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**MULTIPLIERS OF TYPE (p, p) AND MULTIPLIERS OF THE
GROUP L_p -ALGEBRAS**

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Let G be a locally compact group with left Haar measure λ and suppose $1 \leq p < \infty$. The purpose of this paper is to exhibit an isometric isomorphism ω of the Banach algebra M_p of all right multipliers on $L_p = L_p(G, \lambda)$ into the normed algebra m_p of all right multipliers on the group L_p -algebra L_p^t . When G is either commutative or compact, ω is surjective.

A function $f \in L_p$ is said to be p -temperate if

- (1) $h * f(x) = \int_G f(t)h(t^{-1}x)d\lambda(t)$ exists for λ -almost all $x \in G$ whenever h is in L_p ;
- (2) $h * f$ is in L_p for all $h \in L_p$;
- (3) $\sup \{ \|h * f\|_p : h \in L_p, \|h\|_p \leq 1 \} < \infty$.

It was shown in [6], Theorem 1, that $f \in L_p$ is p -temperate if

$$(4) \quad \sup \{ \|h * f\|_p : h \in C_{00}, \|h\|_p \leq 1 \} < \infty$$

where C_{00} denotes the set of all continuous complex-valued functions on G with compact support. The set of all p -temperate functions will be written as L_p^t . Each function $f \in C_{00}$ is in L_p^t and so L_p^t comprises a dense subspace of L_p . For $f \in L_p^t$, the number given by either (3) or (4) will be written as $\|f\|_p^t$. The function $\|\cdot\|_p^t$ so defined is a norm under which L_p^t is a normed algebra. This normed algebra will be referred to as the *group L_p -algebra*.

By a right multiplier on L_p^t will be meant a bounded linear operator T on L_p^t such that

$$(5) \quad T(f * g) = f * T(g) \quad \text{for all } f \text{ and } g \text{ in } L_p^t.$$

The set of all such T , which constitutes a normed algebra under the usual operator norm, will be written as m_p . Write \mathfrak{B}_p for the Banach algebra of all bounded linear operators on L_p . An operator $T \in \mathfrak{B}_p$ is said to be a right multiplier of type (p, p) (see [3]) if

$$(6) \quad T({}_x f) = {}_x T(f) \quad \text{for all } f \in L_p$$

where ${}_x h(y) = h(xy)$ for each function h on G . The set of all such T will be written as M_p . It is a complete sub-algebra of \mathfrak{B}_p .

The group L_p -algebra was utilized in [6] to study a related algebra A_p , of which the Banach algebra of left multipliers was found

to be isomorphic to M_p . The situation is reversed here. For $f \in L_p^t$, an operator W_f in \mathfrak{B}_p is defined by

$$(7) \quad W_f(g) = g * f \quad \text{for all } g \in L_p$$

and, consequently,

$$(8) \quad \|W_f\| = \|f\|_p^t.$$

The closure in \mathfrak{B}_p of the linear span of the set $\{W_{f * g} : f \in L_p^t, g \in C_\infty\}$ will be written as A_p . It is a Banach algebra with a minimal left approximate identity ([6], Theorem 3). Concrete interpretations of both A_p and L_p^t , in the cases where G is either commutative or compact, may be found in [6]. It will be mentioned here only that L_1^t is the group algebra L_1 and that L_2^t is the group Hilbert algebra (see [1] and [2] for example).

PROPOSITION 1. *Let T be in M_p and f and g be in L_p . Then*

- (i) $T(f * g) = f * T(g)$ if $f \in L_1$;
- (ii) $T(g)$ is in L_p^t if g is in L_p^t ;
- (iii) $T(f * g) = f * T(g)$ if g is in L_p^t .

