \(x \in G\) whenever \(h\) is in \(L_p\);
2. \(h*f\) is in \(L_p\) for all \(h \in L_p\);
3. \(\text{sup} \{||h*f||_p; h \in L_p, ||h||_p \leq 1\} < \infty\).

It was shown in [6], Theorem 1, that \(f \in L_p\) is \(p\)-temperate if

4. \(\text{sup} \{||h*f||_p; h \in C_\infty, ||h||_p \leq 1\} < \infty\)

where \(C_\infty\) denotes the set of all continuous complex-valued functions on \(G\) with compact support. The set of all \(p\)-temperate functions will be written as \(L_p^t\). Each function \(f \in C_\infty\) is in \(L_p^t\) and so \(L_p^t\) comprises a dense subspace of \(L_p\). For \(f \in L_p^t\), the number given by either (3) or (4) will be written as \(||f||_p^t\). The function \(||\ ||_p\) so defined is a norm under which \(L_p^t\) is a normed algebra. This normed algebra will be referred to as the group \(L_p\)-algebra.

By a right multiplier on \(L_p\) will be meant a bounded linear operator \(T\) on \(L_p\) such that

5. \(T(f*g) = f*T(g)\) for all \(f\) and \(g\) in \(L_p\).

The set of all such \(T\), which constitutes a normed algebra under the usual operator norm, will be written as \(M_p\). Write \(B_p\) for the Banach algebra of all bounded linear operators on \(L_p\). An operator \(T \in B_p\) is said to be a right multiplier of type \((p, p)\) (see [3]) if

6. \(T(x*f) = x*T(f)\) for all \(f \in L_p\)

where \(xh(y) = h(xy)\) for each function \(h\) on \(G\). The set of all such \(T\) will be written as \(M_p\). It is a complete sub-algebra of \(B_p\).

The group \(L_p\)-algebra was utilized in [6] to study a related algebra \(A_p\), of which the Banach algebra of left multipliers was found...
to be isomorphic to $M_p$. The situation is reversed here. For $f \in L_p$, an operator $W_f$ in $B_p$ is defined by

$$W_f(g) = g \ast f \quad \text{for all } g \in L_p$$

and, consequently,

$$\|W_f\| = \|f\|_p^*.$$ 

The closure in $B_p$ of the linear span of the set $\{W_fg: f \in L_p, g \in C_0\}$ will be written as $A_p$. It is a Banach algebra with a minimal left approximate identity ([6], Theorem 3). Concrete interpretations of both $A_p$ and $L_p^1$, in the cases where $G$ is either commutative or compact, may be found in [6]. It will be mentioned here only that $L_1$ is the group algebra $L_i$ and that $L_1^i$ is the group Hilbert algebra (see [1] and [2] for example).

**Proposition 1.** Let $T$ be in $M_p$ and $f$ and $g$ be in $L_p$. Then

(i) $T(f \ast g) = f \ast T(g)$ if $f \in L_1$;
(ii) $T(g)$ is in $L^p_\infty$ if $g$ is in $L^p$;
(iii) $T(f \ast g) = f \ast T(g)$ if $g$ is in $L^p_\infty$.

**Proof.** Part (i) was proved in the corollary to Theorem 4 in [6]. Let $g$ be in $L_\infty$. By (i),

$$\sup \{ \| h \ast T(g) \|_p^*: h \in C_0, \| h \|_p \leq 1 \} = \sup \{ \| T(h \ast g) \|_p^*: h \in C_0, \| h \|_p \leq 1 \} \leq \| T \| \cdot \| g \|_p^*.$$ 

By (4), this implies that $T(g)$ is in $L_p^1$.

