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**VARIETIES OF IMPLICATIVE SEMI-LATTICES. II**

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## VARIETIES OF IMPLICATIVE SEMI-LATTICES II

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**This paper is concerned with a process of coordinatization of the lattice of varieties of implicative semilattices. Equational descriptions of some elements in each coordinate class, and a complete equational description of one coordinate class are given.**

1. **Introduction.** This paper is a continuation of [8]. Familiarity with [8] and [6] is assumed. After stating some of the consequences of the local finiteness of the variety of implicative semilattices, we describe a system for partitioning the lattice of varieties of implicative semi-lattices into coordinate intervals, and give some results that can be obtained from a study of this coordinatization. Finally, we give equational descriptions for the largest and smallest varieties in each coordinate class, the covers of the smallest variety in each coordinate class and a complete equational description of the coordinate class 4,2.

Recall that an implicative semi-lattice is subdirectly irreducible if and only if it has a single dual atom. In accordance with the usage of [8], this dual atom will be denoted by  $u$ . If in a subdirectly irreducible implicative semi-lattice, the dual atom is deleted, the remaining structure is both a subalgebra and a homomorphic image of the original. Thus every subdirectly irreducible implicative semi-lattice may be thought of as obtained by appending a single dual atom to some already given implicative semi-lattice. If  $L$  is an implicative semilattice, the subdirectly irreducible implicative semi-lattice obtained in this manner will be denoted by  $\hat{L}$ .

2. **Local finiteness.** The following theorem was proven first by A. Diego [2] in a slightly different context. McKay [4] extended the result to implicative semi-lattices. We present a much simpler proof here.

**THEOREM 2.1.** *The variety of implicative semi-lattices is locally finite.*

*Proof.* Let  $F_n$  denote the free implicative semi-lattice on  $n$  generators. The proof proceeds by induction.  $F_1$  has two elements. Assume that  $F_n$  is finite.  $F_{n+1} \leq_s \coprod \hat{L}_i$ , where each  $\hat{L}_i$  is  $n+1$  generated. Hence each  $L_i$  is  $n$  generated. It follows from the induction assumption that there are only a finite number of distinct  $L_i$  each

of which is finite. Therefore the same statement applies to the  $\hat{L}_i$ , and hence  $F_{n+1}$  is finite.

**COROLLARY 2.2.** *Every variety of implicative semi-lattices is generated by its finite sub-directly irreducible members.*

**COROLLARY 2.3.** *If  $f$  is a homomorphism of an implicative semi-lattice  $L$  onto a finite implicative semi-lattice  $M$ , then there exists  $L' \leq_s L$  such that  $f|_{L'}$  is an isomorphism.*

**COROLLARY 2.4.** *The lattice of all varieties of implicative semi-lattices is itself implicative.*

**COROLLARY 2.5.** *If  $L$  is a finite subdirectly irreducible implicative semi-lattice, then the class of all those implicative semi-lattices which do not contain a sub-implicative semi-lattice isomorphic to  $L$  is a variety.*

**3. Coordinates of varieties.** In this section,  $A$  will denote a subdirectly irreducible implicative semi-lattice. Also the term "algebra" will be used in place of "implicative semi-lattice". Let  $\mathcal{C}_n$  denote the variety generated by  $C_n$ , the  $n$  chain, and  $\mathcal{B}_n$  denote the variety generated by  $\hat{B}_n$ , where  $B_n$  is the Boolean algebra with  $n$  atoms. Let  $\overline{\mathcal{C}}_n$  denote the variety of all algebras which do not have  $n + 1$  chains as sub-algebras, and similarly let  $\overline{\mathcal{B}}_n$  denote the variety of all algebras which do not have sub-algebras isomorphic to  $\hat{B}_{n+1}$ . (Throughout  $n$  and  $m$  will denote natural numbers.) Let  $W_{n,m} = \mathcal{C}_n \vee \mathcal{B}_m$ , and  $V_{n,m} = \overline{\mathcal{C}}_n \cap \overline{\mathcal{B}}_m$ . We say that a variety has coordinates  $n, m$  if it is in the interval  $[W_{n,m}, V_{n,m}]$ .