Proof. Part (i) was proved in the corollary to Theorem 4 in [6]. Let g be in L_p^t . By (i),

$$\begin{aligned} & \sup \{ \|h * T(g)\|_p : h \in C_{00}, \|h\|_p \leq 1 \} \\ &= \sup \{ \|T(h * g)\|_p : h \in C_{00}, \|h\|_p \leq 1 \} \leq \|T\| \cdot \|g\|_p^t. \end{aligned}$$

By (4), this implies that $T(g)$ is in L_p^t .

Let again g be in L_p^t and choose a sequence $\{f_n\}$ in C_∞ which converges to f in L_p . Then

$$\begin{aligned} \lim_n \|f_n * g - f * g\|_p &= 0 \text{ and, in view of (ii),} \\ \lim_n \|f_n * T(g) - f * T(g)\|_p &= 0. \text{ Thus, by (i),} \\ f * T(g) &= \lim_n f_n * T(g) = \lim_n T(f_n * g) = T(f * g). \end{aligned}$$

LEMMA 1. *For each nonzero $f \in L_p$, there exists $g \in C_\infty$ for which $g * f \neq 0$.*

Proof. See [4] 20.15.

LEMMA 2. *For each $T \in m_p$ and $V \in A_p$,*

$$\sup \{ \|T \circ V(h)\|_p : h \in L_p^t, \|h\|_p \leq 1 \} \leq \|T\| \cdot \|V\|.$$

Proof. Write D for the set $\{W_f : f \in L_p^t, W_f \in A_p\}$. Then D is a dense subspace of A_p and, by (8), $\|W_f\| = \|f\|_p^t$ for all $W_f \in D$.

Hence, if $\rho' | D \rightarrow \mathfrak{B}_p$ is defined by $\rho'(W_f) = W_{T(f)}$ for all $W_f \in D$, then ρ' is continuous. Let $\rho | A_p \rightarrow \mathfrak{B}_p$ be the unique continuous extension of ρ to A_p . The immediate object is to show that $\rho(V)$ and $T \circ V$ coincide on L_p^t .

Let $h \in L_p^t$ be such that $\|h\|_p \leq 1$ and let $\{f_n\}$ be a sequence in L_p^t such the W_{f_n} is in D for each $n \in N$ and $\lim_n \|W_{f_n} - V\| = 0$. Since A_p is a subset of M_p , the operator V is in M_p and so, by Proposition 1.iii,

$$V \circ W_h(g) = V(g * h) = g * V(h) = W_{V(h)}(g)$$

for all $g \in L_p$; hence, $V \circ W_h = W_{V(h)}$. That $W_{W_{f_n}}(h) = W_{f_n} \circ W_h$ is easy to check. Thus, for each $n \in N$, (8) yields $\|W_{f_n}(h) - V(h)\|_p^t = \|W_{f_n} \circ W_h - V \circ W_h\|$. Hence,

$$\overline{\lim}_n \|W_{f_n}(h) - V(h)\|_p^t \leq \overline{\lim}_n \|W_{f_n} - V\| \cdot \|W_h\| = 0.$$

Consequently,

$$(9) \quad \lim_n \|T(W_{f_n}(h)) - T(V(h))\|_p^t = 0.$$

For each $n \in N$ and $g \in L_p^t$, $W_{T(f_n)}(g) = g * T(f_n) = T(g * f_n) = T \circ W_{f_n}(g)$; hence, $\rho(W_{f_n}) = \rho'(W_{f_n}) = W_{T(f_n)} = T \circ W_{f_n}$. Consequently

$$\overline{\lim}_n \|T \circ W_{f_n} - \rho(V)\| = \lim_n \|\rho(W_{f_n}) - \rho(V)\| = 0.$$