Let again $g$ be in $L_p^1$ and choose a sequence $\{f_n\}$ in $C_\infty$ which converges to $f$ in $L_p$. Then

$$\lim_n \| f_n \ast g - f \ast g \|_p = 0 \quad \text{and, in view of (ii),}$$

$$\lim_n \| f_n \ast T(g) - f \ast T(g) \|_p = 0. \quad \text{Thus, by (i),}$$

$$f \ast T(g) = \lim_n f_n \ast T(g) = \lim_n T(f_n \ast g) = T(f \ast g).$$

**Lemma 1.** For each nonzero $f \in L_p$, there exists $g \in C_\infty$ for which $g \ast f \neq 0$.


**Lemma 2.** For each $T \in m_p$ and $V \in A_p$,

$$\sup \{ \| T \circ V(h) \|_p^*: h \in L_p, \| h \|_p \leq 1 \} \leq \| T \| \cdot \| V \|.$$ 

**Proof.** Write $D$ for the set $\{W_f: f \in L_p, W_f \in A_p\}$. Then $D$ is a dense subspace of $A_p$ and, by (8), $\|W_f\| = \|f\|_p^*$ for all $W_f \in D$. 

Hence, if $\rho' \mid D \to \mathcal{B}p$ is defined by $\rho'(W_f) = W_{T(f)}$ for all $W_f \in D$, then $\rho'$ is continuous. Let $\rho \mid A_p \to \mathcal{B}$ be the unique continuous extension of $\rho$ to $A_p$. The immediate object is to show that $\rho(V)$ and $T \circ V$ coincide on $L^1_p$.

Let $h \in L^1_p$ be such that $\|h\|_p \leq 1$ and let $\{f_n\}$ be a sequence in $L^1_p$ such the $W_{f_n}$ is in $D$ for each $n \in N$ and $\lim_n \|W_{f_n} - V\| = 0$. Since $A_p$ is a subset of $M_p$, the operator $V$ is in $M_p$ and so, by Proposition 1.iii,

$$V \circ W_h(g) = V(g \ast h) = g \ast V(h) = W_{V(h)}(g)$$

for all $g \in L^1_p$; hence, $V \circ W_h = W_{V(h)}$. That $W_{W_{f_n}}(h) = W_{f_n} \circ W_h$ is easy to check. Thus, for each $n \in N$, (8) yields $\|W_{f_n}(h) - V(h)\|_p = \|W_{f_n} \circ W_h - V \circ W_h\|$. Hence,

$$\lim_n \|W_{f_n}(h) - V(h)\|_p \leq \|W_{f_n} - V\| \cdot \|W_h\| = 0.$$

Consequently,

$$(9) \quad \lim_n \|T(W_{f_n}(h)) - T(V(h))\|_p = 0.$$ 

For each $n \in N$ and $g \in L^1_p$, $W_{T(f_n)}(g) = g \ast T(f_n) = T(g \ast f_n) = T \circ W_{f_n}(g)$; hence, $\rho(W_{f_n}) = \rho'(W_{f_n}) = W_{T(f_n)} = T \circ W_{f_n}$. Consequently

$$\lim_n \|T \circ W_{f_n} - \rho(V)\| = \lim_n \|\rho(W_{f_n}) - \rho(V)\| = 0.$$

Thus

$$\lim_n \|T \circ W_{f_n}(h) - [\rho(V)](h)\|_p = 0 \quad \text{and so} \quad \lim_n \|g \ast (T \circ W_{f_n}(h)) - g \ast [\rho(V)](h)\|_p = 0$$

for each $g \in C_\infty$. But, by (9),

$$\lim_n \|g \ast (T \circ W_{f_n}(h)) - g \ast (T(V(h)))\|_p = 0$$

for all $g \in C_\infty$. It follows that $g \ast [\rho(V)](h) = g \ast (T(V(h)))$ for all $g \in C_\infty$. By Lemma 1, this yields that