**LEMMA 3.1.** *If  $A \in V_{n,m}$ , and if  $A$  is finite, then  $|A| \leq 2^{m(n-3)}(2^m + 1)$ , where  $|A|$  denotes the number of elements in  $A$ .*

*Proof.* Since  $A$  is subdirectly irreducible and does not contain  $\hat{B}_{m+1}$  as a subalgebra,  $A$  cannot contain  $B_{m+1}$ . Thus the closed algebra of  $A$  has at most  $m$  atoms. The proof now proceeds by induction. The case  $n = 3$  holds since  $A \in V_{3,m}$  implies  $A = \hat{B}_l$  for some  $l \leq m$ . Assume that the proposition holds for some  $n$ , and let  $A \in V_{n+1,m}$ . Then the dense filter  $D$  of  $A$  is an element of  $V_{n,m}$ . Thus  $|D| \leq 2^{m(n-3)}(2^m + 1)$ . The proposition follows for the  $n + 1$  case since every element of  $A$  is the meet of a closed element and a dense element.

**COROLLARY 3.2.**  *$V_{n,m}$  contains only a finite number of distinct finite subdirectly irreducible algebras.*

**THEOREM 3.3.**  $V_{n,m}$  contains no infinite subdirectly irreducible algebras.

*Proof.* Assume the contrary, and let  $n$  be the least integer for which there is an  $m$  such that  $V_{n,m}$  has an infinite subdirectly irreducible algebra,  $A$ . Now  $A$  is unbounded, since if  $A$  were bounded, the dense filter of  $A$  would be an infinite subdirectly irreducible algebra in  $V_{n-1,m}$ . This reasoning also shows that any principal filter of  $A$  is bounded in size by the bound of Lemma 3.1, and this in turn implies that  $A$  is bounded, which establishes a contradiction.

**COROLLARY 3.4.** If  $V$  is a variety of implicative semi-lattices, then the following are equivalent:

- (i)  $V$  has only finitely many subvarieties.
- (ii)  $V$  is generated by a finite algebra.
- (iii)  $V$  has coordinates  $n, m$  for some natural numbers  $n$  and  $m$ .

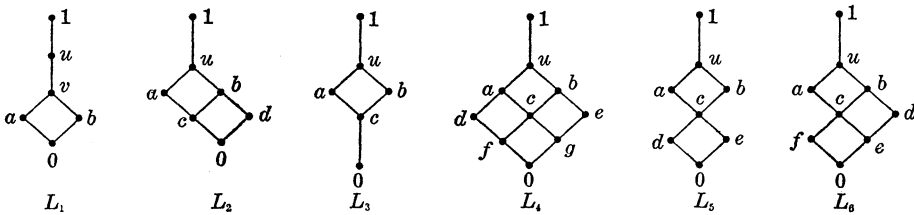


FIGURE 1

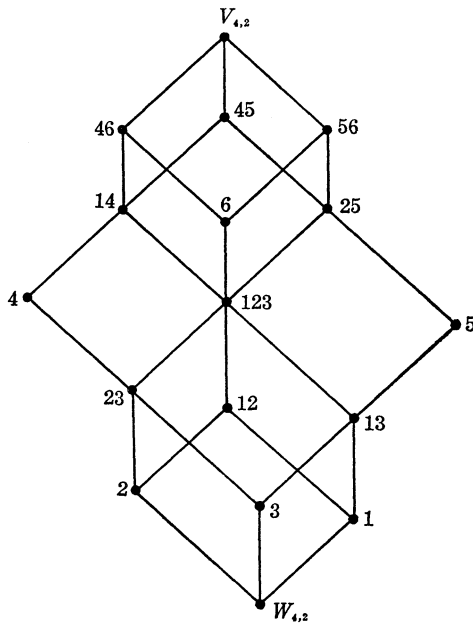


FIGURE 2

In order for  $A$  to be in  $V_{4,2}$ , the closed algebra of  $A$  must be  $B_1$  or  $B_2$ , and the dense filter of  $A$  must be  $\hat{B}_2, C_2$  or  $C_3$ . In [6] a method is given for constructing all algebras having a given closed algebra and a given dense filter. We omit the details, but using this process one finds that the subdirectly irreducible members of  $V_{4,2} - W_{4,2}$  are those shown in Figure 1. We have  $L_1 \leq_s L_5, L_2 \leq_s L_5; L_2, L_3 \leq_s L_4; L_1, L_2, L_3 \leq_s L_6$ ; and these are the only subalgebra relations holding among these six algebras. Thus the interval  $[W_{4,2}, V_{4,2}]$  is as pictured in Figure 2, where the numbers beside a point in the lattice correspond to the indices of the algebras which generate that variety.