Thus

$$\begin{aligned} \lim_n \|T \circ W_{f_n}(h) - [\rho(V)](h)\|_p &= 0 \quad \text{and so} \\ \lim_n \|g * (T \circ W_{f_n}(h)) - g * [\rho(V)](h)\|_p &= 0 \end{aligned}$$

for each $g \in C_\infty$. But, by (9),

$$\lim_n \|g * (T \circ W_{f_n}(h)) - g * (T(V(h)))\|_p = 0$$

for all $g \in C_\infty$. It follows that $g * [\rho(V)](h) = g * (T(V(h)))$ for all $g \in C_\infty$. By Lemma 1, this yields that

$$[\rho(V)](h) = T(V(h)).$$

Now

$$\begin{aligned} \|T \circ V(h)\|_p &= \|[\rho(V)](h)\|_p = \lim_n \|[\rho(W_{f_n})](h)\|_p \\ &= \lim_n \|h * T(f_n)\|_p \leq \|h\|_p \cdot \overline{\lim}_n \|T(f_n)\|_p^t \\ &\leq (\text{since } \|h\|_p \leq 1 \text{ and because of (8)}) \\ &\quad \|T\| \cdot \overline{\lim}_n \|f_n\|_p^t = \|T\| \cdot \overline{\lim}_n \|W_{f_n}\| = \|T\| \cdot \|V\|. \end{aligned}$$

PROPOSITION 2. For each $T \in m_p$, $V \in A_p$, and $f \in L_p^t$,

$$\|T(V(f))\|_p \leq \|T\| \cdot \|V(f)\|_p.$$

Proof. Let ε be any positive number. Since A_p is a Banach algebra with a minimal left approximate identity, Cohen's factorization theorem ([5] 32.26) implies that there exist P and S in A_p such that $\|P\| = 1$, $\|S - V\| < \varepsilon$, and $V = PS$. Thus, $\|S(f)\|_p \leq \|V(f)\|_p + \varepsilon \cdot \|f\|_p$ and, by Lemma 2,

$$\begin{aligned} \|T(V(f))\|_p &= \|T \circ P(S(f))\|_p \\ &\leq \|T\| \cdot \|P\| \cdot \|S(f)\|_p = \|T\| (\|V(f)\|_p + \varepsilon \|f\|_p). \end{aligned}$$

It follows that $\|T(V(f))\|_p \leq \|T\| \cdot \|V(f)\|_p$.

LEMMA 3. The set $\{V(f): f \in L_p^t, V \in A_p\}$ is a dense subspace of L_p .

Proof. Let ε be a positive number and g be in L_p . Choose $f \in C_0$ such that $\|g - f\|_p < \varepsilon/2$. If $\{V_\alpha\}$ is a minimal left approximate identity for A_p , it follows from [6], Lemma 3, that $\lim_\alpha \|V_\alpha(f) - f\|_p = 0$. Thus, for some index α , $\|V_\alpha(f) - f\|_p < \varepsilon/2$ and so $\|V_\alpha(f) - g\|_p < \varepsilon$.

LEMMA 4. Let V be in \mathfrak{B}_p and D a dense subset of L_p such that $V(h*f) = h*V(f)$ for all $h \in C_0$ and $f \in D$. Then V is in M_p .

Proof. Let x be in G . By [4] 20.15, there is a net $\{f_\alpha\}$ in C_0 such that $\lim_\alpha \|{}_x h - f_\alpha * h\|_p = 0$ for all $h \in L_p$. It follows that $\lim_\alpha \|V({}_x h) - V(f_\alpha * h)\|_p = 0$ and $\lim_\alpha \|{}_x V(h) - f_\alpha * V(h)\|_p = 0$. Hence, for $h \in D$

$$\|V({}_x h) - {}_x V(h)\|_p = \lim_\alpha \|V(f_\alpha * h) - f_\alpha * V(h)\|_p = \lim_\alpha 0$$

by the hypothesis for V . Since D is dense in L_p , V is in M_p .

THEOREM 1. Define $\omega: M_p \rightarrow m_p$ by letting $\omega_T(f) = T(f)$ for each $T \in M_p$ and $f \in L_p^t$. Then ω is an isometric isomorphism of M_p into m_p . Furthermore, if T is any operator in m_p , then there exists some $S \in M_p$ such that, for all $V \in A_p$ and $f \in L_p^t$, $\omega_S(V(f)) = T(V(f))$.