$$[\rho(V)](h) = T(V(h)).$$

Now

$$\|T \circ V(h)\|_p = \|[\rho(V)](h)\|_p = \lim_n \|[\rho(W_{f_n})](h)\|_p$$

$$= \lim_n \|h \ast T(f_n)\|_p \leq \|h\|_p \cdot \lim_n \|T(f_n)\|_p$$

$$\leq (\text{since } \|h\|_p \leq 1 \text{ and because of (8))}$$

$$\|T\| \cdot \lim_n \|f_n\|_p = \|T\| \cdot \lim_n \|W_{f_n}\| = \|T\| \cdot \|V\|.$$
**PROPOSITION 2.** For each $T \in m_p$, $V \in A_p$, and $f \in L^t_p$,

$$\| T(V(f)) \|_p \leq \| T \| \cdot \| V(f) \|_p.$$  

**Proof.** Let $\varepsilon$ be any positive number. Since $A_p$ is a Banach algebra with a minimal left approximate identity, Cohen's factorization theorem ([5] 32.28) implies that there exist $P$ and $S$ in $A_p$ such that $\| P \| = 1$, $\| S - V \| < \varepsilon$, and $V = PS$. Thus, $\| S(f) \|_p \leq \| V(f) \|_p + \varepsilon \cdot \| f \|_p$ and, by Lemma 2,

$$\| T(V(f)) \|_p = \| T_0 P(S(f)) \|_p$$

$$\leq \| T \| \cdot \| P \| \cdot \| S(f) \|_p = \| T \| (\| V(f) \|_p + \varepsilon \| f \|_p).$$

It follows that $\| T(V(f)) \|_p \leq \| T \| \cdot \| V(f) \|_p$.

**LEMMA 3.** The set $\{V(f): f \in L^t_p, V \in A_p\}$ is a dense subspace of $L_p$.

**Proof.** Let $\varepsilon$ be a positive number and $g$ be in $L_p$. Choose $f \in C_\infty$ such that $\| g - f \|_p < \varepsilon/2$. If $\{V_a\}$ is a minimal left approximate identity for $A_p$, it follows from [6], Lemma 3, that $\lim_{a} \| V_a(f) - f \|_p = 0$. Thus, for some index $a$, $\| V_a(f) - f \|_p < \varepsilon/2$ and so $\| V_a(f) - g \|_p < \varepsilon$.

**LEMMA 4.** Let $V$ be in $S_p$ and $D$ a dense subset of $L_p$ such that $V(h \ast f) = h \ast V(f)$ for all $h \in C_\infty$ and $f \in D$. Then $V$ is in $M_p$.

**Proof.** Let $x$ be in $G$. By [4] 20.15, there is a net $\{f_a\}$ in $C_\infty$ such that $\lim_{a} \| x - f_a \ast h \|_p = 0$ for all $h \in L_p$. It follows that $\lim_{a} \| V_a(h) - V(f_a \ast h) \|_p = 0$ and $\lim_{a} \| x V(h) - f_a \ast V(h) \|_p = 0$. Hence, for $h \in D$

$$\| V_x(h) - V(h) \|_p = \lim_{a} \| V(f_a \ast h) - f_a \ast V(h) \|_p = \lim \| 0$$

by the hypothesis for $V$. Since $D$ is dense in $L_p$, $V$ is in $M_p$.

**THEOREM 1.** Define $\omega \mid M_p \to m_p$ by letting $\omega_T(f) = T(f)$ for each $T \in M_p$ and $f \in L^t_p$. Then $\omega$ is an isometric isomorphism of $M_p$ into $m_p$. Furthermore, if $T$ is any operator in $m_p$, then there exists some $S \in M_p$ such that, for all $V \in A_p$ and $f \in L^t_p$, $\omega(V(f)) = T(V(f))$.

**Proof.** That $\omega$ is well-defined follows from Proposition 1. That $\omega$ is an isomorphism is evident when it is noted that $L^t_p$ is a dense subset of $L_p$.