For  $n \leq 4$  and  $m \leq 2$ , it is clear that the varieties  $W_{n+1,m}, W_{n,m+1}$ , and  $W_{n,m} \vee \{L_i\}^e$  for  $i = 1, 2, 3$  cover  $W_{n,m}$ . ( $\{L\}^e$  is the variety generated by  $L$ .) It is also clear that any other cover of  $W_{n,m}$  would have to be a subvariety of  $V_{n,m}$ . We now show that there are no additional covers of  $W_{n,m}$ .

**DEFINITION 3.5.** For  $B, D \leq_s L$ , we say  $B$  is fixed with respect to  $D$  if  $d*b = b$  for  $b \in B$ , and  $d \in D$ . We say that  $D$  is total with respect to  $B$  if  $b*d \in D$  for  $b \in B, d \in D$ . Let  $B \nabla D = \{b \wedge d \mid b \in B, d \in D\}$ .

It was shown in [5] that  $B \nabla D$  is a subalgebra of  $L$  if  $B$  is fixed with respect to  $D$  and  $D$  is total with respect to  $B$ .

**THEOREM 3.6.** *If  $L$  is a subdirectly irreducible implicative semi-lattice, and if  $C_4 \leq_s L$ , then either  $L$  is a chain or  $L_i \leq_s L$  for some  $i = 1, 2, 3$ .*

*Proof.* First, consider the case where  $L$  is bounded. If the dense filter of  $L$  is not a chain, then it contains  $\hat{B}_2$  as a subalgebra, and thus  $L_3 \leq_s L$ . Hence, we may assume that the dense filter of  $L$  is a chain. If the closed algebra of  $L$  is simple, then  $L$  is also a chain. Therefore we may assume that the closed algebra of  $L$  contains a subalgebra  $\{1, b, b', 0\}$ , where  $b'$  is the complement of  $b$  in the closed algebra. Now either  $b*d = 1$  for every dense element  $d$ , or there is a dense element  $d < 1$  such that  $b*d = d$ . If  $b*d = d$ , then  $b'*d = 1$ . Thus in either case, we have a subalgebra  $D = \{1, u, d\}$  of the dense filter of  $L$  such that  $B$  is fixed with respect to  $D$  and  $D$  is total with respect to  $B$ . Hence  $B \nabla D \leq_s L$ . We may assume that  $b' \leq d$ . If  $b \leq d$ , then  $B \nabla D = L_1$ . If  $b \not\leq d$ , then  $B \nabla D = L_2$ . Now suppose that  $L$  is not bounded and that  $L_i \not\leq_s L$  for any  $i = 1, 2, 3$ . Let  $a, b \in L$ , and let  $d$  be the least element of some example of  $C_4$  in  $L$ . Then from consideration of the bounded case, it follows that the principal filter generated by  $a \wedge b \wedge d$  is a chain. Thus  $a$  and  $b$  are comparable and so  $L$  is a chain.

COROLLARY 3.7. For  $n \geq 4$  and  $m \geq 2$ ,  $W_{n,m}$  has exactly five covers.

COROLLARY 3.8.  $\mathcal{E}_n \vee \overline{\mathcal{E}_3}$  and  $\mathcal{B}_m \vee \overline{\mathcal{B}_2}$  have exactly three covers.

4. Identities. If  $g(x_1, \dots, x_n)$  is an implicative semi-lattice term and if  $L$  is an implicative semi-lattice, then we say that  $g(x_1, \dots, x_n)$  holds in  $L$ , or simply that  $g$  holds in  $L$ , provided the equation  $g(x_1, \dots, x_n) = 1$  holds in  $L$ . If this is not the case we say that  $g$  fails in  $L$ . We let  $V(g)$  denote the variety of all implicative semi-lattices in which  $g$  holds. We are interested here only in subdirectly irreducible implicative semi-lattices, and we let  $u$  denote the dual atom in any such algebra. If there exist elements  $a_1, \dots, a_n \in L$  such that  $g(a_1, \dots, a_n) = u$ , then we say that  $g$   $u$ -fails in  $L$ . If  $g$   $u$ -fails in every subdirectly irreducible algebra in which it fails, then we say that  $g$  has property  $U$ .