Proof. That ω is well-defined follows from Proposition 1. That ω is an isomorphism is evident when it is noted that L_p^t is a dense subset of L_p .

Let T be an arbitrary element of m_p . It follows from Proposition 2 and Lemma 3 that there exists a unique operator S in \mathfrak{B}_p such that $S(V(f)) = T(V(f))$ for all $V \in A_p$ and $f \in L_p^t$. For $h \in C_0$, $V \in A_p$, and $f \in L_p^t$, Proposition 1 implies

$$\begin{aligned} S(h*V(f)) &= S(V(h*f)) = T(V(h*f)) \\ &= T(h*V(f)) = h*T(V(f)) = h*S(V(f)) . \end{aligned}$$

By Lemmas 3 and 4, this implies that S is in M_p . Consequently, $\omega_s(V(h)) = S(V(h)) = T(V(h))$ for all $h \in L_p^t$ and $V \in A_p$.

To complete this proof, it will now suffice to show that ω is an isometry. Let T be in M_p . Let f be in L_p^t and ε a positive number. Choose $g \in L_p^t$ for which $\|g\|_p \leq 1$ and $\|\omega_T(f)\|_p^t < \|g*\omega_T(f)\|_p + \varepsilon$. By Proposition 1.iii, $T(g*f) = g*T(f)$; this means that $T \circ W_f(g) = g*\omega_T(f)$. Hence,

$$\|\omega_T(f)\|_p^t < \|T \circ W_f(g)\| + \varepsilon \leq \|T\| \cdot \|W_f\| + \varepsilon .$$

By (8), this implies $\|\omega_T(f)\|_p^t \leq \|T\| \cdot \|f\|_p^t$. Hence

$$\|\omega_T\| \leq \|T\| .$$

On the other hand, Proposition 2 and Lemma 3 imply

$$\begin{aligned} \|T\| &= \sup \{ \|T(V(h))\|_p : V \in A_p, h \in L_p^t, \|V(h)\|_p \leq 1 \} \\ &= \sup \{ \|\omega_T(V(h))\|_p : V \in A_p, h \in L_p^t, \|V(h)\|_p \leq 1 \} \leq \|\omega_T\| . \end{aligned}$$

This proves that $\|T\| = \|\omega_T\|$.

THEOREM 2. *Let ω be as in Theorem 1 and G be either commutative or compact. Then ω is surjective.*

Proof. Let T be any operator in m_p . By Theorem 1, there is an operator S in M_p for which $T(V(f)) = \omega_s(V(f))$ for all $V \in A_p$ and $f \in L_p^t$.

If G is compact, then $L_p^t = L_p$. It follows from the Hewitt-Curtis-Figa Talamanca factorization theorem ([5] 32.22) that each $h \in L_p^t$ is of the form $V(f)$ for some $V \in A_p$ and $f \in L_p^t$. Hence, $T = \omega_s$.

Suppose now that G is commutative (not necessarily compact). Assume that there existed $h \in L_p^t$ such that $\omega_s(h) \neq T(h)$. Then Lemma 1 implies that $g*(\omega_s - T)(h) \neq 0$ for some $g \in C_{00}$. Let $\{h_n\}$ be a sequence in C_{00} for which $\lim_n \|h_n - h\|_p = 0$. Then

$$\begin{aligned} &\|g*(\omega_s - T)(h)\|_p \\ &= \|(\omega_s - T)(g*h)\|_p = \|(\omega_s - T)(h*g)\|_p \\ &= \|h*(\omega_s - T)(g)\|_p = \lim \|h_n*(\omega_s - T)(g)\|_p \\ &= \|\lim_n \|(\omega_s - T)(h_n*g)\|_p = \lim_n \|(\omega_s - T)(W_{h_n}(g))\|_p = 0 \end{aligned}$$

a contradiction. Thus, $\omega_s = T$.

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