Let $T$ be an arbitrary element of $m_p$. It follows from Proposition 2 and Lemma 3 that there exists a unique operator $S$ in $S_p$ such that $S(V(f)) = T(V(f))$ for all $V \in A_p$ and $f \in L^t_p$. For $h \in C_\infty$, $V \in A_p$, and $f \in L^t_p$, Proposition 1 implies
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\[ S(h \ast V(f)) = S(V(h \ast f)) = T(V(h \ast f)) \]
\[ = T(h \ast V(f)) = h \ast T(V(f)) = h \ast S(V(f)). \]

By Lemmas 3 and 4, this implies that \(S\) is in \(M_p\). Consequently, \(\omega_s(V(h)) = S(V(h)) = T(V(h))\) for all \(h \in L'_p\) and \(V \in A_p\).

To complete this proof, it will now suffice to show that \(\omega\) is an isometry. Let \(T\) be in \(M_p\). Let \(f\) be in \(L'_p\) and \(\varepsilon\) a positive number. Choose \(g \in L'_p\) for which \(\|g\|_p \leq 1\) and \(\|\omega_r(f)\|_p < \|g \ast \omega_r(f)\|_p + \varepsilon\).

By Proposition 1.iii, \(T(g \ast f) = g \ast T(f)\); this means that \(T \circ W_r(g) = g \ast \omega_r(f)\). Hence,

\[ \|\omega_r(f)\|_p < \|T \circ W_r(g)\|_p + \varepsilon \leq \|T\| \cdot \|W_r\| + \varepsilon. \]

By (8), this implies \(\|\omega_r(f)\|_p \leq \|T\| \cdot \|f\|_p\). Hence

\[ \|\omega_r\| \leq \|T\|. \]

On the other hand, Proposition 2 and Lemma 3 imply

\[ \|T\| = \sup \{\|T(V(h))\|_p : V \in A_p, h \in L'_p, \|V(h)\|_p \leq 1\} \]
\[ = \sup \{\|\omega_r(V(h))\|_p : V \in A_p, h \in L'_p, \|V(h)\|_p \leq 1\} \leq \|\omega_r\|. \]

This proves that \(\|T\| = \|\omega_r\|\).

**Theorem 2.** Let \(\omega\) be as in Theorem 1 and \(G\) be either commutative or compact. Then \(\omega\) is surjective.

**Proof.** Let \(T\) be any operator in \(m_p\). By Theorem 1, there is an operator \(S\) in \(M_p\) for which \(T(V(f)) = \omega_s(V(f))\) for all \(V \in A_p\) and \(f \in L_p\).

If \(G\) is compact, then \(L'_p = L_p\). It follows from the Hewitt-Curtis-Figa Talamanca factorization theorem ([5] 32.22) that each \(h \in L'_p\) is of the form \(V(f)\) for some \(V \in A_p\) and \(f \in L_p\). Hence, \(T = \omega_s\).

Suppose now that \(G\) is commutative (not necessarily compact). Assume that there existed \(h \in L'_p\) such that \(\omega_s(h) \neq T(h)\). Then Lemma 1 implies that \(g \ast (\omega_s - T)(h) \neq 0\) for some \(g \in C_{\infty}\). Let \(\{h_n\}\) be a sequence in \(C_{\infty}\) for which \(\lim_n \|h_n - h\|_p = 0\). Then

\[ \|g \ast (\omega_s - T)(h)\|_p \]
\[ = \|(\omega_s - T)(g \ast h)\|_p = \|(\omega_s - T)(h \ast g)\|_p \]
\[ = \|h \ast (\omega_s - T)(g)\|_p = \lim_n \|h_n \ast (\omega_s - T)(g)\|_p \]
\[ = \lim_n \|\omega_s - T)(h_n \ast g)\|_p = \lim_n \|\omega_s - T)(W_{h_n}(g))\|_p = 0 \]

a contradiction. Thus, \(\omega_s = T\).
REFERENCES


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WASHINGTON STATE UNIVERSITY
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