We let  $a + b$  denote the psuedo-join (see [7]) of the elements  $a$  and  $b$  (i.e.  $a + b = ((a * b) * b) \wedge ((b * a) * a)$ ). In general this is not an associative operation, and when not indicated otherwise, we intend for the grouping to be to the left (i.e.  $a + b + c = (a + b) + c$ ). If  $a$  and  $b$  are comparable elements, then  $a + b$  is the larger of the two.

LEMMA 4.1. If  $a_1 \geq a_i$  for  $i = 2, \dots, n$ , then

$$a_1 + a_2 + \dots + a_n = a_1 .$$

We should note that this lemma depends on our convention of association.

DEFINITION 4.2. If  $g_1(x_1, \dots, x_n)$  and  $g_2(x_1, \dots, x_m)$  are terms, then we let

$$(g_1 \oplus g_2)(x_1, \dots, x_{n+m}) = g_1(x_1, \dots, x_n) + g_2(x_{n+1}, \dots, x_{n+m})$$

and

$$(g_1 \wedge g_2)(x_1, \dots, x_{n+m}) = g_1(x_1, \dots, x_n) \wedge g_2(x_{n+1}, \dots, x_{n+m}) .$$

LEMMA 4.3. If  $g_1$   $u$ -fails in  $L$  and if  $g_2$  fails in  $L$ , then  $g_1 \oplus g_2$   $u$ -fails in  $L$ . Thus if  $g_1$  has property  $U$ , then so does  $g_1 \oplus g_2$ .

LEMMA 4.4. If  $g_1$  has property  $U$ , then  $V(g_1) \vee V(g_2) = V(g_1 \oplus g_2)$ .

*Proof.* By [2, Lemma 4.1] any subdirectly irreducible member,  $L$ , of  $V(g_1) \vee V(g_2)$  is in  $V(g_1) \cup V(g_2)$ . Thus  $g_1$  holds in  $L$  or  $g_2$  holds in  $L$ . Hence  $g_1 \oplus g_2$  holds in  $L$ .

On the other hand, if  $L$  is any subdirectly irreducible not in

$V(g_1) \vee V(g_2)$ , then  $g_1$  and  $g_2$  both fail in  $L$ . Thus  $g_1$   $u$ -fails in  $L$ ; so  $g_1 \oplus g_2$  fails in  $L$ .

LEMMA 4.5.  $V(g_1) \wedge V(g_2) = V(g_1 \wedge g_2)$ . Furthermore, if  $g_1$  and  $g_2$  both have property  $U$ , then so does  $g_1 \wedge g_2$ .

The main idea in the following theorem is present in a similar theorem for Heyting algebras due to Alan Day [1].

THEOREM 4.6. Letting  $t^*$  denote  $t^*(x_1 \wedge \dots \wedge x_{n+1})$  and  $l_{ij}$  denote  $x_i^{**} * x_j^{**}$ , we have

$$\overline{\mathcal{B}}_n = V(P_n)$$

where

$$P_n(x_1, \dots, x_{n+2}) = x_{n+2} + l_{12} + l_{21} + \dots + l_{n+1,n}$$

where each  $l_{ij}$  with  $i \neq j$  and  $i, j \leq n + 1$  occurs exactly once. Also,  $P_n$  has property  $U$ .

*Proof.* Let  $a_1, \dots, a_{n+1}$  be the atoms of  $\hat{B}_{n+1}$ . Then  $a_i^{**} = a_i$  and  $a_i^{**} * a_j^{**} < 1$  if  $i \neq j$ . Thus  $P_n(a_1, \dots, a_{n+1}, u) = u$ . Hence  $V(P_n) \subseteq \overline{\mathcal{B}}_n$ .

Suppose now that  $L$  is any subdirectly irreducible member of  $\overline{\mathcal{B}}_n$  and that  $P_n(a_1, \dots, a_{n+2}) < 1$  in  $L$ . Then  $a_1^{**}, \dots, a_{n+1}^{**}$  are pairwise incomparable closed elements in the principal filter generated by  $a_1 \wedge \dots \wedge a_{n+1}$ . Thus  $\hat{B}_{n+1} \leq_s L$ , a contradiction. Hence  $P_n$  holds in  $L$ .

In [8] terms were given which characterize the varieties  $\mathcal{C}_n$  and  $\overline{\mathcal{C}}_n$ . Denote these terms by  $q_n$  and  $r_n$ , respectively. It is easy to see that  $q_n$  and  $r_n$  have property  $U$ .

COROLLARY 4.7.  $V_{n,m} = V(P_m \wedge r_n)$ . In particular,  $\mathcal{B}_m = V(P_m \wedge r_3)$ .

COROLLARY 4.8.  $W_{n,m} = V(q_n \oplus (P_m \wedge r_3))$ .

We now turn our attention to the varieties of the interval  $[W_{4,2}, V_{4,2}]$ . First we shall give an indexed list of identities which can be used to describe these varieties. Note that for a term  $t$ ,  $t^*$  is as defined in Theorem 4.6.

$$\begin{aligned}
 g_1 &= x_4 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_3)) + (x_1 * x_2) + (x_2 * x_1) \\
 g_{12} &= x_4 + (x_1 * x_2) + (x_2 * x_1) + (x_1 \wedge x_2) * + (x_1^{**} * x_1) + (x_2^{**} * x_2) \\
 g_{23} &= x_4 + (x_4 * x_3) + (x_1 * x_2) + (x_2 * x_1) + (x_3 + (x_3 * x_1)) + (x_3 + (x_3 * x_2)) \\
 g_2 &= g_{12} \wedge g_{23} \\
 g_3 &= x_4 + (x_4 * x_3) + (x_1 * x_2) + (x_2 * x_1) + (x_3 + (x_3 * (x_1 \wedge x_2))) \\
 g_{13} &= g_1 \oplus g_3 \\
 g_{123} &= g_{12} \oplus g_3 \\
 g_4 &= x_4 + (x_4 * x_3) + ((x_3 \wedge x_1) * (x_3 \wedge x_2)) + ((x_3 \wedge x_2) * (x_3 \wedge x_1)) \\
 &\quad + ((x_3 + (x_3 * (x_3 \wedge x_1))) + (x_3 + (x_3 * (x_3 \wedge x_2)))) \\
 g_{14} &= g_1 \oplus g_4 \\
 g_5 &= x_4 + (x_1 * x_2) + (x_2 * x_1) + (x_1 * x_3) + (x_3 * x_1) + (x_2 * x_3) \\
 g_{25} &= g_2 \oplus g_5 \\
 g_{45} &= g_4 \oplus g_5 \\
 g_{56} &= x_4 + (x_1 * x_2) + (x_2 * x_1) + (x_1 * x_3) + (x_3 * x_1) + (x_2 * x_3) + (x_3 * x_2) \\
 g_{46} &= x_5 + (x_1 * x_2) + (x_2 * x_1) + (x_1 \wedge x_2 \wedge x_3) * (x_1 \wedge x_2 \wedge x_4) \\
 &\quad + (x_1 \wedge x_2 \wedge x_4) * (x_1 \wedge x_2 \wedge x_3) \\
 &\quad + (x_1 + ((x_1 \wedge x_2) + (x_1 \wedge x_2 \wedge x_3))) \\
 &\quad + (x_1 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_4))) \\
 &\quad + (x_2 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_3))) \\
 &\quad + (x_2 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_4))) .
 \end{aligned}$$

**THEOREM 4.9.** For  $i, j = 1, \dots, 6$  let  $h_i = g_i \wedge P_4 \wedge r_3$ ,  $h_{ij} = g_{ij} \wedge P_4 \wedge r_3$ ,  $h_{123} = g_{123} \wedge P_4 \wedge r_3$ .

Then

- (i)  $\{L_i\}^e = V(h_i)$
- (ii)  $\{L_i, L_j\}^e = V(h_{ij})$  for  $\{i, j\} = \{1, 3\}, \{1, 2\}, \{2, 3\}, \{1, 4\}, \{2, 5\}, \{4, 5\}, \{4, 6\}$ , and  $\{5, 6\}$ .
- (iii)  $\{L_1, L_2, L_3\}^e = V(h_{123})$ .

**COROLLARY 4.10.** For  $i, j$  as in the previous theorem and  $n > 4$ ,  $m > 2$  we have

- (i)  $\{L_i\}^e \vee W_{n,m} = V(h_i \oplus (q_n \oplus (P_m \wedge r_3)))$ ,
- (ii)  $\{L_i, L_j\}^e \vee W_{n,m} = V(h_{ij} \oplus (q_n \oplus (P_m \wedge r_3)))$ ,
- (iii)  $\{L_1, L_2, L_3\}^e \vee W_{n,m} = V(h_{123} \oplus (q_n \oplus (P_m \wedge r_3)))$ .

In some cases the identities given can be simplified somewhat, but these were chosen for convenience in presentation.

*Proof.* The proof amounts to showing that each of the indexed polynomials  $g$  is valid in the corresponding variety of the diagram



of figure 2 and its subvarieties, and that it fails elsewhere in the diagram. Note that each of these identities has property  $U$ . We shall establish the validity of three of the more complicated identities only.

(1)  $g_{12}$  holds in  $L_1$  and  $L_2$ , but fails in  $L_3$ : If  $g_{12}(a_1, \dots, a_4) < 1$  in  $L_1$ , then  $a_1$  and  $a_2$  are incomparable and  $(a_1 \wedge a_2)^* = 1$ , a contradiction.

If  $g_{12}(a_1, \dots, a_4) < 1$  in  $L_2$ , then we must have  $\{a_1, a_2\} = \{a, b\}$  and  $a_1 \wedge a_2 \wedge a_3 = 0$ . However,  $a^{***}a = 1$  then yields a contradiction.

In  $L_3$  we have

$$g_{12}(a, b, 0, u) = u + b + a + (c*0) + (1*a) + (1*b) = u.$$

(2)  $g_4$  holds in  $L_4$  but fails in  $L_1$ : In  $L_1$  we have  $g_4(a, b, v, u) = u + v + b + a + ((u + a) + (v + b)) = u$ .

If  $g_4(a_1, \dots, a_4) < 1$  in  $L_4$ , then  $a_3 < u$ . In fact  $a_3 = a, b$ , or  $c$  since there must be a pair of incomparable elements below  $a_3$ . If  $a_3 = a$  we have  $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, c\}, \{d, g\}$ , or  $\{f, g\}$ . If  $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, c\}$ , then  $a_3 + (a_3*c) = a + b = 1$ . If  $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, g\}$ , then  $a_3 + (a_3*g) = a + e = 1$ . If  $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{f, g\}$ , then we get the same contradiction as in the preceding case. The case  $a_3 = b$  is completely analogous. If  $a_3 = c$ , then  $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{f, g\}$ . Then  $(a_3 + (a_3*f)) + (a_3 + (a_3*g)) = (c + d) + (c + e) = a + b = 1$ , a contradiction.

(3)  $g_{46}$  holds in  $L_4$  and  $L_6$ , but fails in  $L_5$ : If  $g_{46}(a_1, \dots, a_6) < 1$ , then  $a_1$  and  $a_2$  are incomparable and there must be a pair of incomparable elements,  $a_1 \wedge a_2 \wedge a_3$  and  $a_1 \wedge a_2 \wedge a_4$ , which are less than  $a_1 \wedge a_2$ . Thus in  $L_4$  we would have to have  $\{a_1, a_2\} = \{a, b\}$  and  $\{a_1 \wedge a_2 \wedge a_3, a_1 \wedge a_2 \wedge a_4\} = \{f, g\}$ . However, we have  $a + ((a \wedge b)*g) = 1$  which would give a contradiction. In  $L_6$  we would have to have  $\{a_1, a_2\} = \{a, b\}$  and  $\{a_1 \wedge a_2 \wedge a_3, a_1 \wedge a_2 \wedge a_4\} = \{f, e\}$ . This would lead to a contradiction, however, since  $a + ((a \wedge b)*e) = 1$ .

In  $L_5$  we have

$$g_{46}(a, b, d, e, u) = u + b + a + e + d \\ + (a + d) + (a + e) + (b + d) + (b + e) = u.$$

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